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Weak Convergence to Fractional Brownian Motion and to the Rosenblatt Process*

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1. Introduction

In this paper we determine the limit under weak convergence of certain normalized partial sums of stationary random variables that exhibit long run non-periodic dependence. The dependence structure of geophysical phenomena (Mandelbrot and Van Ness (1968)) motivated this research. Because of the relatively strong dependence, the limit process is never Brownian motion, nor is it necessarily Gaussian.

Let $\{X_i, i=1,2,\dots\}$ be a stationary Gaussian sequence with $EX_i=0$ and $EX_i^2=1$. Let $G(X_i)$ have mean 0 and finite variance. We study the weak limit, as $N\to\infty$, of the random functions

$$Z_N(t)=\frac{1}{d_N}\sum_{i=1}^{[Nt]} G(X_i)$$

where $0\leqq t\leqq 1$ and d_N^2 is asymptotically proportional to $\operatorname{Var}\sum_{i=1}^{N} G(X_i)$. The weak convergence is understood to hold in $\mathscr{D}([0,1])$, the space of all functions on $[0,1]$ whose discontinuities are at most of the first kind.

In order for $Z_N(t)$ to converge weakly to a non-degenerate limit $\bar{Z}(t)$ which is continuous in probability, it is necessary that $d_N^2\sim N^{2H}L(N)$ as $N\to\infty$ for some constant H and some slowly varying function at infinity L (Lamperti (1962)). Here, the symbol $\sim$ denotes asymptotic equivalence.

The case of independent $G(X_i)$ is well known. In this instance $H=\frac{1}{2}$ and $Z_N(t)$ converges weakly to Brownian motion (Donsker's theorem). Sun (1965) has shown that the random variable $Z_N(1)$ remains asymptotically normal when $\{X_i\}$ has an absolutely continuous spectrum,

$$\sum_{k=0}^{\infty}(EX_1X_{1+k})^2<\infty \quad \text{and} \quad \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}EX_iX_j$$

is finite. Notice that the random variables $G(X_i)$ are allowed to be dependent but H is still equal to $\frac{1}{2}$.

We shall focus on values of H satisfying $\frac{1}{2}<H<1$. We assume $EX_iX_{i+k}\sim k^{-D}L(k)$ as $k\to\infty$ for some slowly varying function L and some constant $D>0$. $H>\frac{1}{2}$ arises when $D<\frac{1}{m}$, where m, the *Hermite rank* of G, is the index of the first

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non zero coefficient in the Hermite polynomials expansion of $G(X_i)$ (see Section 3). Under these assumptions, $\sum_{k=1}^{\infty} EG(X_i)\,G(X_{i+k})=\infty$ and the sequence $\{G(X_i)\}$ is so strongly dependent that the limit of $Z_N(t)$ may not be Gaussian. That limit, when it exists, is an a.s. continuous process depending essentially on m (Section 2). To determine this limiting process it is sufficient to study the convergence of the finite-dimensional distributions of $Z_N(t)$, when $G(X_i)=H_m(X_i)$, where H_m denotes the Hermite polynomial of order m (Section 4).

When $m=1$ (Section 5), $Z_N(t)$ converges weakly to the fractional Brownian motion process $B_H(t)$ with parameter $\frac{1}{2}<H=1-\dfrac{D}{2}<1$. This limiting process is Gaussian with zero mean and $E\,|B_H(t_2)-B_H(t_1)|^2=|t_2-t_1|^{2H}$. The process is defined for $0<H<1$. It is Brownian motion when $H=\frac{1}{2}$. For a detailed treatment of $B_H(t)$, see Mandelbrot and Van Ness (1968).

When $m=2$ (Section 6), $Z_N(t)$ converges weakly to the non Gaussian "Rosenblatt process". This fact extends a result obtained by Rosenblatt (1961).

Partial results for $m\geqq 3$ are given in Taqqu (1972). The limiting moments of $Z_N(t)$ are determined there for all fixed $m\geqq 3$. They are finite and not those of a Gaussian process. Whether they characterize a unique process $\bar{Z}_m(t)$ is still an open problem. Of special interest would be the representation of (one of the possible) $\bar{Z}_m(t)$ in terms of $B_H(t)$ or of Brownian motion. For a suggestion, see Taqqu (1972).

Weak convergence of $Z_N(t)$ has direct practical applications. It validates one of the uses of the statistic "R/S" introduced by Hurst and developed by Mandelbrot for investigating long run non-periodic statistical dependence of time series. The theory of R/S is described in Mandelbrot (1975). Applications arise in hydrology (Mandelbrot and Wallis (1968), (1969), Taqqu (1970)), in geophysics (Mandelbrot and McCamy (1970)), and in economics (Mandelbrot (1972)).

Davydov (1970) has studied the polygonal line process obtained by connecting with straight lines the points of discontinuity of

$$Z_N(t)=\frac{1}{d_N}\sum_{i=1}^{[Nt]}\sum_{k=-\infty}^{+\infty} c_{k-i}\,\xi_k.$$

Here $\{\xi_k\}$ is a sequence of independent, identically distributed random variables with 0 mean, and the c_k satisfy $\sum_{k=-\infty}^{+\infty} c_k^2<\infty$. When $E\,|\xi_k|^{2r}<\infty$, $r\geqq 2$, and the c_k are such that

$$\operatorname{Var}\left(\sum_{i=1}^{N}\sum_{k=-\infty}^{+\infty} c_{k-i}\,\xi_k\right)\sim N^{2H}L(N)\quad \text{as } N\to\infty,\quad \text{with } \frac{1}{r+2}<H<1,$$

the polygonal line process induced by $Z_N(t)$ converges weakly to $B_H(t)$ as $N\to\infty$. To prove that the process $Z_N(t)$ itself converges weakly to $B_H(t)$, tranpose Davydov's argument to $\mathscr{D}([0,1])$. Recently, Gisselquist (1973), extending results of Spitzer (1969), proved that the finite-dimensional distributions of some collision process converge to those of $B_H(t)$ with $\frac{1}{4}\leqq H\leqq\frac{1}{2}$. Note that the increments of $B_H(t)$ are negatively correlated when $0<H<\frac{1}{2}$, are independent when $H=\frac{1}{2}$, and are positively correlated when $\frac{1}{2}<H<1$. In Section 5 we give an example of a sequence $Z_N(t)$ converging to $B_H(t)$ for $0<H<\frac{1}{2}$.

2. Sufficient Conditions for Weak Convergence

The purpose of this section is to state sufficient conditions for a sequence $Z_N(t)$ of random functions of $\mathscr{D}([0,1])$ to converge weakly as $N \to \infty$ to a process $\bar{Z}(t)$ endowed with the following properties $\Pi(H)$, for some $0 < H \leqq 1$.

Properties $\Pi(H)$ of $\bar{Z}(t)$. 1. $\bar{Z}(0)=0$ a.s.

2. $\bar{Z}(t)$ has strictly stationary increments, that is the random function $M_h(t) = \bar{Z}(t+h) - \bar{Z}(t)$, $h \geqq 0$, is strictly stationary.

3. $\bar{Z}(t)$ is semi-stable of order H, that is

$$P\{\bar{Z}(c\,t_1) \leqq x_1, \bar{Z}(c\,t_2) \leqq x_2, \ldots, \bar{Z}(c\,t_p) \leqq x_p\}$$
$$= P\{c^H \bar{Z}(t_1) \leqq x_1, c^H \bar{Z}(t_2) \leqq x_2, \ldots, c^H \bar{Z}(t_p) \leqq x_p\}.$$

4. $E\bar{Z}(t)=0$ and $E|\bar{Z}(t)|^\gamma < \infty$ for $\gamma \leqq \frac{1}{H}$.

5. $\bar{Z}(t)$ is separable and a.s. continuous.

Remarks. Brownian motion starting at 0 is endowed with properties $\Pi(\frac{1}{2})$. The concept of semi-stability was introduced by Lamperti (1962). Mandelbrot and Van Ness (1968) call it "self similarity" when it appears in conjunction with stationary increments, as it does here.

Conventions. 1. A regularly varying function $N^\rho L(N)$ of the integer N with positive exponent ρ is assumed to possess a regularly varying extension $x^\rho L(x)$ defined for all $x \geqq 0$, which is bounded on finite intervals.

2. Empty sums are equal to 0.

The main result of this section is

Theorem 2.1. *Suppose the sequence $Z_N(t)$, $N=1,2,\ldots$ of random functions of $\mathscr{D}([0,1])$ satisfies*

(i) $$Z_N(t) = \frac{S_{[Nt]}}{N^{2H} L(N)}$$

with $0 < H \leqq 1$, L slowly varying, and $S_N = \sum_{i=1}^N Y_i$ for some strictly stationary sequence $\{Y_i\}$ with 0 mean and finite variance;

(ii) $$ES_N^2 = O(N^{2H} L(N)) \quad \text{as } N \to \infty;$$

(iii) $$E|S_N|^{2a} = O((ES_N^2)^a) \quad \text{as } N \to \infty \quad \text{for some } a > \frac{1}{2H};$$

(iv) *the finite-dimensional distributions of $Z_N(t)$ converge as $N \to \infty$.*

Then the sequence $Z_N(t)$ converges weakly as $N \to \infty$ to a process $\bar{Z}(t)$ endowed with the properties $\Pi(H)$ and whose finite-dimensional distributions are the limit of those of $Z_N(t)$. Furthermore, $E|\bar{Z}(t)|^\gamma < \infty$ for $\gamma < 2a$.

Remark. When $\frac{1}{2} < H \leqq 1$, the choice $a=1$ satisfies condition (iii).

The rest of this section is devoted to the proof of Theorem 2.1. We first prove the following lemma which extends a result of Davydov (1970).

Lemma 2.1. *A sequence* $Z_N(t)$, $N=1,2,\dots$ *of* $\mathscr{D}([0,1])$ *that satisfies condition* (i), (ii) *and* (iii) *of Theorem* 2.1 *is tight.*

Proof. Let $1 \geqq t_2 \geqq t \geqq t_1 \geqq 0$, $a > \frac{1}{2H}$ and

$$J_N(a, t_2, t, t_1) = E\,|Z_N(t_2) - Z_N(t)|^a\,|Z_N(t) - Z_N(t_1)|^a.$$

Using Schwarz inequality and the stationary of the Y_i,

$$J_N(a, t_2, t, t_1) \leqq \frac{1}{(N^{2H} L(N))^a} (E\,|S_{[Nt_2]-[Nt]}|^{2a})^{\frac{1}{2}} (E\,|S_{[Nt]-[Nt_1]}|^{2a})^{\frac{1}{2}}.$$

Introduce the regularly varying function $U(x) = x^{Ha} L^{a/2}(x)$ and let $\alpha_N(t_2, t_1) = \frac{[Nt_2]-[Nt_1]}{N}$.

By hypothesis (ii) and (iii) there is a positive constant K and an integer N_1 such that for $N > N_1$,

$$J_N(a, t_2, t, t_1) \leqq K \frac{U(N\alpha_N(t_2, t))}{U(N)} \frac{U(N\alpha_N(t, t_1))}{U(N)}.$$

Since the exponent Ha is positive and $U(x)$ is bounded in finite intervals,

$$\lim_{N\to\infty} \frac{U(N\alpha_N(t_2, t))}{U(N)} = (t_2 - t)^{Ha}$$

holds uniformly on the interval $0 \leqq \alpha_N \leqq 1$ (see de Haan (1970), p. 21). It follows that for some $N_2 > N_1$ and some positive constant C,

$$J_N(a, t_2, t, t_1) \leqq C^{2Ha}(t_2 - t)^{Ha}(t - t_1)^{Ha} \leqq (C\,t_2 - C\,t_1)^{2Ha}.$$

Notice that the exponent Ha is greater than $\frac{1}{2}$ because $a > \frac{1}{2H}$. Then by Theorems 15.4 and 15.6 of Billingsley (1968) the random functions $Z_N(t)$, $N=1,2,\dots$ of $\mathscr{D}([0,1])$ are tight. □

Proof of Theorem 2.1. Convergence of the finite-dimensional distributions (condition (iv)) and tightness (preceding lemma) ensure weak convergence of $Z_N(t)$ to some limiting process $\bar{Z}(t)$. (Billingsley (1968), Theorem 15.1, p. 124). Choose a separable version. $\bar{Z}(0)=0$ trivially since $Z_N(0)=0$. The increments of $\bar{Z}(t)$ must be strictly stationary because of the strict stationarity of the sequence $\{Y_i\}$. Semi-stability follows from Lamperti (1962).

We now investigate the asymptotic behavior of the moments of $Z_N(1)$. By condition (ii) and (iii), there is an $a > \frac{1}{2H}$ such that

$$E\,|Z_N(1)|^{2a} \sim \frac{E\,|S_N^{2a}|}{(N^{2H} L(N))^a} = \frac{O((ES_N^2)^a)}{(N^{2H} L(N))^a} = O(1)$$

as $N \to \infty$, and hence $\sup_N E\,|Z_N(1)|^{2a} < \infty$. The sequence $|Z_N(1)|^\gamma$, $N=1,2,\dots$ with $\gamma < 2a$ is uniformly integrable and $E\,|\bar{Z}(1)|^\gamma = \lim_{N\to\infty} E\,|Z_N(1)|^\gamma < \infty$. Let $\gamma_0 = a + \frac{1}{2H}$

and note that $\frac{1}{H}<\gamma_0<2a$. It follows that

$$E|\bar{Z}(t)|^\gamma=t^\gamma E|\bar{Z}(1)|^\gamma<\infty \qquad \text{for } \gamma\leqq\frac{1}{H}<2a.$$

Moreover, $E\bar{Z}(t)=0$ because $EZ_N(1)=0$.

We now prove the a.s. continuity of $\bar{Z}(t)$. A sufficient condition for continuity is that for all $0\leqq t_1, t_2\leqq 1$,

$$E|\bar{Z}(t_2)-\bar{Z}(t_1)|^\gamma\leqq|F(t_2)-F(t_1)|^\alpha$$

holds for some $\gamma\geqq 0$, $\alpha>1$, and some continuous, non decreasing function $F(t)$ (Billingsley (1968) p. 97). The condition is satisfied with $\gamma=\gamma_0$, $\alpha\equiv\gamma_0 H>1$ and $F(t)=Ct$, $C>0$, since by stationarity and semi-stability

$$E|\bar{Z}(t_2)-\bar{Z}(t_1)|^{\gamma_0}=(t_2-t_1)^{\gamma_0 H}E|\bar{Z}(1)|^{\gamma_0}<\infty. \quad \square$$

3. The Hermite Rank *m*

Let $\{X_i\}$ be a normalized stationary Gaussian sequence, and let $r(k)\equiv EX_i X_{i+k}$, $k=1,2,\ldots$, be its correlation kernel.

What conditions are to be imposed on a function G and on the sequence of correlations $r(k)$ in order for Var $\left(\sum_{i=1}^N G(X_i)\right)$ to be asymptotically proportional to $N^{2H}L(N)$ as $N\to\infty$, for $\frac{1}{2}<H<1$? This section provide some answers.

We first introduce the notion of *Hermite rank.*

Let X denote an $N(0,1)$ random variable and define

$$\mathscr{G}=\{G\colon EG(X)=0,\ EG^2(X)<\infty\}.$$

$\mathscr{G}$ is then a subset of

$$\mathbb{L}^2\left(\mathbb{R}^1,\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right)\equiv\left\{G\colon \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}G^2(x)\exp\left(-\frac{x^2}{2}\right)dx<\infty\right\}.$$

The Hermite polynomials

$$H_q(x)=(-1)^q e^{\frac{x^2}{2}}\frac{d^q}{dx^q}e^{-\frac{x^2}{2}}$$

$(q=0,1,2,\ldots)$ form a complete orthogonal system of functions in $\mathbb{L}^2\left(\mathbb{R}^1,\frac{e^{-x^2/2}}{\sqrt{2\pi}}\right)$. They satisfy $EH_l(X)H_q(X)=\delta_{lq}q!$. For any $G\in\mathscr{G}$, introduce

$$J(q)=EG(X)H_q(X).$$

The series $\sum_{q=0}^\infty\frac{J(q)}{q!}H_q(x)$ converges to $G(x)$ in $\mathbb{L}^2\left(\mathbb{R}^1,\frac{e^{-x^2/2}}{\sqrt{2\pi}}\right)$.

Define

$$m=\min_{q=0,1,2,\ldots}(q\colon J(q)\neq 0)$$

and call this number the *Hermite rank* of G. m is always positive since $J(0)=EG(X)=0$.

For *example*, odd powers of X have Hermite rank 1. Even powers of X with their mean subtracted have Hermite rank 2. Usually, an odd G has Hermite rank 1 and an even G with its mean subtracted has Hermite rank 2. The Hermite polynomial H_m has Hermite rank m.

We may now give a preliminary answer to the question stated in the beginning of this section.

Proposition 3.1. *Let $G \in \mathscr{G}$ and suppose that for large k the sequence $r(k)$ is non-negative and monotone decreasing.*

Then, $\operatorname{Var}\left(\sum_{i=1}^{N} G(X_i)\right)$ *is regularly varying with exponent* $2H$, $\frac{1}{2}<H<1$, *as* $N \to \infty$, *if and only if* $r(k)$ *is regularly varying with exponent* $-D=\dfrac{2H-2}{m}$ *as* $k\to\infty$, *where m is the Hermite rank of G.*

The proof follows straightforwardly from Lemma 3.1 and Theorem 3.1 below. Proposition 3.1 is inserted here to motivate the following definitions of the class of functions $\mathscr{G}_m$ and of the class of Gaussian stationary sequences $(m)(D, L(\cdot))$. Notice first that $0<D<\dfrac{1}{m}$ when $\frac{1}{2}<H<1$.

Definition A. The class $\mathscr{G}_m$.

$$\mathscr{G}_m=\{G\colon G\in\mathscr{G},\ G \text{ has Hermite rank } m\}.$$

Hence

$$\mathscr{G}=\mathscr{G}_\infty\cup\mathscr{G}_1\cup\mathscr{G}_2\cup\cdots \quad \text{with } \mathscr{G}_i\cap\mathscr{G}_j=\phi \quad \text{if } i\neq j$$

and where $\mathscr{G}_\infty \equiv \{G(x)\equiv 0\}$.

Definition B. The class $(m)(D, L(\cdot))$.

For any positive integer *m*, $\{X_i\}\in(m)(D,L(\cdot))$ if $r(k)\sim k^{-D}L(k)$ as $k\to\infty$ with $0<D<\dfrac{1}{m}$ and *L* slowly varying.

For *example,*

$$r(k)=\frac{1}{1+|k|^D} \quad \text{and} \quad r(k)=\tfrac{1}{2}\left\{\left||k|+1\right|^{-D}-2|k|^{-D}+\left||k|-1\right|^{-D}\right\} \quad \text{with } D>0$$

provide bona-fide correlation kernels satisfying $r(k)\sim k^{-D}L(k)$ for some *L*.

Notice that $(m_2)(D,L(\cdot))\subset(m_1)(D,L(\cdot))$ for $m_2>m_1$.

It is useful to introduce the following, less restrictive class of sequences $\{X_i\}$.

Definition B′. The class $(m)'(H, L(\cdot))$.

For any positive integer *m*, $\{X_i\}\in(m)'(H,L(\cdot))$ if

(i) $$\lim_{k\to 0} r(k)=0,$$

(ii) $$\sum_{i=1}^{N}\sum_{j=1}^{N}(r(i-j))^m \sim N^{2H}L(N) \qquad \text{as } N\to\infty,$$

(iii) $$\sum_{i=1}^{N}\sum_{j=1}^{N}|r(i-j)|^m = 0\left(N^{2H}L(N)\right) \qquad \text{as } N\to\infty$$

with $\frac{1}{2}<H<1$ and *L* slowly varying.

Notice that assumption (iii) is automatically satisfied when *m* is even.

The following lemma relates definitions *B* and *B′*.

Lemma 3.1.

$$\{X_i\}\in(m)(D, L(\cdot)) \Rightarrow \{X_i\}\in(m)'\left(1-\frac{mD}{2}, \frac{2L^m(\cdot)}{(1-mD)(2-mD)}\right).$$

Conversely, suppose that $r(k)$ *is monotone decreasing for large* k. *Then*

$$\{X_i\}\in(m)'(H, L(\cdot)) \Rightarrow \{X_i\}\in(m)\left(\frac{2-2H}{m}, [H(2H-1)L(\cdot)]^{1/m}\right).$$

Proof.

$$\sum_{i=1}^{N}\sum_{j=1}^{N} r^m(i-j)=r^m(0)+\sum_{s=1}^{N-1}\left[r^m(0)+2\sum_{k=1}^{s} r^m(k)\right].$$

Suppose $\{X_i\}\in(m)(D, L(\cdot))$. Adapt Karamata's theorem (Feller (1971), *p.* 281) to get

$$\sum_{i=1}^{N}\sum_{j=1}^{N} r^m(i-j)\sim\frac{2}{(-mD+1)(-mD+2)}N^{-mD+2}L^m(N)$$

as $N\to\infty$. Furthermore,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}|r(i-j)|^m\sim\sum_{i=1}^{N}\sum_{j=1}^{N} r^m(i-j)$$

as $N\to\infty$ because $|r(k)|\sim r(k)$ as $k\to\infty$ and $(1-mD)(2-mD)>0$.

Conversely, suppose $\{X_i\}\in(m)'(H, L(\cdot))$ and that $r(k)$ is monotone decreasing for large k. Adapt the lemma in Feller (1971), p. 446, to get

$$r^m(k)\sim\frac{2H(2H-1)}{2}k^{2H-2}L(k) \quad \text{as } k\to\infty. \quad \square$$

The adequacy of the class $(m)'(H, L(\cdot))$ is apparent in the next theorem.

Theorem 3.1. *Let* $G\in\mathcal{G}_m$ *for some* $m>1$

1. *If* $\{X_i\}\in(m)'(H, L(\cdot))$, *then*

$$\operatorname{Var}\left(\sum_{i=1}^{N} G(X_i)\right)\sim\frac{J^2(m)}{m!}N^{2H}L(N), \quad N\to\infty,\ \tfrac{1}{2}<H<1 \tag{3.1}$$

where $J(m)=EG(X)H_m(X)$.

2. *If the sequence* $r(k)$ *is non-negative for large* k *and converges as* $k\to\infty$, *then* (3.1) *entails* $\{X_i\}\in(m)'(H, L(\cdot))$.

Proof. Expand $EG(X_i)G(X_j)$ as a power series in $r(i-j)$.

$$EG(X_i)G(X_j)=\sum_{q=m}^{\infty}\frac{J^2(q)}{q!}r^q(i-j)$$

where $J(q)=EG(X)H_q(X)$. (See Rozanov (1967), p. 182.) Also introduce

$$G^*(X_i)=G(X_i)-\frac{J(m)}{m!}H_m(X_i).$$

Then

$$\operatorname{Var}\left(\sum_{i=1}^{N} G(X_i)\right)=\sum_{i=1}^{N}\sum_{j=1}^{N} EG(X_i)G(X_j)=S_1(N)+S_2(N)$$

where

$$S_1(N)=\frac{J^2(m)}{m!}\sum_{i=1}^{N}\sum_{j=1}^{N} r^m(i-j)$$

and

$$S_2(N)=\sum_{i=1}^{N}\sum_{j=1}^{N} EG^*(X_i)G^*(X_j).$$

1. Suppose $\{X_i\}\in(m)'(H, L(\cdot))$. Then

$$S_1(N)\sim\frac{J^2(m)}{m!}N^{2H}L(N)$$

follows from (ii) of definition B'. Hence, to establish (3.1) it is sufficient to prove that $S_2(N)=o(N^{2H}L(N))$ as $N\to\infty$.

$$\sum_{i=1}^{N}\sum_{j=1}^{N}|EG^*(X_i)G^*(X_j)|\leqq\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{q=m+1}^{\infty}\frac{J^2(q)}{q!}|r(i-j)|^q$$
$$\leqq C'\sum_{i=1}^{N}\sum_{j=1}^{N}|r(i-j)|^{m+1}$$

where

$$C'=\sum_{q=m+1}^{\infty}\frac{J^2(q)}{q!}=E(G^*(X))^2<\infty.$$

Since $|r(i-j)|\to 0$ as $|i-j|\to\infty$ ((i) of Definition B'), there exists for arbitrary $\varepsilon>0$, a number $\delta(\varepsilon)>0$ such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N}|r(i-j)|^{m+1}\leqq\sum_{\substack{i=1\\|i-j|\leqq\delta}}^{N}\sum_{j=1}^{N}1+\varepsilon\sum_{i=1}^{N}\sum_{j=1}^{N}|r(i-j)|^m$$
$$\sim C(\varepsilon)N+\varepsilon O(N^{2H}L(N))$$

as $N\to\infty$, for some constant C depending on ε ((iii) of Definition B'). Since $H>\frac{1}{2}$, as $N\to\infty$,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}|EG^*(X_i)G^*(X_j)|=o(N^{2H}L(N)).$$

2. Suppose now that (3.1) holds. We first prove $\lim_{\tau\to\infty} r(k)=0$. Suppose *ab absurdum* that $\lim_{k\to\infty} r(k)=\lambda>0$. Then for $\varepsilon_1>0$, small enough for $\lambda-\varepsilon_1$ to be positive, there is a δ_1 such that $r(k)>\lambda-\varepsilon_1$ for all $k>\delta$.

$$\operatorname{Var}\left(\sum_{i=1}^{N}G(X_i)\right)=\sum_{\substack{i=1\\|i-j|\leqq\delta_1}}^{N}\sum_{j=1}^{N}\sum_{q=m}^{\infty}\frac{J^2(q)}{q!}r^q(i-j)$$
$$+\sum_{\substack{i=1\\|i-j|>\delta_1}}^{N}\sum_{j=1}^{N}\sum_{q=m}^{\infty}\frac{J^2(q)}{q!}r^q(i-j).$$

The first term on the right hand side is $O(N)$. The second term however is greater than

$$\sum_{\substack{i=1\\|i-j|>\delta_1}}^{N}\sum_{j=1}^{N}\sum_{q=m}^{\infty}\frac{J^2(q)}{q!}(\lambda-\varepsilon_1)^q \geqq \frac{J^2(m)}{m!}(\lambda-\varepsilon_1)^m \sum_{\substack{i=1\\|i-j|>\delta_1}}^{N}\sum_{j=1}^{N} 1$$

$$> C_1(\varepsilon_1)O(N^2)+C_1'(\varepsilon_1)O(N)$$

where C_1 and C_1' are positive constants depending on ε_1. This contradicts (3.1), since $H<1$.

It remains to prove that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} r^m(i-j)\sim N^{2H}L(N).$$

(Condition (iii) of Definition B′ follows straightforwardly.) For any $\varepsilon_2>0$, there is a $\delta_2>0$ such that $0\leqq r(k)<\varepsilon_2$ for all $k>\delta_2$. A derivation similar to part 1 of this proof yields

$$|S_2(N)|\leqq C_2(\varepsilon_2)O(N)+\varepsilon_2 S_1(N), \quad \text{as } N\to\infty$$

for some positive constant C_2 depending on ε_2. But by (3.1),

$$S_1(N)+S_2(N)\sim\frac{J^2(m)}{m!}N^{2H}L(N) \quad \text{as } N\to\infty,$$

with $H>\frac{1}{2}$. Therefore, as $N\to\infty$, $S_2(N)=o(S_1(N))$, and

$$\sum_{i=1}^{N}\sum_{j=1}^{N} r^m(i-j)=\frac{m!}{J^2(m)}S_1(N)\sim N^{2H}L(N). \quad \square$$

The proof of the first part of Theorem 3.2 provides

Corollary 3.1. *Suppose* $G\in\mathscr{G}_m$ *and* $\{X_i\}\in(m)'(H,L(\cdot))$. *Then the sequence*

$$G^*(X_i)=G(X_i)-\frac{J(m)}{m!}H_m(X_i), \quad i=1,2,\ldots$$

satisfies

$$\sum_{i=1}^{N}\sum_{j=1}^{N}|EG^*(X_i)G^*(X_j)|=o(N^{2H}L(N)) \quad \textit{as } N\to\infty.$$

4. The Reduction Theorem

We now suppose $G\in\mathscr{G}_m$ for some $m\geqq 1$ and $\{X_i\}\in(m)'(H,L(\cdot))$ (alternatively, $\{X_i\}\in(m)(D,L(\cdot))$), and study the weak convergence of

$$Z_N(t)=\frac{1}{d_N}\sum_{i=1}^{[Nt]}G(X_i) \tag{4.1}$$

as $N\to\infty$, where d_N^2 is asymptotically proportional to $\operatorname{Var}(\sum_{i=1}^{N}G(X_i))$. In fact, according to Theorem 3.1, d_N^2 may be chosen asymptotic to $N^{2H}L(N)$ for all $G\in\mathscr{G}_m$. A typical member of $\mathscr{G}_m$ is the Hermite polynomial H_m. When $G(X_i)=H_m(X_i)$, we use the special notation

$$Z_{N,m}(t)=\frac{1}{d_N}\sum_{i=1}^{[Nt]}H_m(X_i). \tag{4.2}$$

The following lemma indicates that $Z_N(t)$ and $\frac{J(m)}{m!} Z_{N,m}(t)$ have the same limit in distribution.

Lemma 4.1. *Let $G \in \mathscr{G}_m$ and $\{X_i\} \in (m)'(H, L(\cdot))$. Define $Z_{N,m}(t)$ as in (4.2), $Z_N(t)$ as in (4.1), and let $d_N^2 \sim N^{2H} L(N)$ as $N \to \infty$.*

If the limit in distribution of $(Z_{N,m}(t_1), Z_{N,m}(t_2), \ldots, Z_{N,m}(t_p))$ exists (we denote it $(\bar{Z}_m(t_1), \bar{Z}_m(t_2), \ldots, \bar{Z}_m(t_p))$), then

$$(Z_N(t_1), Z_N(t_2), \ldots, Z_N(t_p)) \xrightarrow{\mathscr{D}} \left(\frac{J(m)}{m!} \bar{Z}_m(t_1), \frac{J(m)}{m!} \bar{Z}_m(t_2), \ldots, \frac{J(m)}{m!} \bar{Z}_m(t_p)\right).$$

($\mathscr{D}$ denotes convergence in distribution).

Proof. Let $a_1, a_2, \ldots, a_p$ be p arbitrary real numbers. The lemma's hypothesis is equivalent to

$$W_{N,m}(p) \equiv \sum_{u=1}^{p} a_u Z_{N,m}(t_u) \xrightarrow{\mathscr{D}} \bar{W}_m(p) \equiv \frac{J(m)}{m!} \sum_{u=1}^{p} a_u \bar{Z}_m(t_u).$$

Similarly, the lemma's conclusion is equivalent to

$$W_N(p) \equiv \sum_{u=1}^{p} a_u Z_N(t_u) \xrightarrow{\mathscr{D}} \bar{W}(p) \equiv \frac{J(m)}{m!} \sum_{u=1}^{p} a_u \bar{Z}_m(t_u).$$

Since

$$W_N(p) = \frac{J(m)}{m!} W_{N,m}(p) + \left(W_N(p) - \frac{J(m)}{m!} W_{N,m}(p)\right),$$

it is sufficient to prove that

$$W_N(p) - \frac{J(m)}{m!} W_{N,m}(p) = \sum_{u=1}^{N} a_u \frac{1}{d_N} \sum_{i=1}^{[Nt_u]} G^*(X_i) \tag{4.3}$$

converges to zero in probability as $N \to \infty$ (Billingsley (1968), Th. 4.1), where $G^*(X_i)$ is defined as in Corollary 3.1. But

$$E\left(W_N(p) - \frac{J(m)}{m!} W_{N,m}(p)\right)^2$$
$$\leqq \sum_{u=1}^{p} \sum_{v=1}^{p} |a_u a_v| \frac{1}{d_N^2} \sum_{i=1}^{N} \sum_{i=1}^{N} |EG^*(X_i) G^*(X_j)| = o(1)$$

(Corollary 3.1). Convergence in probability of (4.3) follows from Tchebycheff inequality. □

The only contribution to the limit of $Z_N(t)$ is then due to the first non zero term in the Hermite expansion of $G(X_i)$, $i = 1, 2, \ldots$, namely $\frac{J(m)}{m!} H_m(X_i)$, $i = 1, 2, \ldots$.

Combining the statements of Lemma 4.1 and Theorem 2.1, we obtain

Reduction Theorem 4.1. *Let $G \in \mathscr{G}_m$ for some $m \geqq 1$ and suppose $\{X_i\} \in (m)'(H, L(\cdot))$. Define $Z_{N,m}(t)$ as in (4.2) and $Z_N(t)$ as in (4.1), choosing $d_N^2 \sim N^{2H} L(N)$, as $N \to \infty$.*

If, as $N \to \infty$, the finite-dimensional distributions of $Z_{N,m}(t)$ converge, then $Z_N(t)$ converges weakly in $\mathscr{D}([0, 1])$ to some process $\frac{J(m)}{m!} \bar{Z}_m(t)$ endowed with the pro-

perties $\Pi(H)$ listed in Section 2. The finite-dimensional distributions of $Z_m(t)$ are the limit of those of $Z_{N,m}(t)$.

Corollary 4.1. *The reduction theorem holds as well when $\{X_i\} \in (m)(D, L(\cdot))$ and $d_N^2 \sim N^{-mD+2} L^m(N)$, as $N \to \infty$.*

Proof of Corollary. By Lemma 3.1, $(m)(D, L(\cdot)) \subset (m)'(H, L_1(\cdot))$ where $2H = -mD+2$ and $L_1(x) \sim CL^m(x)$ as $x \to \infty$ for some positive constant C. □

We may now limit our attention to functions $G(X_i) = H_m(X_i)$ and concentrate on the convergence of the finite-dimensional distributions of $Z_{N,m}(t)$.

5. The Case $m=1$

It is easy to evaluate the limiting finite-dimensional distributions of $Z_{N,1}(t)$ because $H_1(X) = X$. The limiting process $\bar{Z}_1(t)$ turns out to be proportional to $B_H(t)$, the fractional Brownian motion process with parameter H.

Definition. $B_H(t)$, defined for $0 < H < 1$, is a Gaussian process endowed with the properties $\Pi(H)$. In particular, $EB_H(t) = 0$ and $EB_H^2(t) = t^{2H}$.

$B_H(t)$ is Brownian motion when $H = \frac{1}{2}$. Mandelbrot and Van Ness (1968) provide a representation of $B_H(t)$ as a weighted integral of Brownian motion. The existence of $B_H(t)$ follows also from Theorem 2.1 and the following lemma (valid for $0 < H < 1$):

Lemma 5.1. *Let $\{X_i\}$ be a stationary Gaussian sequence with mean 0 and correlations $r(i-j) = EX_iX_j$. Assume*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} r(i-j) \sim KN^{2H} L(N) \tag{5.1}$$

as $N \to \infty$, with $0 < H < 1$, L slowly varying, and K, a positive constant.

Then

$$Z_{N,1}(t) = \frac{1}{d_N} \sum_{i=1}^{[Nt]} X_i$$

with $d_N^2 \sim N^{2H} L(N)$, converges weakly as $N \to \infty$ to $\sqrt{K}\, B_H(t)$.

Proof. Since $X_1, \ldots, X_N$ are jointly Gaussian so are $Z_{N,1}(t_1), \ldots, Z_{N,1}(t_p)$. Let $C_N(t_i, t_j) = EZ_{N,1}(t_i) Z_{N,1}(t_j)$ and $C(t_i, t_j) = \lim_{N \to \infty} C_N(t_i, t_j)$. As $N \to \infty$, the characteristic function of $(Z_{N,1}(t_1), \ldots, Z_{N,1}(t_p))$ converges to that of

$$(\bar{Z}_1(t_1), \ldots, \bar{Z}_1(t_p)) \quad \text{where} \quad \bar{Z}_1(t_1), \ldots, \bar{Z}_1(t_p)$$

are jointly Gaussian with mean 0 and covariance matrix $((C(t_i, t_j)))$, $i, j = 1, \ldots, p$.

Let

$$S_{[Nt_i]} = \sum_{u=1}^{[Nt_i]} X_u.$$

$$C_N(t_i, t_j) = \frac{1}{N^{2H} L(N)} \frac{1}{2} \{ES_{[Nt_i]}^2 + ES_{[Nt_j]}^2 - ES_{|[Nt_i]-[Nt_j]|}^2\}.$$

By hypothesis,

$$ES^2_{[Nt_i]}=\sum_{u=1}^{[Nt_i]}\sum_{v=1}^{[Nt_i]} r_{uv} \sim KN^{2H}L([Nt_i])t_i^{2H}$$

as $N\to\infty$, so that

$$C(t_i,t_j)=K\tfrac{1}{2}\{t_i^{2H}+t_j^{2H}-|t_i-t_j|^{2H}\}.$$

Hence

$$(Z_N(t_1),\dots,Z_N(t_p))\xrightarrow{\mathscr{D}}(\sqrt{K}\,B_H(t_1),\dots,\sqrt{K}\,B_H(t_p))$$

since $B_H(t_1),\dots,B_H(t_p)$ are jointly Gaussian with mean 0 and covariance matrix $\left(\left(\frac{1}{K}C(t_i,t_j)\right)\right)$.

We now check tightness. Since S_N is Gaussian, ES_N^{2k} is proportional to $(ES_N^2)^k$ for all $k\geqq 1>\frac{1}{2H}$. The sequence $\{Z_N(t)\}$ is then tight (Lemma 2.1). Therefore, as $N\to\infty$, $Z_N(t)$ converges weakly to $\sqrt{K}\,B_H(t)$ (Billingsley (1968), Th. 15.1). □

Remark. Condition (5.1) of the preceding lemma is satisfied for $\frac{1}{2}<H<1$ when $r(k)\sim k^{2H-2}L(k)$ (see Lemma 3.1); and for $0<H<\frac{1}{2}$, when $r(k)\sim -k^{2H-2}L(k)$ as $k\to\infty$ with $r(0)+2\sum_{k=1}^{\infty} r(k)=0$ (adapt Feller (1971), p. 281 and apply the proof of Lemma 3.1).

We now restrict H to the interval $(\frac{1}{2},1)$ and state the main result of this section as a straightforward consequence of the preceding lemma and the reduction Theorem 4.1.

Theorem 5.1. *Suppose* $G\in\mathscr{G}_1$ *and* $\{X_i\}\in(1)'(H,L(\cdot))$. *Then*

$$Z_N(t)=\frac{1}{d_N}\sum_{i=1}^{[Nt]}G(X_i)$$

with $d_N^2\sim N^{2H}L(N)$, *converges weakly as* $N\to\infty$ *to* $J(1)B_H(t)$ *where* $J(1)=EXG(X)$.

Corollary 5.1. *The same result holds if* $\{X_i\}\in(1)(D,L(\cdot))$ *and*

$$d_N^2\sim\frac{2}{(1-D)(2-D)}N^{2-D}L(N)$$

as $N\to\infty$.

6. The Case $m=2$

The limiting finite-dimensional distributions of $Z_{N,2}(t)$ are more complicated than those of $Z_{N,1}(t)$ because $H_2(X)=X^2-1$. We shall restrict ourselves here to $\{X_i\}\in(2)(D,L(\cdot))$. Let us introduce the following notations.

If $t^{(p)}=(t_1,t_2,\dots,t_p)$ with $0\leqq t_1\leqq t_2\leqq\dots\leqq t_p\leqq 1$ and $p\geqq 1$ and if $s^{(p)}=(s_1,s_2,\dots,s_p)$ where $s_1,s_2,\dots,s_p$ are non-negative integers, then $(a^{(k)})\leftrightarrow(t^{(p)},s^{(p)})$, that is, $a^{(k)}=(a_1,a_2,\dots,a_k)$ where $k=s_1+s_2+\dots+s_p$ and where the first s_1 parameters a_i are equal to t_1, the next s_2 parameters a_i are equal to $t_2,\dots,$ the last s_p parameters a_i are equal to t_p.

The main result of this section is

Theorem 6.1. *For any* $G\in\mathscr{G}_2$ *and* $\{X_i\}\in(2)(D,L(\cdot))$,

$$Z_N(t)=\frac{1}{d_N}\sum_{i=1}^{[Nt]}G(X_i)$$

with $d_N \sim N^{1-D} L(N)$, *converges weakly as* $N \to \infty$ *to a process* $\frac{J(2)}{2} \bar{Z}_2(t)$ *where* $J(2) = EX^2 G(X)$. $\bar{Z}_2(t)$ *has the properties* $\Pi(H)$ *with* $H = 1-D$, *and the characteristic function of the random vector* $(\bar{Z}_2(t_1), \bar{Z}_2(t_2), \dots, \bar{Z}_2(t_p))$ *admits the following representation valid for small values of* $|u_1|, |u_2|, \dots, |u_p|$:

$$\phi(u_1, u_2, \dots, u_p)$$

$$= \exp \left\{ \frac{1}{2} \sum_{k=2}^{\infty} \frac{(2i)^k}{k} \sum_{\substack{s_1, \dots, s_p \geq 0 \\ s_1 + \dots + s_p = k}} \frac{p!}{s_1! s_2! \dots s_p!} u_1^{s_1} u_2^{s_2} \dots u_p^{s_p} S_D(a^{(k)}) \right\} \tag{6.1}$$

where $a^{(k)} \leftrightarrow (t^{(p)}, s^{(p)})$, *and*

$$S_D(a^{(k)}) = \int_0^{a_1} dx_1 \int_0^{a_2} dx_2 \dots \int_0^{a_k} dx_k |x_1 - x_2|^{-D} |x_2 - x_3|^{-D}$$
$$\dots |x_{k-1} - x_k|^{-D} |x_k - x_1|^{-D}.$$

The theorem follows from the reduction Theorem 4.1 and the following proposition that generalizes a result from Rosenblatt (1961).

Proposition 6.1. *Let* $\{X_i\} \in (2)(D, L(\cdot))$ *and*

$$Z_{N,2}(t) = \frac{1}{d_N} \sum_{i=1}^{[Nt]} (X_i^2 - 1)$$

with $d_N \sim N^{1-D} L(N)$ *as* $N \to \infty$.

Then, the finite-dimensional distributions of $Z_{N,2}(t)$ *converge, and the limiting characteristic function of* $(Z_{N,2}(t_1), Z_{N,2}(t_2), \dots, Z_{N,2}(t_p))$ *admits the representation* (6.1) *for small values of its arguments.*

Proof. Let $u_1, u_2, \dots, u_p$ be $p \geq 1$ arbitrary real numbers and let

$$\phi_N(u_1, u_2, \dots, u_p) = E \exp \left\{ i \sum_{l=1}^{p} u_l \frac{1}{d_N} \sum_{j=1}^{[Nt_l]} (X_j^2 - 1) \right\}.$$

Let z be a complex variable. We prove first that $\Phi_N(z) \equiv \phi_N(u_1 z, u_2 z, \dots, u_p z)$ converges as $N \to \infty$ to $\Phi(z) \equiv \phi(u_1 z, u_2 z, \dots, u_p z)$ in some neighborhood of the origin. We then show that $\Phi(z)$ is analytic around the origin.

For any $l = 1, \dots, p$, let $D_N(t_l)$ be the $N \times N$ diagonal matrix whose $[Nt_l]$ first diagonal elements are equal to 1 and the others to 0. $D_N(1)$ is then the $N \times N$ identity matrix. Introduce also

$$D_N(u^{(p)}, t^{(p)}) = \sum_{l=1}^{p} u_l D_N(t_l)$$

where $u^{(p)} = (u_1, \dots, u_p)$ and $t^{(p)} = (t_1, \dots, t_p)$. Then

$$\Phi_N(z) = \Phi_N^{(0)}(z) \, \Phi_N^{(+)}(z)$$

where

$$\Phi_N^{(0)}(z) = \exp\{-i z d_N^{-1} \operatorname{Tr} D_N(u^{(p)}, t^{(p)})\}$$

and

$$\Phi_N^{(+)}(z) = (2\pi)^{-N/2} |R_N|^{-\frac{1}{2}} \int d^N x \exp\{-\tfrac{1}{2} x' [R_N^{-1} - 2 i z d_N^{-1} D_N(u^{(p)}, t^{(p)})] x\}$$

where R_N is the covariance matrix of $X=(X_1, X_2, \ldots, X_N)$ and where a prime denotes a transpose. The integration is over the real N-dimensional space $\mathbb{R}^N$.

$$\begin{aligned}\Phi_N^{(+)}(z)&=|R_N|^{-\frac{1}{2}}\,|R_N^{-1}-2\,i\,z\,d_N^{-1}\,D_N(u^{(p)}, t^{(p)})|^{-\frac{1}{2}}\\&=|D_N(1)-2\,i\,z\,d_N^{-1}\,D_N(u^{(p)}, t^{(p)})\,R_N|^{-\frac{1}{2}}\end{aligned}$$

Let $\lambda_{n,N}$, $n=1,2,\ldots,N$ be the N eigenvalues of the matrix $D_N(u^{(p)}, t^{(p)})\,R_N$. Then

$$\begin{aligned}\Phi_N^{(+)}(z)&=\prod_{n=1}^{N}(1-2\,i\,z\,d_N^{-1}\,\lambda_{n,N})^{-\frac{1}{2}}\\&=\exp\left\{-\tfrac{1}{2}\sum_{n=1}^{N}\log(1-2\,i\,z\,d_N^{-1}\,\lambda_{n,N})\right\}\end{aligned}$$

where log stands for the principal determination of the logarithm. However,

$$\sum_{n=1}^{N}\lambda_{n,N}=\operatorname{Tr} D_N(u^{(p)}, t^{(p)})\,R_N=\operatorname{Tr} D_N(u^{(p)}, t^{(p)})$$

since R_N has the elements 1 in its diagonal. Hence

$$\begin{aligned}\Phi_N(z)&=\Phi_N^{(0)}(z)\,\Phi_N^{(+)}(z)\\&=\exp\left\{\tfrac{1}{2}\sum_{n=1}^{N}[-2\,i\,z\,d_N^{-1}\,\lambda_{n,N}-\log(1-2\,i\,z\,d_N^{-1}\,\lambda_{n,N})]\right\}.\end{aligned}$$

Now restrict z to the neighborhood $|z|<\varepsilon$, where $\varepsilon>0$ is small enough for

$$\sum_{n=1}^{N}[-2\,i\,z\,d_N^{-1}\,\lambda_{n,N}-\log(1-2\,i\,z\,d_N^{-1}\,\lambda_{n,N})]=\sum_{k=2}^{\infty}\sum_{n=1}^{N}\frac{(2\,i\,z\,d_N^{-1}\,\lambda_{n,N})^k}{k}$$

to hold for all $N\geqq 1$.

Such an $\varepsilon>0$ exists because

$$\begin{aligned}\frac{1}{d_N^2}\sum_{n=1}^{N}\lambda_{n,N}^2&=\frac{1}{d_N^2}\operatorname{Tr}\big(D_N(u^{(p)}, t^{(p)})\,R_N\big)^2\\&\leqq\frac{1}{d_N^2}\operatorname{Tr}\left(\sum_{l=1}^{p}u_l\,D_N(1)\,R_N\right)^2\\&\leqq\left(\sum_{l=1}^{p}|u_l|\right)^2\frac{1}{d_N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}|r(i-j)|^2\\&\to\left(\sum_{l=1}^{p}|u_l|\right)^2\frac{1}{(1-2D)(1-D)}\end{aligned}$$

as $N\to\infty$ (refer to the proof of Lemma 3.1).

Hence, for $|z|<\varepsilon$,

$$\Phi_N(z)=\exp\left\{\tfrac{1}{2}\sum_{k=2}^{\infty}\frac{(2\,i\,z)^k}{k}\,d_N^{-k}\sum_{n=1}^{N}\lambda_{n,N}^k\right\}.$$

But

$$\sum_{n=1}^{N}\lambda_{n,N}=\operatorname{Tr}\big(D_N(u^{(p)}, t^{(p)})\,R_N\big),$$

and hence

$$\sum_{n=1}^{N} \lambda_{n,N}^k = \operatorname{Tr}\big(D_N(u^{(p)}, t^{(p)})\, R_N\big)^k$$
$$= \operatorname{Tr}\left(\sum_{l=1}^{p} u_l\, D_N(t_l)\, R_N\right)^k$$
$$= \sum_{\substack{s_1, \ldots, s_p \geqq 0 \\ s_1 + \cdots + s_p = k}} \frac{p!}{s_1! \ldots s_p!} u_1^{s_1} \cdots u_p^{s_p} \operatorname{Tr} \prod_{l=1}^{p} (D_N(t_l)\, R_N)^{s_l}.$$

At this point we use the notation $a^{(k)} \leftrightarrow (t^{(p)}, s^{(p)})$ introduced at the beginning of this section. Let $\rho_{l,N}(i,j)$, $i,j=1, \ldots, N$ be the elements of the matrix $D_N(a_l)\, R_N$. Then $\rho_{l,N}(i,j) = r(|i-j|)$ if $i \leqq [N a_l]$ and $\rho_{l,N}(i,j)=0$ if $[N a_l] < i \leqq N$. Hence

$$d_N^{-k} \operatorname{Tr} \prod_{l=1}^{p} (D_N(t_l)\, R_N)^{s_l}$$
$$= d_N^{-k} \operatorname{Tr} \prod_{l=1}^{k} D_N(a_l)\, R_N$$
$$= d_N^{-k} \sum_{i_1, i_2 \ldots i_N = 1}^{N} \rho_{1,N}(i_1, i_2)\, \rho_{2,N}(i_2, i_3) \ldots \rho_{k-1,N}(i_{k-1}, i_k)\, \rho_{k,N}(i_k, i_1)$$
$$= d_N^{-k} \sum_{i_1=1}^{[Na_1]} \sum_{i_2=1}^{[Na_2]} \cdots \sum_{j_k=1}^{[Na_k]} r(|i_1 - i_2|)\, r(|i_2 - i_3|) \ldots r(|i_{k-1} - i_k|)\, r(|i_k - i_1|).$$

This last expression converges as $N \to \infty$ to $S_D(a^{(k)})$.

Hence $\lim_{N \to \infty} \Phi_N(z) = \Phi(z)$ for $|z| < \varepsilon$.

We now prove that $\Phi(z)$ is analytic for $|z| < \varepsilon'$ where $0 < \varepsilon' \leqq \varepsilon$.

First, note that $S_D(a^{(k)}) \leqq C^k$ for $C = \left[\frac{2}{1-2D}\right]^{\frac{1}{2}}$. To see this replace all t's in $S_D(a^{(k)})$ by 1, use Schwarz inequality to break the chain of integrands, and make the change of variables $y_i = x_i - x_{i+1}$, $i = 1, \ldots, k-1$, to obtain

$$S_D(a^{(k)}) \leqq \left\{ \int_{-1}^{1} |y|^{-2D}\, dy \right\}^{\frac{1}{2}} \left\{ \int_{-1}^{1} \cdots \int_{-1}^{1} dy_1 \ldots dy_{k-1}\, |y_1|^{-2D} \ldots |y_{k-1}|^{-2D} \right\}^{\frac{1}{2}} = C^k.$$

Hence

$$|\Phi(z)| \leqq \exp\left\{ \sum_{k=2}^{\infty} \frac{1}{k} \left(2\, C\, |z| \sum_{l=1}^{p} |u_l|\right)^k \right\}$$

which converges for small enough $|z|$.

An analytic continuation argument (see Lukacs (1970), th. 7.1.1) ensures that $\Phi(z)$ agrees with a unique characteristic function for all real values of z. □

Remark. When $p=1$ and $t=1$, ϕ becomes

$$\phi(u) = \exp\left\{ \tfrac{1}{2} \sum_{k=2}^{\infty} \frac{(2\, i\, u)^k}{k} S_D(a^{(k)}) \right\} \tag{6.2}$$

where now $a^{(k)}$ is the k-th dimensional vector $(1, 1, \ldots, 1)$. The expression (6.2) first appears in Rosenblatt (1961). It is therefore suggested that $\bar{Z}_2(t)$ be called the "Rosenblatt process".

$\phi(u)$ is expressed in (6.2) as the expansion of a characteristic function in terms of its cumulants κ_k. Here $\kappa_1 = 0$ (because the mean is 0) and $\kappa_k = (k-1)!\, 2^{k-1}\, S_D(a^{(k)})$ for $k = 2, 3, \ldots$. $\phi(u)$ could equivalently have been expressed in terms of moments. This suggests that the Propositon 6.1 can be proved by a method of moments. This is done in Taqqu (1972). Note that $\phi(u)$ differs from the characteristic function of the chi-square distribution by $S_D(a^{(k)}) \neq 1$.

It would be interesting to find a representation of $\overline{Z}_2(t)$ in terms of a double integral of Brownian motion or in terms of fractional Brownian motion. A detailed study of the distribution of $(\overline{Z}_2(t_1), \ldots, \overline{Z}_2(t_p))$ should yield important information.

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