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Asymptotic Expansions in the Central Limit Theorem for Compound and Markov Processes

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Summary. For independent identically distributed bivariate random vectors (X_1, Y_1), (X_2, Y_2), ... and for large t the distribution of $X_1+\ldots+X_{N(t)}$ is approximated by asymptotic expansions. Here $N(t)$ is the counting process with lifetimes $Y_1, Y_2, \ldots$. Similar expansions are derived for multivariate X_1. Furthermore, local asymptotic expansions are valid for the distribution of $f(X_1)+\ldots+f(X_N)$ when N is large and nonrandom, and X_i, $i=1,2,\ldots$, is a discrete strongly mixing Markov chain.

1. Introduction and Summary

Let (X, Y), (X_1, Y_1), (X_2, Y_2), ... be a sequence of independent identically distributed bivariate random vectors, where Y is nonnegative. For positive t define

$$N(t)=\max\{k\colon Y_1+\ldots+Y_k\leqq t\}$$

and the randomly stopped sum

$$S(t)=X_1+\ldots+X_{N(t)}.$$

The stochastic process $S(t)$, $t>0$, is called a compound process whenever X and Y are stochastically independent. If in addition the distribution of Y is exponential, then we have a compound Poisson process which is often used as a model for the aggregate claims process. In the following we shall drop the assumption that the waiting times – the Y's – and the claim amounts – the X's – are stochastically independent, and that X is univariate.

Consider a sequence (X, Y), (X_1, Y_1), (X_2, Y_2), ... of independent identically distributed $p+1$-variate random vectors, X p-variate, $Y\geqq 0$, and define $N(t)$ and $S(t)$ as above. The process $S(t)$, $t\geqq 0$ will be called compound process with dependence. For large t, the distribution of $S(t)$ is approximately normal (Theorem 2.2) and can be approximated by asymptotic expansions. The latter will be done for $p\geqq 2$ under the assumption that the covariance matrix of

(X, Y) is nonsingular. We assume that (X, Y) has a lattice distribution or satisfies a Cramér condition in one of its arguments. For $p=1$ we shall also discuss the case of a singular covariance matrix of (X, Y). On the other hand, we consider Markov chains $\xi_1, \xi_2, \ldots$ on a countable state space I which satisfy a strong mixing condition. For $f: I \to \mathbb{Z}$ we derive asymptotic expansions for

$$(1.1) \qquad P\{f(\xi_1)+\ldots+f(\xi_N)=z\}, \qquad z \in \mathbb{Z}.$$

The main tool for our proof of higher order approximations for compound processes with dependence and for (1.1) are local approximations for the joint distribution of $(X_1+\ldots+X_N, Y_1, \ldots, Y_N)$, N large, where (X_i, Y_i), $i=1, 2, \ldots$, are iid random vectors. This idea was used by Bolthausen (1980, 1982) in his proof for the Berry-Esseen bound for functionals of Markov chains. Asymptotic expansions for (1.1) can also be obtained from Götze and Hipp (1983). For this, however, we have to assume that $\xi_1, \xi_2, \ldots$ satisfies a strong mixing condition with exponentially decreasing mixing coefficients. The Bolthausen method used here works also in the case of polynomially decreasing mixing coefficients.

Laws of large numbers and the central limit theorem for univariate $S(t)$ can be found in Smith (1955, 1958). Asymptotic expansions for compound Poisson processes can be found in v. Chossy and Rappl (1983).

It is not possible to compute reasonable absolute error bounds for our approximations. The fact that the error of approximation has the right order indicates that the approximations are good for moderate to large N or t.

In Sect. 2 we state and prove the results for compound processes with dependence. The same is done in Sect. 3 for discrete Markov chains. Section 4 contains auxiliar results.

2. Results for Compound Processes with Dependence

Throughout this section we assume that

$$EY > 0$$

and that for some integer $s \geqq 2$

$$(2.1) \qquad E\|X\|^s + EY^s < \infty.$$

Write C for the covariance matrix of

$$X^{(j)} - YEX^{(j)}/EY, \qquad j=1, \ldots, p$$

and $U(t)$ for the p-vector with components

$$U_j(t) = (t/EY)^{-1/2}(S(t)^{(j)} - tEX^{(j)}/EY), \qquad j=1, \ldots, p.$$

The following theorem seems to be well known.

(2.2) **Theorem.** *Assume that C is nonsingular. Then for t tending to infinity, the distribution of $U(t)$ converges weakly to the p-variate normal law with mean zero and covariance matrix C.*

Proof. Let $o_P(1)$ denote a generic random variable which converges to zero in probability when t tends to infinity. For $j=1, \ldots, p$ define

$$W_j(t)=(t/EY)^{-1/2} \sum' \{X_\nu^{(j)}-EX^{(j)}-(Y_\nu-EY)\,EX^{(j)}/EY\}$$

where the sum extends over $1 \leqq \nu \leqq t/EY$. It suffices to show that for $j=1, \ldots, p$

$$U_j(t)=W_j(t)+o_P(1).$$

We have

$$t^{-1/2}\sum_{\nu=1}^{N(t)} Y_\nu = t^{1/2}+o_P(1)$$

and hence

$$t^{-1/2}\sum_{\nu=1}^{N(t)} (Y_\nu-EY)=t^{-1/2}(t-N(t)\,EY)+o_P(1)$$

Lemma (4.1) yields that

$$t^{-1/2}\sum_{\nu=1}^{N(t)} (Y_\nu-EY)=t^{-1/2}\sum'(Y_\nu-EY)+o_P(1)$$

and

$$t^{-1/2}\sum_{\nu=1}^{N(t)} (X_\nu^{(j)}-EX^{(j)})=t^{-1/2}\sum'(X_\nu^{(j)}-EX^{(j)})+o_P(1).$$

Hence

$$U_j(t)=W_j(t)+o_P(1).$$

Higher order approximations for the distribution of $U(t)$ will now be derived under the assumption that the covariance matrix C_0 of (X, Y) is nonsingular, that (2.1) holds for some $s \geqq 3$, and that one of the following conditions is satisfied.

(2.3) The distribution of (X, Y) satisfies the Cramér condition, i.e. for all positive e there exists a positive number d such that

$$\|u_1\|+|u_2| \geqq e \quad \text{implies} \quad |E\exp(iu_1^T X+iu_2 Y)| \leqq 1-d.$$

(2.4) (X, Y) has a lattice distribution with minimal lattice $\mathbb{Z}^{p+1}$, the set of all $p+1$-vectors with integral components, i.e. for fixed $z \in \mathbb{Z}^{p+1}$ with $P\{(X, Y)=z\}>0$, $\mathbb{Z}^{p+1}$ – as an additive group – is generated by the support of $(X, Y)-z$.

(2.5) Y has a lattice distribution with minimal lattice $\mathbb{Z}$, and the joint distribution of (X, Y) satisfies a uniform Cramér condition in its first argument: For all positive e there exists a positive number d such that

$$\|u_1\| \geqq e \quad \text{implies} \quad |E\exp(iu_1^T X+iu_2 Y)| \leqq 1-d.$$

(2.6) X has a lattice distribution with minimal lattice $\mathbb{Z}^p$, and the joint distribution of (X, Y) satisfies a uniform Cramér condition in its second argument: For all positive e there exists a positive number d such that

$$|u_1| \geqq e \quad \text{implies} \quad |E \exp(iu_1^T X + iu_2 Y)| \leqq 1-d.$$

Under conditions (2.4) and (2.5) the random function $S(t)$ is constant for t in $[r, r+1)$, r an integer. Under these conditions the distribution of $S(t)$ will be approximated for integral t only. The approximations derived under conditions (2.3) and (2.6) are valid for all nonnegative t.

(2.7) **Theorem.** *Assume that* (2.1) *holds for some integral* $s \geqq 3$, *and that the covariance matrix of* (X, Y) *is nonsingular. Then there exist polynomials* q_r *and* q_r^*, $r=0, \ldots, s-3$, *yielding higher order approximations for the distribution of* $U(t)$ *in all cases* (2.3)–(2.6) *in the following sense.*

(i) *If* (2.3) *holds, then uniformly for convex measurable* $A \subset \mathbb{R}^p$

$$P\{U(t) \in A\} = \int_A \varphi_C(z) \sum_{r=0}^{s-3} t^{-r/2} q_r(z)\, dz + O(t^{-(s-2)/2}). \tag{2.8}$$

Here φ_C *is the density of the p-variate normal law with zero mean and covariance matrix C.*

(ii) *If condition* (2.5) *holds, then* (2.8) *is true for integral t when* q_r *is replaced by* q_r^*, $r=0, \ldots, s-3$.

(iii) *If* (2.6) *holds, then uniformly for* $A \subset \mathbb{R}^p$

$$P\{U(t) \in A\} = M_{t,s}(A) + O(t^{-(s-2)/2}) \tag{2.9}$$

where $M_{t,s}$ *is the finite signed discrete measure defined by*

$$M_{t,s}\{u\} = t^{-p/2} n_{t,s}(u),$$
$$u = ((t/EY)^{-1/2}(z_j - tEX^{(j)}/EY)_{j=1,\ldots,p}), \qquad z_1, \ldots, z_p \in \mathbb{Z}.$$

Here $n_{t,s}(z) = \varphi_C(z) \sum_{r=0}^{s-3} t^{-r/2} q_r(z)$.

(iv) *If condition* (2.4) *holds, then* (2.9) *is true for integral t and*

$$n_{t,s}(z) = \varphi_C(z) \sum_{r=0}^{s-3} t^{-r/2} q_r^*(z).$$

We have in particular for $p=1$

$$q_0(z) \equiv q_0^*(z) \equiv 1$$
$$q_1(z) = \tfrac{1}{2}\operatorname{cov}(X, Y)(z/\sigma^2 - z^3/\sigma^4)$$
$$+ \tfrac{1}{2}\operatorname{var}(Y)\, EX(-2z/\sigma^2 + z^3/\sigma^4) - \tfrac{1}{2} EX\, z/\sigma^2$$
$$- \tfrac{1}{6} E(X - Y\cdot EX)^3 (z^3/\sigma^6 - 3z/\sigma^4)$$
$$q_1^*(z) = q_1(z) + \tfrac{1}{2} EX\, z/\sigma^2$$

where $\sigma^2 = \operatorname{var}(X) - 2EX \operatorname{cov}(X, Y) + (EX)^2 \operatorname{var}(Y)$ *is the asymptotic variance of* $U(t)$, *and* $EY = 1$ *for simplicity.*

Proof. In this proof we shall use R and R' as generic functions in A and t satisfying

$$\sup\{|R(A, t)| : A \subset \mathbb{R}^p \text{ measurable}\} = O(t^{-(s-2)/2})$$

and

$$\sup\{|R'(A, t)| : A \subset \mathbb{R}^p \text{ measurable, convex}\} = O(t^{-(s-2)/2}),$$

respectively.

(a) Truncation. For $t \geqq 0$ define the vectors ${}^tX, {}^tX_1, {}^tX_2, \ldots$ by

$${}^tX_\nu = X_\nu 1_{\{\|X_\nu\| \leqq t^{1/2}\}}, \qquad \nu = 1, 2, \ldots$$

and

$${}^tY, {}^tY_1, {}^tY_2, \ldots \quad \text{by} \quad {}^tY_\nu = Y_\nu 1_{\{Y_\nu \leqq t^{1/2}\}}.$$

Here 1_A is the indicator function of the set A. Let $S'(t)$ and $N'(t)$ be defined as $S(t)$ and $N(t)$ with ${}^tX_j$ and ${}^tY_j$ instead of X_j and Y_j. Then

$$P\{S(t) \in A\} = P\{S'(t) \in A\} + R. \tag{2.10}$$

In order to prove (2.10) it suffices to show that for $j = 1, \ldots, p$

$$P\{{}^tX_1^{(j)} + \ldots + {}^tX_{N(t)}^{(j)} \neq X_1^{(j)} + \ldots + X_{N(t)}^{(j)}\} + P\{N(t) \neq N'(t)\} = R.$$

Lemma (4.2) implies that there exists a positive constant c such that

$$P\{N(t) > ct\} = R.$$

With Chebyshev's inequality we obtain

$$P\{{}^tX_1^{(j)} + \ldots + {}^tX_{N(t)}^{(j)} \neq X_1^{(j)} + \ldots + X_{N(t)}^{(j)}\} = R.$$

Recall that $N'(t) \geqq N(t)$, and inequality can hold if for some $k \leqq N(t) + 1$ we have

$$Y_k \neq {}^tY_k.$$

Hence

$$P\{N'(t) \neq N(t)\} \leqq (ct + 1) P\{Y \neq {}^tY\} = R.$$

This proves (2.10).

(b) Representation. For $m = 0, 1, \ldots$ and measurable subsets $A \subset \mathbb{R}^p$ define

$$Q(m, t, A) = P\{(({}^tX_1^{(j)} + \ldots + {}^tX_m^{(j)})_{j=1, \ldots, p}) \in A \text{ and}$$

$$t - {}^tY_{m+1} < {}^tY_1 + \ldots + {}^tY_m \leqq t\}.$$

Our starting point is the relation

$$P\{S'(t) \in A\} = \sum_{m=0}^{\infty} Q(m, t, A).$$

Lemma (4.2) implies that

$$P\{S'(t)\in A\}=\sum' Q(m,t,A)+R$$

where the dashed sum extends over m with

$$(m-t/EY)^2\leqq \operatorname{var}({}^tY)(s-2)\,t\log t/(E^tY)^2.$$

(c) Approximation. Write Q for the distribution of ${}^tY$. Then

$$Q(m,t,A)=\int P\{(({}^tX_1^{(j)}+\ldots+{}^tX_m^{(j)})_{j=1,\ldots,p})\in A \text{ and}$$

$$t-x<{}^tY_1+\ldots+{}^tY_m\leqq t\}\,Q(dx).$$

(c1) Assume that condition (2.3) holds. Lemma (4.3) implies that uniformly for convex $A\subset\mathbb{R}^p$ and $x>0$

$$\begin{aligned}&P\{{}^tX_1+\ldots+{}^tX_m\in A \text{ and } t-x<{}^tY_1+\ldots+{}^tY_m\leqq t\}\\&\quad=\iint m^{-(p+1)/2}\,g_m(m^{-1/2}(u-mE^tX),m^{-1/2}(v-mE^tY))\\&\qquad\cdot 1_A(u)\,du\,1_{(t-x,t]}(v)\,dv\\&\qquad+(1+x)(1+(m^{-1/2}(t-mE^tY))^2)^{-1}\,O(m^{-(s-1)/2})\\&\qquad+O(m^{-(s+1)/2}).\end{aligned}$$

Summation over m renders an error of order $O(t^{-(s-2)/2})$. We thus obtain

$$P\{S'(t)\in A\}=\sum' Q_1(m,t,A)+R'$$

where

$$\begin{aligned}Q_1(m,t,A)=E\iint m^{-(p+1)/2}\,&g_m(m^{-1/2}(u-mE^tX),m^{-1/2}(v-mE^tY))\\&\cdot 1_A(u)\,du\,1_{(t-{}^tY,t]}(v)\,dv.\end{aligned}$$

(c2) Assume now that condition (2.5) holds and that t is an integer. We have

$$Q(m,t,A)=\sum^* Q_l^*(m,t,A)$$

where for integral l

$$Q_l^*(m,t,A)=P\{{}^tX_1+\ldots+{}^tX_m\in A,\,{}^tY_1+\ldots+{}^tY_m=l\}\,P\{Y>t-l\}$$

and the stared sum extends over integral l with

$$t-t^{1/2}<l\leqq t.$$

Lemma (4.5) implies that uniformly for $l\in\mathbb{Z}$ and convex $A\subset\mathbb{R}^p$

$$\begin{aligned}Q_l^*(m,t,A)=\{&m^{-(p+1)/2}\int_A g_m(m^{-1/2}(u-mE^tX),m^{-1/2}(l-mE^tY))\,du\\&+(1+|m^{-1/2}(t-l-mE^tY)|^2)^{-1}\,O(m^{-(s-1)/2})\}\,P\{Y>t-l\}.\end{aligned}$$

Summing over m and then over l we obtain with

$$\sum P\{Y>l\}<\infty$$

that

$$P\{S'(t)\in A\}=\sum' Q_2(m, t, A)+R'$$

where

$$Q_2(m, t, A)=\sum^* m^{-(p+1)/2}\int_A g_m(m^{-1/2}(u-mE^tX), m^{-1/2}(l-mE^tY))\,du\,P\{Y>t-l\}.$$

($c3$) Assume that condition (2.4) is satisfied. Then by Lemma (4.4) we have uniformly for $u\in\mathbb{Z}^p$, $v\in\mathbb{Z}$

$$\begin{aligned}Q_v^*(m, t, \{u\})=\{&m^{-(p+1)/2}\, g_m(m^{-1/2}(u-mE^tX), m^{-1/2}(v-mE^tY))\\ &+(1+\|m^{-1/2}(u-mE^tX)\|^{2p+2}+|m^{-1/2}(v-mE^tY)|^{2p+2})^{-1}\\ &*O(m^{-(s+p-1)/2})\}\,P\{Y>t-v\}.\end{aligned}$$

The facts that for positive a, b

$$(1+a^s)(1+b^s)\leqq 3(1+a^{2s}+b^{2s})$$

and

$$\sum P\{Y>l\}<\infty$$

imply that summation over u and then over v yields an error term of the right order, i.e.

$$P\{S'(t)\in A\}=\sum' Q_3(m, t, A)+R'$$

where

$$\begin{aligned}Q_3(m, t, A)=&\sum_{u\in A\cap\mathbb{Z}^p}\sum^* m^{-(p+1)/2}\\ &\cdot g_m(m^{-1/2}(u-mE^tX), m^{-1/2}(v-mE^tY))\,P\{Y>t-v\}.\end{aligned}$$

($c4$) Assume now that condition (2.6) holds. Then Lemma (4.6) implies that uniformly for $u\in\mathbb{Z}^p$ and m in the range of summation

$$\begin{aligned}Q(m, t, \{u\})=&\int\int_{t-x}^{t} m^{-(p+1)/2}\, g_m(m^{-1/2}(u-mE^tX), m^{-1/2}(v-mE^tY))\,dv\,Q(dx)\\ &+O(t^{-(s+p+1)/2})\\ &+O(t^{-(s+p-1)/2})(1+\|m^{-1/2}(u-mE^tX)\|^{p+1})^{-1}\\ &\cdot\int\max\{(1+(m^{-1/2}(t-x-mE^tY))^2)^{-1},\\ &\cdot(1+(m^{-1/2}(t-mE^tY))^2)^{-1}\}\,Q_0(dx)\end{aligned}$$

where Q_0 is the distribution of ${}^tY$. Since

$$\sup_{x\in\mathbb{R}} t^{-1/2}\sum_m(1+(m^{-1/2}(t-x-mE^tY))^2)^{-1}<\infty$$

we obtain that summation over u and m renders an error term of the right order, i.e.

$$P\{S'(t)\in A\}=\sum' Q_4(m, t, A)+R$$

where

$$Q_4(m, t, A) = \sum_{u \in A \cap \mathbb{Z}^p} \int \int_{t-x}^{t} m^{-(p+1)/2} \cdot g_m(m^{-1/2}(u - mE^t X), m^{-1/2}(v - mE^t Y))\, dv\, Q_0(dx).$$

(d) Expansion. We shall now derive higher order approximations for $Q_j(m, t, A)$, $j = 1, \ldots, 4$. For smooth $g: \mathbb{R}^2 \to \mathbb{R}$ and nonnegative integer v write $D_2^v g(u, v) = \frac{\partial^v}{\partial v^v} g(u, v)$. Let $Q_j'(m, t, A)$ be defined as $Q_j(m, t, A)$, where

$$g_m(m^{-1/2}(u - mE^t X), m^{-1/2}(v - mE^t Y))$$

is replaced by its Taylor expansion of order $s-2$ at the point

$$(m^{-1/2}(u - mE^t X), m^{-1/2}(t - mE^t Y)).$$

Then for $j = 1, \ldots, 4$

$$\sum{}' Q_j(m, t, A) = \sum{}' Q_j'(m, t, A) + R. \tag{2.11}$$

In order to prove (2.11) we use the relation

$$\begin{aligned} &|Q_j(m, t, A) - Q_j'(m, t, A)| \\ &\quad \leqq c \int m^{-(s+p-1)/2} \sup \{|D_2^{s-1} g_m(m^{-1/2}(u - mE^t X), \\ &\qquad m^{-1/2}(\xi - mE^t Y))| : t - t^{1/2} \leqq \xi \leqq t\}\, du\, E Y^s, \end{aligned}$$

where c is a positive constant not depending on m or t. From the special structure of g_m we obtain that there exists a positive ε such that for all $u \in \mathbb{R}^p$, $v \in \mathbb{R}$

$$|D_2^{s-1} g_m(u, v)| \leqq \varepsilon^{-1} \exp(-\varepsilon \|u\|^2 - \varepsilon v^2).$$

Furthermore, if $A'(t)$ is the range of summation for m,

$$\sup \{\int m^{-1/2} \exp(-\varepsilon m^{-1} \|u - mE^t X\|^2)\, du : m \in A'(t), t \geqq 0\} < \infty.$$

For the proof of (2.11) it suffices to show that

$$\sum{}' m^{-1/2} \sup \{\exp(-\varepsilon m^{-1} (\xi - mE^t Y)^2) : t - t^{1/2} \leqq \xi \leqq t\} < \infty. \tag{2.12}$$

For $j \in \mathbb{Z}$ define

$$A_j = \{m \in A'(t) : t + j t^{1/2} \leqq mE^t Y < t + (j+1)\, t^{1/2}\}.$$

Then $\# A_j \leqq t^{1/2}$, and for $m \in A_j$, $|j| \geqq 2$, and $t - t^{1/2} \leqq \xi \leqq t$

$$|m^{-1/2}(\xi - mE^t Y)| \geqq (t/EY + t\delta_t)^{-1/2}\, t^{1/2} (|j| - 2),$$

where $\delta_t = \sup \{(m - t/EY)/t : m \in A'(t)\}$.

Now $\lim \delta_t = 0$, $E^t Y > 0$, and

$$\sum_{j \in \mathbb{Z}} \exp(-\delta(|j|-2)) < \infty \qquad \textit{for all } \delta > 0$$

implies (2.12).

In each of the $Q_j'(m, t, A)$ replace $E^t Y$ and $E^t X$ by EY and EX, respectively, and denote the new terms by $Q_j''(m, t, A)$. Since

$$E^t Y = EY + O(t^{-(s-1)/2})$$

and

$$E^t X = EX + O(t^{-(s-1)/2})$$

we obtain that for $j = 1, \ldots, 4$

$$\sum' Q_j'(m, t, A) = \sum' Q_j''(m, t, A) + R.$$

The terms $Q_j''(m, t, A)$ are linear combinations of expressions of the following kind:

$$m^{-r/2} \int_A m^{-(p+1)/2} (\varphi_{C_0} H)(m^{-1/2}(u - mEX), m^{-1/2}(t - mEY))\, du \tag{2.13}$$

for $j = 1$ and $j = 3$, and

$$m^{-r/2} \sum_{u \in A \cap \mathbb{Z}^p} m^{-(p+1)/2} (\varphi_{C_0} H)(m^{-1/2}(u - mEX), m^{-1/2}(t - mEY)) \tag{2.14}$$

for $j = 2$ and $j = 4$.

Here, φ_{C_0} is the $p+1$-variate normal density with mean zero and covariance matrix C_0, and $H(u, v)$ is a polynomial. The coefficients of this linear combination and of H do not depend on m or t.

For m in the range of summation we have

$$m/t = 1 - (t-m)/t = 1 + o(t^{-1/2} \log t).$$

Denote

$$a = t^{-1/2}(m - t)$$
$$b = t^{-1/2}(u - tEX).$$

Then

$$\begin{aligned} m^{-1/2} &= t^{-1/2}(1 + t^{-1/2} a)^{-1/2} \\ &= t^{-1/2} \sum_{l \neq 0}^{s-2} \binom{-1/2}{l} t^{-1/2} a^l + o(t^{-(s-2)/2}). \end{aligned} \tag{2.15}$$

Without increasing the order of the error we may replace $m^{-1/2}$ by the expansion in $t^{-1/2}$ and a given on the r.h.s. of (2.13) and (2.14), and use Taylor expansions. The resulting approximations for (2.13) is a linear combination of terms of the following kind:

$$t^{-j/2} \int_A \varphi_{C_0}(b - aEX, -a)\, a^k b^l t^{-(p+1)/2}\, du \tag{2.16}$$

and (2.14) is approximated by a linear combination of terms of the kind

(2.17) $$t^{-j/2} \sum_{u\in A\cap\mathbb{Z}^p} \varphi_{C_0}(b-aEX, -a)\, a^k b^l t^{-(p+1)/2}.$$

Here, for $l\in\mathbb{Z}^p_+$ and $b\in\mathbb{R}^p$ we write $b^l=\prod_1^p b_\nu^{l_\nu}$. Using the Euler-Maclaurin summation formula ([2], p. 258, Theorem A.4.3) we see that the sum $\sum'$ over m of the terms (2.16) and (2.17) can be replaced by

$$t^{-j/2} \iint \varphi_{C_0}(b-yEX, -y)\, y^k b^l t^{-p/2} 1_A(u)\, du\, dy$$

and

$$t^{-j/2} \sum_{u\in A\cap\mathbb{Z}^p} \int t^{-(p+1)/2} \varphi_{C_0}(b-yEX, -y)\, y^k b^l\, dy,$$

respectively. Let (U, V) be a $p+1$-variate random vector with Lebesgue-density $(u, v)\to\varphi_{C_0}(u-vEX, -v)$, and h the continuous Lebesgue-density of U. Then

$$d(b) := \int y^q \varphi_{C_0}(b-yEX, -y)\, dy/h(b)$$

is the q^{th} conditional moment of V, given $U=b$.

The conditional distribution of V, given $U=b$, is normal with mean $a_{12}^T a_{11}^{-1} b$ and variance $a_{22}-a_{12}^T a_{11}^{-1} a_{21}$, where $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the covariance matrix of (U, V). Hence d is a polynomial. Since U has a normal distribution with mean zero and covariance matrix C, this proves the first part of Theorem (2.7).

In order to compute q_0 and q_1 for $p=1$ we expand

$$\sum_j (j!)^{-1}(-1)^j \sum_k t^{-(k+j+1)/2} \sum_l \binom{-(k+j+2)/2}{l}$$
$$t^{-1/2} \frac{EY^{j+1}}{j+1} \int y^l D_2^j(\varphi_{C_0} p_k) \left(\sum_q \binom{-1/2}{q} t^{-q/2} y^q (b-yEX, -y)\right) dy$$

up to an error of order $o(t^{-1})$. The relevant expression is

$$\begin{aligned} &t^{-1/2} \int \varphi_{C_0}((1-t^{-1/2} y/2)(b-yEX, -y))\, dy \\ &\quad -t^{-1} \int y\varphi_{C_0}(b-yEX, -y)\, dy + t^{-1} \int (\varphi_{C_0} p_1)(b-yEX, -y)\, dy \\ &\quad -\tfrac{1}{2} t^{-1} EY^2 \int D_2 \varphi_{C_0}(b-yEX, -y)\, dy =: I_1+I_2+I_3+I_4. \end{aligned}$$

Recall that for $q=0, 1, 2, 3$

$$\int y^q \varphi_{C_0}(b-yEX, -y)\, dy = \varphi_\sigma(b)\, H_q(b)$$

where

$$H_0\equiv 1, \quad H_1(b)=a_{12} a_{11}^{-1} b, \quad H_2=a_{22}-a_{12}^2 a_{11}^{-1}+(a_{12} a_{11}^{-1} b)^2,$$
$$H_3(b)=3a_{12} a_{11}^{-1} b(a_{22}-a_{12}^2 a_{11}^{-1})+(a_{12} a_{11}^{-1} b)^3,$$

φ_σ is the normal density with mean zero and variance σ^2, and – with $C_0=(\sigma_{ij})$ –

$$a_{11}=\sigma^2, \quad a_{12}=-\sigma_{12}+EX\sigma_{22}, \quad a_{22}=\sigma_{22}.$$

Define

$$C_0^{-1}=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Then

$$\begin{aligned}
&I_1+I_2+I_4 \\
&\quad =t^{-1/2}\varphi_\sigma(b)+t^{-1}\int\{y(b-yEX)(A_{11}(b-yEX)-A_{12}y)/2 \\
&\qquad -y^2(A_{12}(b-yEX)-A_{22}y)/2-y \\
&\qquad\quad +(\sigma_{22}+1)(A_{12}(b-yEX), -A_{22}y)/2\}\varphi_{C_0}(b-yEX, -y)\,dy+o(t^{-1}) \\
&\quad =t^{-1/2}\varphi_\sigma(b)(1+t^{-1/2}\{\tfrac{1}{2}(\sigma_{22}+1)A_{12}b] \\
&\qquad +H_1(b)[-1-\tfrac{1}{2}(\sigma_{22}+1)(A_{12}EX+A_{22})+\tfrac{1}{2}A_{11}b^2] \\
&\qquad +b\cdot H_2(b)[-A_{12}-A_{11}EX]+\tfrac{1}{2}H_3(b)[A_{22}+2EXA_{12}+(EX)^2A_{11}]\}) \\
&\qquad +o(t^{-1}) \\
&\quad =t^{-1/2}\varphi_\sigma(b)(1+t^{-1/2}\{\tfrac{1}{2}(\sigma_{22}+1)b[A_{12}-a_{12}a_{11}^{-1}(A_{22}+EXA_{12})] \\
&\qquad +a_{12}a_{11}^{-1}b[-1+\tfrac{1}{2}A_{11}b^2]-(a_{22}-a_{12}^2a_{11}^{-1})b(A_{12}+A_{11}EX) \\
&\qquad -(a_{12}a_{11}^{-1})^2b^3(A_{12}+A_{11}EX) \\
&\qquad +\tfrac{3}{2}a_{12}a_{11}^{-1}b(a_{22}-a_{12}^2a_{11}^{-1})(A_{22}+2EXA_{12}+(EX)^2A_{11}) \\
&\qquad +\tfrac{1}{2}(a_{12}a_{11}^{-1})^3b^3(A_{22}+2EXA_{12}+(EX)^2A_{11})\}) \\
&\qquad +o(t^{-1}).
\end{aligned}$$

The relations

$$\begin{aligned}
A_{12}-a_{12}a_{11}^{-1}(A_{22}+EXA_{12})&=-EX/\sigma^2, \\
a_{12}a_{11}^{-1}&=(a_{22}-a_{12}^2a_{11}^{-1})(A_{12}+EXA_{11}) \\
\tfrac{1}{2}A_{11}a_{12}a_{11}^{-1}-(a_{12}a_{11}^{-1})^2(A_{12}+A_{11}EX)& \\
+\tfrac{1}{2}(a_{12}a_{11}^{-1})^3(A_{22}+2EXA_{12}+(EX)^2A_{11})&=\tfrac{1}{2}a_{12}a_{11}^{-2},
\end{aligned}$$

and $(a_{22}-a_{12}^2a_{11}^{-1})^{-1}=A_{22}+2EXA_{12}+(EX)^2A_{11}$ yield

$$\begin{aligned}
I_1+I_2+I_4=t^{-1/2}\varphi_\sigma(b)(1+t^{-1/2}\{-\tfrac{1}{2}(\sigma_{22}+1)bEX/\sigma^2 \\
-\tfrac{1}{2}a_{12}b/\sigma^2+\tfrac{1}{2}a_{12}b^3/\sigma^4\})+o(t^{-1}).
\end{aligned}$$

Since I_3 is the $n^{-1/2}$-term of the formal Edgeworth expansion for $\sum_1^n X_i-(EX)\sum_1^n Y_i$, this proves the formulas for q_0 and q_1. Under conditions (2.4) and (2.6) the factors $EY^{j+1}/(j+1)$ must be replaced by $\sum l^j P\{Y>l\}$, i.e. EY^2 must be replaced by $\operatorname{var}(Y)$. This gives the asserted formulas for q_0^* and q_1^*.

(2.18) *Remark.* Up to now, the counting process $N(t)$, $t\geqq 0$, was univariate. However, Theorem (2.7) yields approximations for special compound processes with multivariate counting processes. Let, e.g., $N(t)=(N_1(t), N_2(t))$, $t \geqq 0$, be a bivariate Poisson process which can be represented as

$$N_1(t)=Z(t)+V(t)$$
$$N_2(t)=Z(t)+W(t)$$

with stochastically independent Poisson processes $V(t)$, $W(t)$, $Z(t)$ (see Aitken (1944), Holgate (1964), and Dwass and Teicher (1957)).

Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. bivariate random vectors, $X=(X^{(1)}, X^{(2)})$, such that $X^{(1)}$ and $X^{(2)}$ are stochastically independent. The bivariate compound Poisson process

$$S(t)=((X_1^{(j)}+\ldots+X_{N_j(t)}^{(j)})_{j=1,2})$$

can be represented as

$$S(t)=S^*(t)+S^{**}(t)$$

with $S^*(t)=X_1+\ldots+X_{Z(t)}$, the processes $S^*(t)$ and $S^{**}(t)$ are stochastically independent, and the components of the bivariate process $S^{**}(t)$ are stochastically independent. Notice that the counting process for $S^*(t)$ is univariate. Hence Theorem (2.7) yields approximations for the distributions of $S^*(t)$ and $S^{**}(t)$. Convolution of these approximations renders approximations for the distribution of $S(t)$.

If $p=1$ and C_0 is singular, then either

Y is constant, or
X is constant, or
$X=aY+b$ with $a, b \in \mathbb{R}$, $a \neq 0$, $\operatorname{var}(Y)>0$.

If Y is constant, then $N(t)$ is nonrandom. Higher order expansions for the distribution of $S(t)$ with nonrandom $N(t)$ can be found in Bhattacharya and Ranga Rao (1976). If X is constant, then $S(t)\equiv 0$ or there exists $a \neq 0$ such that $S(t)=aN(t)$. For this case we include the following

(2.19) **Theorem.** *If $Y \geqq 0$, $EY>0$, $EY^s<\infty$ for some $s \geqq 3$, and if*

a) *Y satisfies Cramér's condition, then uniformly for $k=0, 1, 2, \ldots$*

$$P\{N(t)=k\}=\psi_{t,s}(t^{-1/2}(k-t/EY))+O(t^{-(s-1)/2})(1+|t^{-1/2}(k-t/EY)|^s)^{-1}.$$

Here

$$\psi_{t,s}(x)=t^{-1/2}\varphi_\sigma(x)\sum_{j=0}^{s-3} t^{-j/2} p_j(x)$$

where $\sigma^2=\operatorname{var}(Y)/E^3 Y$ and p_j, $j=0, \ldots, s-3$, are polynomials.

b) *If Y has a lattice distribution with minimal lattice $\mathbb{Z}$, then uniformly for $k \in \mathbb{Z}$ and integral t*

$$P\{N(t)=k\}=\psi^*_{t,s}(t^{-1/2}(k-t/EY))$$
$$+O(t^{-(s-1)/2})(1+|t^{-1/2}(k-t/EY)|^{s/2})^{-1}$$

where

$$\psi^*_{t,s}(x) = t^{-1/2}\,\varphi_\sigma(x)\sum_{j=0}^{s-3} p^*_j(x)\,t^{-j/2}$$

with polynomials p^*_j, $j=0,\dots,s-3$.

We have in particular with $EY=1$, $\sigma^2=\operatorname{var}(Y)$, $\mu_3=E(Y-1)^3$, *and* $z=x/\sigma$:

$$p_0 \equiv p^*_0 \equiv 1,$$

$$p_1(x) = \sigma\left(-z+\tfrac{1}{2}z^3-\tfrac{1}{2}z\sigma^2\right)+\tfrac{1}{6}(3z-z^3)\,\mu_3/\sigma^3$$

and

$$p^*_1(x) = p_1(x)+\tfrac{1}{2}z.$$

The proof of this theorem can be done following the pattern of proof for Theorem (2.7).

(2.20) **Theorem.** *Assume that* $X=Y$, *that* $Y\geqq 0$ *satisfies* $EY>0$, $EY^s<\infty$, *and*

a) *Cramér's condition. Then uniformly for* $r\geqq 0$

$$P\{S(t)<t-r\} = \int x\,1_{(r,\infty)}(x)\,Q(dx)+O(t^{-(s-2)/2})$$

where Q *is the distribution of* Y.

b) *If* Y *has a lattice distribution with minimal lattice* $\mathbb{Z}$, *then uniformly for integral* $t\geqq 0$ *and* $r\in\mathbb{Z}$

$$P\{S(t)<t-r\} = \int (x+1)\,1_{(r,\infty)}\,Q(dx)+O(t^{-(s-2)/2}).$$

Proof. Apply (4.8.a) and (4.8.c).

(2.21) *Remark.* If $X=aY+b$ with $b\neq 0$ and $a\in\mathbb{R}$, then by Theorem 2.2 the distribution of

$$t^{-1/2}(S(t)-t(a+b))$$

converges weakly to the normal distribution with zero mean and variance $b^2\operatorname{var}(Y)$, provided $EY=1$. Higher order asymptotic expansions could not be derived with the methods developed in the proof of Theorem (2.7).

3. Results for Discrete Markov Chains

Consider an irreducible discrete Markov chain $\xi_0, \xi_1, \xi_2, \dots$ on a countable state space I with transition matrix $(p_{ij})_{i,j\in I}$. Write P_μ for the joint distribution of $\xi_0, \xi_1, \xi_2, \dots$ when μ is the initial distribution, and E_μ for the corresponding expectation. If μ is concentrated at some $i\in I$ then we shall use the symbols P_i and E_i.

Let $0\in I$ be a fixed element of S. Define

$$T_0=\min\{k\geqq 0\colon \xi_k=0\},\qquad T_m=\min\{k>T_{m-1}\colon \xi_k=0\},$$

and $Y_m = T_m - T_{m-1}$, $m=1,2,\dots$.

For $f\colon I\to\mathbb{Z}$ we are interested in the asymptotic behavior of

$$P_\mu\{f(\xi_1)+\ldots+f(\xi_N)=z\},$$

N large and $z\in\mathbb{Z}$.

Define $X_m=\sum\{f(\xi_k)\colon T_{m-1}<k\leqq T_m\}$, $m=1,2,\ldots$. The Markov property implies that – under P_0 – the sequence $(X_1,Y_1),(X_2,Y_2),\ldots$ is idd.

If $EY_1<\infty$ then there exists an initial distribution π – the stationary distribution – for which P_π is stationary. The asymptotic behavior of $P_\mu\{f(\xi_1)+\ldots+f(\xi_N)=z\}$ is described by the following approximation in variational norm.

(3.1) **Theorem.** *Let μ be an initial distribution, $s\geqq 3$ an integer, and assume that $E_0(\sum\{|f(\xi_\nu)|\colon 1\leqq\nu\leqq T_1\})^s<\infty$, $E_0 Y_1^s<\infty$, $E_\mu T_0^{s-2}<\infty$, and $E_\mu(\sum\{|f(\xi_\nu)|\colon 1\leqq\nu\leqq T_0\})^{s-2}<\infty$. If, under P_0, the random vector (X_1,Y_1) has a lattice distribution with minimal lattice $\mathbb{Z}^2$, then for some polynomials $q_0,\ldots,q_{s-3}$ we have*

$$\sum_{z\in\mathbb{Z}}\left|P_\mu\left\{\sum_{\nu=1}^N f(\xi_\nu)=z\right\}-N^{-1/2}\varphi_\sigma(x_N)\sum_{r=0}^{s-3}N^{-r/2}q_r(x_N)\right|=O(N^{-(s-2)/2}).$$

Here, $x_N=N^{-1/2}(z-NM)$, $M=E_\pi f(\xi_1)$,

$$\sigma^2=E_\pi(f(\xi_1)-M)^2+2\sum_{\nu=2}^{\infty}E_\pi(f(\xi_1)-M)(f(\xi_\nu)-M)>0$$

and φ_σ is the univariate normal density with mean zero and variance σ^2.

Proof. Our starting point is the well known relation (see [3, p. 61])

$$\begin{aligned}P_\mu&\left\{\sum_{\nu=1}^N f(\xi_\nu)=z\right\}\\&=\sum_{l,m,n=0}^{N}\sum_{x,y\in\mathbb{Z}}P_\mu\left\{\sum_{\nu=1}^{l}f(\xi_\nu)=x,\ Y_0=l\right\}\\&\quad\cdot P_0\left\{\sum_{\nu=1}^{m}X_\nu=z-x-y,\ \sum_{\nu=1}^{m}Y_\nu=N-n-l\right\}\\&\quad\cdot P_0\left\{\sum_{\nu=1}^{n}f(\xi_\nu)=y,\ Y_1>n\right\}.\end{aligned}$$

Notice first that for arbitrary positive ε we can omit all terms corresponding to m with $|m-N/E_0Y_1|\geqq\varepsilon N$. Furthermore, all terms corresponding to $l\geqq N^{1/2}$ or $n\geqq N^{1/2}$ can be neglected. The total error of these omissions is of order $O(N^{-(s-2)/2})$. Finally, we can omit all terms corresponding to $|x|\geqq N^{1/2}$ or $|y|\geqq N^{1/2}$ without increasing the order of the error. In order to prove the last statement consider first the sum over $|x|\geqq N^{1/2}$. Let $\sum^*$ denote the sum over $|m-N/E_0Y_1|\leqq\varepsilon N$. Using local approximations for

$$P_0\left\{\sum_{\nu=1}^{m}Y_\nu=N-n-l\right\}$$

we easily see that

$$\sum^* P_0\left\{\sum_{\nu=1}^{m} Y_\nu = N-n-l\right\}$$

is uniformly bounded for all N, n, and l. Hence

$$\begin{aligned}
&\sum^* \sum_{l,n=0}^{N} \sum_{y,z\in\mathbb{Z}} \sum_{|x|\geqq N^{1/2}} P_\mu\left\{\sum_{\nu=1}^{l} f(\xi_\nu)=x, Y_0=l\right\}\\
&\quad\cdot P_0\left\{\sum_{\nu=1}^{m} X_\nu = z-x-y, \sum_{\nu=1}^{m} Y_\nu = N-n-l\right\} P_0\left\{\sum_{\nu=1}^{n} f(\xi_\nu)=y, Y_1>n\right\}\\
&\leqq K \sum_{l,n=0}^{N} \sum_{y\in\mathbb{Z}} \sum_{|x|\geqq N^{1/2}} P_\mu\left\{\sum_{\nu=1}^{l} f(\xi_\nu)=x, Y_0=l\right\}\\
&\quad\cdot P_0\left\{\sum_{\nu=1}^{n} f(\xi_\nu)=y, Y_1>n\right\}\\
&\leqq K P_\mu\left\{\left|\sum_{\nu=1}^{Y_0} f(\xi_\nu)\right|\geqq N^{1/2}\right\} E_0 Y_1\\
&\leqq K E_\mu\left|\sum_{\nu=1}^{Y_0} f(\xi_\nu)\right|^{s-2} E_0 Y_1 N^{-(s-2)/2}
\end{aligned}$$

with a constant K not depending on N. Similarly,

$$\begin{aligned}
&\sum^* \sum_{l,n=0}^{N} \sum_{x,z\in\mathbb{Z}} \sum_{|y|\geqq N^{1/2}} P_\mu\left\{\sum_{\nu=1}^{l} f(\xi_\nu)=x, Y_0=l\right\}\\
&\quad\cdot P_0\left\{\sum_{\nu=1}^{m} X_\nu = z-x-y, \sum_{\nu=1}^{m} Y_\nu = N-n-1\right\} P_0\left\{\sum_{\nu=1}^{n} f(\xi_\nu)=y, Y_1>n\right\}\\
&\leqq K \sum_{n=0}^{N} P_0\left\{\left|\sum_{\nu=1}^{n} f(\xi_\nu)\right|\geqq N^{1/2}, Y_1>n\right\}\\
&\leqq K N^{-s/2} \sum_{n=0}^{N} E_0\left(\sum |f(\xi_\nu)|: 1\leqq \nu\leqq Y_1\right)^s = O(N^{-(s-2)/2}).
\end{aligned}$$

We shall now approximate the terms

$$P_0\left\{\sum_{\nu=1}^{m} X_\nu = z-x-y, \sum_{\nu=1}^{m} Y_\nu = N-n-l\right\} \tag{3.2}$$

in the remaining sum by local asymptotic expansions. In order to apply our Lemma 4.4 the random variables X_ν and Y_ν have to be truncated at $N^{1/2}$. Define

$${}^N X_\nu = X_\nu 1_{\{|X_\nu|\leqq N^{1/2}\}} \quad\text{and}\quad {}^N Y_\nu = Y_\nu 1_{\{Y_\nu\leqq N^{1/2}\}},\ \nu=1,\dots,N,$$

and in the above sum replace the terms (3.2) by

$$P_0\left\{\sum_{\nu=1}^{m} {}^N X_\nu = z-x-y, \sum_{\nu=1}^{m} {}^N Y_\nu = N-n-1\right\}. \tag{3.3}$$

The error made by this replacement is of order $O(N^{-(s-2)/2})$. Our assumptions for the joint distribution of (X_1, Y_1) under P_0 are curtailed for a possible application of Lemma 4.4. We obtain that the terms (3.3) can be replaced by

$$(3.4)\qquad m^{-1} g_m(m^{-1/2}(z-x-y-mE_0 X_1), m^{-1/2}(N-n-1-mE_0 Y_1)).$$

The error induced by this replacement is bounded by

$$\sum{}^* \sum_{l,n=0}^{N} \sum_{x,y,z\in\mathbb{Z}} P_\mu\left\{\sum_{\nu=1}^{l} f(\xi_\nu)=x,\, Y_0=l\right\} P_0\left\{\sum_{\nu=1}^{n} f(\xi_\nu)=y,\, Y_1>n\right\}$$
$$\cdot O(N^{-s/2})(1+N^{-1}(z-x-y-mE_0 X_1)^2)^{-1}(1+N^{-1}(N-n-l-mE_0 Y_1)^2)^{-1}.$$

The factors

$$N^{-1/2}\sum_{z\in\mathbb{Z}}(1+N^{-1}(z-x-y-mE_0 X_1)^2)^{-1}$$

and

$$N^{-1/2}\sum_{m=1}^{N}(1+N^{-1}(N-n-l-mE_0 Y_1)^2)^{-1}$$

remain bounded uniformly in x, y, m and N, n, l, respectively. Hence the total error induced by replacing (3.3) by (3.4) is of order $O(N^{-(s-2)/2})$.

Now we replace the terms (3.4) by their Taylor expansions at $(m^{-1/2}(z-mE_0 X_1), m^{-1/2}(N-mE_0 Y_1))$ of order $s-2$. As in the proof for Theorem 2.7, part *d*, we see that this replacement yields an error of at most the order $O(N^{-(s-2)/2})$. The terms of this expansion are of the following kind:

$$\sum{}^*\sum{}'\sum_{x,y} x^{r_1} y^{r_2} l^{r_3} n^{r_4} P_\mu\left\{\sum_{\nu=1}^{l} f(\xi_\nu)=x,\, Y_0=l\right\}$$

$$(3.5)\qquad \cdot P_0\left\{\sum_{\nu=1}^{n} f(\xi_\nu)=y,\, Y_1>n\right\} m^{-1} H(\bar z,\bar m)\,\varphi_D(\bar z,\bar m)\, m^{-r/2}.$$

Here, $\sum'$ extends over $0\leqq l,\ n\leqq N^{1/2}$, the sum over x, y extends over $|x|, |y|\leqq N^{1/2}$. H is a polynomial, $\bar z=m^{-1/2}(z-mE_0 X_1)$, $\bar m=m^{-1/2}(N-mE_0 Y_1)$, D is the non singular covariance matrix of (X_1, Y_1) under P_0, and φ_D is the bivariate normal density with zero mean and covariance matrix D. Now we may extend the domain of summation to $x, y\in\mathbb{Z}$, and $n, l\in\mathbb{Z}$. This extension yields an error which is at most of the order $O(N^{-(s-2)/2})$. The sums over x, y, l, and n yield moments of the type

$$E_\mu Y_0^{r_3}(\sum\{f(\xi_\nu)\colon 1\leqq\nu\leqq Y_0\})^{r_1}$$

and

$$\sum_{n\geqq 0} n^{r_4} E_0\left(\sum_{\nu=1}^{n} f(\xi_\nu)\right)^{r_2} 1_{\{Y_1>n\}}.$$

Notice that the last sum converges for $r_2+r_4\leqq s$ since it is bounded by $E_0 Y_1^{r_4}(\sum\{|f(\xi_\nu)|\colon 1\leqq\nu\leqq Y_1\})^{r_2}$. As before, define

$$a=N^{-1/2}(m-N/E_0 Y_1)$$
$$b=N^{-1/2}(z-NE_0 X_1/E_0 Y_1).$$

It suffices to consider only those m and z for which $|a| \leqq \log N$ and $|b| \leqq \log N$. Using

$$m/N = (E_0 Y_1)^{-1} + N^{-1/2} a$$
$$\bar{m} = -(m/N)^{-1/2} E_0 Y_1 a$$
$$\bar{z} = (m/N)^{-1/2} (b - E_0 X_1 a)$$

we can replace all powers of $m^{-1/2}$ by powers of $N^{-1/2}$, and polynomials in $\bar{z}, \bar{m}$ by polynomials in a and b. After this replacement – which does not influence the order of the error – the terms (3.5) are substituted by linear combinations of terms of the following kind:

$$(3.6) \qquad N^{-r/2} \sum_m N^{-1} a^{r_1} b^{r_2} \varphi_D((E_0 Y_1)^{-1}(b - E_0 X_1 a, -a)).$$

Here the sum extends over all m for which $|a| \leqq \log N$. Now replace the sum by the sum over $m \in \mathbb{Z}$ and use the Euler-Maclaurin formula. Then (3.6) is replaced by a linear combination of terms of the kind

$$(3.7) \qquad N^{-r/2} b^{r_2} N^{-1/2} \int a^{r_1} \varphi_D((E_0 Y_1)^{-1}(b - E_0 X_1 a, -a))\, da.$$

As in the proof of Theorem 2.7, part d, we use that the terms (3.7) can be written as follows:

$$N^{-r/2} p_r(b) \varphi_\sigma(b)$$

with a polynomial p_r. To complete the proof we have to show that $M = E_0 X_1 / E_0 Y_1$ and $\sigma^2 > 0$ is the one defined in the assertion.

We have $E_0 X_1 = E_0 Y_1 E_\pi f(\xi_1)$ by Wald's identity (see, e.g., [12, p. 67, Theorem 1.10]).

The variance σ^2 equals $C^T D C$ with $C = (1, -E_0 X_1 / E_0 Y_1)^T$ which is positive by assumption. Since $s \geqq 3$, σ^2 must equal $\lim_N N^{-1} E_0 \left[\sum_{\nu=1}^{N} (f(\xi_\nu) - M) \right]^2$ which is the σ^2 defined in the assertion.

For applications the conditions of Theorem 3.1 have to be replaced by conditions on the Markov chain and on f which are easily checked or at least more familiar. It is known that the moment condition for Y_1 can be replaced by a strong mixing condition of the chain. For details see [3], Theorem 2 and the Corollaries. The lattice condition of our Theorem 3.1 can be replaced by a lattice condition for the possible values of f on paths in the chain. To be more specific, a sequence $i_0, i_1, \ldots, i_{n-1}$ of elements of I with $P_{i_0}\{\xi_k = i_k,\ k = 0, \ldots, n-1\} > 0$ will be called a path from i_0 to i_{n-1} of length n. A path $i_0, \ldots, i_{n-1}$ avoids $i_n \in I$ if $i_k \neq i_n$, $k = 0, \ldots, n-1$. The number $\sum\{f(i_k): 0 \leqq k \leqq n-1\}$ is called the f-value of the path $i_0, \ldots, i_{n-1}$.

A simplified version of our Theorem 3.1 now reads as follows:

(3.8) **Corollary.** *Let $\alpha(n)$, $n = 1, 2, \ldots$, be the strong mixing coefficients of the stationary chain. Assume that $f: I \to \mathbb{Z}$ is bounded, $\sum n^{s-2} \alpha(n) < \infty$, that $p(i, i) > 0$ for some $i \in I$, and that there exists a 0-avoiding path from i to i with f-value $3f(i)$*

$+1$ [*or* $3f(i)-1$]. *Then for some polynomials* $q_0, \dots, q_{s-3}$ *we have*

$$\sum_{z\in\mathbb{Z}}\left|P_0\left\{\sum_{\nu=1}^{N} f(\xi_\nu)=z\right\}-N^{-1/2}\varphi_\sigma(x_N)\sum_{r=0}^{s-3}N^{-r/2}q_r(x_N)\right|=O(N^{-(s-2)/2})$$

with x_N, M, *and* σ^2 *as in Theorem 3.1.*

Proof. All moment conditions of Theorem 3.1 follow from the strong mixing condition and boundedness of f. We have to show that, under P_0, (X_1, Y_1) has a lattice distribution with minimal lattice $\mathbb{Z}^2$. To this aim we have to find a sufficiently large subset of the support L of (X_1, Y_1). Since the chain is recurrent there exists a shortest path from 0 to i with length n_1 and f-value x_1, say, and a shortest path from i to 0 with length n_2 and f-value x_2, say. The condition $p(i,i)>0$ implies that for all $k=0,1,2,\dots$ we have

$$(x_1+x_2-f(i)+kf(i), n_1+n_2-1+k)\in L.$$

These values correspond to paths which are constructed taking the shortest path from 0 to 0 via i and pausing k times in state i. If instead of pausing we fit into this path the 0-avoiding path k times, we arrive at

$$(x_1+x_2+k(f(i)+1), n_1+n_2+k(n_3-2))\in L, \qquad k=1,2,\dots$$

where n_3 is the length of the 0-avoiding path. We have to show that $\mathbb{Z}^2$ is generated by $L-L$. Since the set $L-L$ remains unchanged if f is replaced by $f+c$ where c is an integral constant, we may assume w.l.g. that $f(i)=0$. Then $(0,1)\in L-L$ and $(1, n_3-2)\in L-L$ implies the assertion. The case that the 0-avoiding path has f-value $3f(i)-1$ can be dealt with similarly.

(3.9) *Remark.* Theorem 3.1 can be extended to more general processes $\xi_0, \xi_1, \dots$. All we need is the existence of a sequence of stopping times $T_0, T_1, T_2, \dots$ with the following properties (i)–(iii). Define

$$L=\sum\{f(\xi_\nu):\ 1\leqq\nu\leqq T_0\}$$
$$X_i=\sum\{f(\xi_\nu):\ T_{i-1}<\nu\leqq T_i\}, \qquad i=1,2,\dots$$
$$Y_i=T_i-T_{i-1}, \qquad i=1,2,\dots$$

i) The vectors (X_i, Y_i), $i=1,2,\dots$ are iid;

ii) L and (X_i, Y_i), $i=1,2,\dots$ are independent;

iii) (X_1, Y_1) has a lattice distribution with minimal lattice $\mathbb{Z}^2$. Under these assumptions and under certain moment conditions, expansions for $P\{f(\xi_1)+\dots+f(\xi_N)=z\}$ are valid. Stopping times $T_0, T_1, \dots$ with properties (i) and (ii) exist for general renewal sequences $\xi_0, \xi_1, \dots$. It is not clear, however, whether property (iii) can easily be checked for general T_i.

4. Auxiliar Results

Our first lemma is an immediate consequence of Kolmogorov's inequality.

(4.1) **Lemma.** *Let* $X_1, X_2, \dots$ *be a sequence of independent identically distributed random variables with mean zero and finite variance, and let* N *be a positive* *integer valued random variable. Then for all positive integers k, m and for any positive number e*

$$P\{|X_1+\ldots+X_N-(X_1+\ldots+X_k)|\geqq e\}\leqq P\{|N-k|\geqq m\}+2mEX_1^2/e^2.$$

Let now $Y, Y_1, Y_2, \ldots$ be a sequence of independent identically distributed nonnegative random variables with positive mean and finite moment of order $s\geqq 3$. For $t\geqq 0$ define

$$N(t)=\max\{k\colon\ Y_1+\ldots+Y_k\leqq t\}.$$

(4.2) **Lemma.**

$$P\{|N(t)-t/EY|\geqq(\operatorname{var}(Y)(s-2)t\log t)^{1/2}/EY\}=o(t^{-(s-2)/2}).$$

Proof. For positive integer r we have $N(t)\geqq r$ iff $Y_1+\ldots+Y_r\leqq t$. The lemma follows from moderate deviation results for the partial sums of the Y's. See [11].

Let $X, X_1, X_2, \ldots$ be a sequence of independent identically distributed p-variate random vectors, assume that for some integers $s>2$

$$E\|X\|^s<\infty$$

and that the covariance matrix D of X is nonsingular. For $t>0$ let ${}^tX, {}^tX_1, {}^tX_2, \ldots$ be the random vectors truncated at $t^{1/2}$, i.e.

$${}^tX_\nu=X_\nu 1_{\{\|X_\nu\|\leqq t^{1/2}\}}.$$

Fix positive constants ε and c, and define

$$A(t)=\{m=0,1,\ldots\colon |m-ct|\leqq\varepsilon t\}.$$

For $m\in A(t)$ let

$$H(m)={}^tX_1+\ldots+{}^tX_m.$$

For nonnegative integral p-vector $\nu=(\nu_1,\ldots,\nu_p)$ with $\nu_1+\ldots+\nu_p\leqq s$ write χ_ν for the cumulant of X of order ν, and introduce the functions

$$u\to\tilde{P}_r(iu\colon\{\chi_\nu\})$$

as in [2], p. 51. Furthermore, let

$$f_m(u)=\sum_{r=0}^{s-3} m^{-r/2}\tilde{P}_r(iu\colon\{\chi_\nu\})\exp(-\tfrac{1}{2}u^T Du).$$

Let g_m be the Lebesgue-density of the finite signed measure with characteristic function f_m. We can write $g_m(u)$ as follows:

$$g_m(u)=\varphi_D(u)\sum_{r=0}^{s-3} m^{-r/2}p_{r,m}(u)$$

where φ_D is the Lebesgue-density of the p-variate normal law with mean zero and covariance matrix D, and $p_{r,m}$ are polynomials with coefficients which remain bounded as long as m varies in $\{m \geqq 1\}$. Write M for the mean of ${}^tX$. Let p_1, p_2 be positive integers with $p_1+p_2=p$. Consider the decomposition $X=(X', X'')$ of X into random vectors X' and X'' which are p_1- and p_2-variate, respectively. Let $H'(m)$, $H''(M)$, M', M'' be the corresponding decompositions of $H(m)$ and M.

(4.3) **Lemma.** *Assume that $E\|X\|^s<\infty$, that D is nonsingular, and that X satisfies Cramér's condition*

$$\limsup_{u\to\infty} |E\exp(iu^T X)|<1.$$

Then uniformly for convex measurable $A\subset\mathbb{R}^{p_1}$, $B\subset\mathbb{R}^{p_2}$ and $m\in A(t)$

$$\begin{aligned} &P\{m^{-1/2}(H(m)-mM)\in A\times B\}\\ &\quad=\int_{A\times B} g_m(u)\,du+O(t^{-(s-2)/2})\,\lambda''(B)(1+d(0,\partial B)^{p_2+1})^{-1}+O(t^{-(s+1)/2}). \end{aligned}$$

Here λ'' is the p_2-variate Lebesgue-measure, ∂B is the boundary of B, and $d(x,A)$ is the Euclidean distance between the point x and the set A.

(4.4) **Lemma.** *Assume that X has a lattice distribution with minimal lattice $\mathbb{Z}^p$. If $E\|X\|^s<\infty$ for some $s\geqq 3$ and D is nonsingular, then uniformly for $z\in\mathbb{Z}^p$ and $m\in A(t)$ and for arbitrary fixed $a>0$:*

$$\begin{aligned} &P\{H(m)=z\}-m^{-p/2}g_m(m^{-1/2}(z-mM))\\ &\qquad\qquad=O(t^{-(s+p-2)/2})*(1+\|m^{-1/2}(z-mM)\|^a)^{-1}. \end{aligned}$$

(4.5) **Lemma.** *Assume that $E\|X\|^s<\infty$, $s\geqq 3$, that D is nonsingular, that X'' has a lattice distribution with minimal lattice $\mathbb{Z}^{p_2}$, and X satisfied the following condition: For all $e>0$ there exists a positive constant d such that for $u\in\mathbb{R}^{p_1}$, $v\in\mathbb{R}^{p_2}$ with $\|u\|\geqq e$*

$$|E\exp(iu^T X'+iv^T X'')|\leqq 1-d.$$

Then uniformly for convex measurable $A\subset\mathbb{R}^{p_1}$, $z\in\mathbb{Z}^{p_2}$, and $m\in A(t)$ and for arbitrary fixed $a>0$

$$\begin{aligned} &P\{m^{-1/2}(H'(m)-mM')\in A \text{ and } H''(m)=z\}\\ &\quad=m^{-p_2/2}\int_A g_m(u, m^{-1/2}(z-mM''))\,du\\ &\qquad+O(t^{-(s+p_2-2)/2})(1+\|m^{-1/2}(z-mM'')\|^a)^{-1}. \end{aligned}$$

(4.6) **Lemma.** *Assume that $s>p_2$, $E\|X\|^s<\infty$, that D is nonsingular, that X' has a lattice distribution with minimal lattice $\mathbb{Z}^{p_1}$, and X satisfies the following condition: For all $e>0$ there exists a positive constant d such that for $u\in\mathbb{R}^{p_1}$ and $v\in\mathbb{R}^{p_2}$ with $\|v\|\geqq e$*

$$|E\exp(iu^T X'+iv^T X'')|\leqq 1-d.$$

Then uniformly for all convex measurable $A \subset \mathbb{R}^{p_2}$, $z \in \mathbb{Z}^{p_1}$, and $m \in A(t)$, and for arbitrary fixed $a > 0$

$$\begin{aligned}
&P\{H'(m) = z \text{ and } m^{-1/2}(H''(m) - mM'') \in A\} \\
&\quad = n^{-p_1/2} \int_A g_m(m^{-1/2}(z - mM'), v)\, dv + O(t^{-(s+p_1+p_2)/2}) \\
&\qquad + O(t^{-(s+p_1-2)/2})(1 + d(0, \partial A)^a)^{-1} \lambda''(A)(1 + \|m^{-1/2}(z - mM')\|^a)^{-1}.
\end{aligned}$$

The proof of Lemmas (4.3) and (4.6) is based on the following

(4.7) **Lemma.** *Let $g: \mathbb{R}^p \to \mathbb{R}$ be measurable such that*

$$\int (1 + \|x\|^{p_1+1}) |g(x)|\, dx < \infty.$$

Then for all measurable subsets $A \subset \mathbb{R}^{p_1}$ and $B \subset \mathbb{R}^{p_2}$ ($p = p_1 + p_2$),

$$\Big| \int_{A \times B} g(x)\, dx \Big| \leqq c(p) \max_\beta \int |D^\beta \hat{g}(u)|\, du\, \lambda''(B)$$

where $c(p)$ is a constant depending on p only, $\hat{g}$ is the Fourier-transform of g, the maximum is taken over all nonnegative integral p-vectors $\beta = (\beta_1, \ldots, \beta_p)$ with $\beta_1 + \ldots + \beta_p \leqq p_1 + 1$, and D^β is the operator

$$\partial^{\beta_1 + \ldots + \beta_p} / (\partial x_1^{\beta_1} \ldots \partial x_p^{\beta_p}).$$

If $p_1 = 0$, then we have

$$\Big| \int_B g(x)\, dx \Big| \leqq c(p) \int |\hat{g}(u)|\, du\, \lambda''(B).$$

The proof of Lemma (4.7) is the proof of Lemma 11.6 in [2], p. 98–99.

We sketch the proofs of Lemmas (4.3)–(4.6). The proofs are rather technical; they are based on well known methods which are described in Bhattacharya and Ranga Rao's monograph (1976). Lemma (4.3) follows from (20.12), p. 209, (20.17), p. 210, Lemma (4.7), and the proof of (20.41), p. 210–214 in [2]. Lemma (4.4) follows from (22.32) and (22.33) in [2], p. 236. Notice that the proof given there works for arbitrary positive integral s_0 (which is defined in (22.22), p. 234). Lemma (4.5) follows from Theorem 9.10 in [2] (expansions for the characteristic function of $H(m)$), and from relation (2.20) in [9]. Also here an arbitrary exponent a can be used since the random vectors under consideration are truncated.

Now we give a more detailed proof for Lemma (4.6). Let $h_m(u, v)$ be the characteristic function of $n^{-1/2}(H(m) - mM)$ and $f_m(u, v)$ the characteristic function of the Edgeworth-expansion with density $g_m(u, v)$. Define

$$L(m) = \{m^{-1/2}(z - mM') \colon z \in \mathbb{Z}^{p_1}\}.$$

For $m \in A(t)$ and $u \in L(m)$ let $S_m(u)$ and $T_m(u)$ be the finite (signed) measures defined by

$$A \to P\{m^{-1/2}(H'(m) - mM') = u \text{ and } m^{-1/2}(H''(m) - mM'') \in A\}$$

$$\text{and} \quad A \to m^{-p_1/2} \int_A g_m(u, v)\, dv,$$

respectively. It suffices to show that for all polynomials $p(u)$, $q(v)$ we have uniformly for convex measurable $A \subset \mathbb{R}^{p_2}$, $m \in A(t)$, and $u \in L(m)$

$$p(u)\left|\int_A q(v)(S_m(u)-T_m(u))(dv)\right| = O(t^{-(s+p_1-2)/2})\lambda''(A) + O(t^{-(s+p_1+p_2)/2}).$$

We may assume w.l.g. that $p(u)>0$. We apply Corollary 11.5 in [2], p. 97–98, for $p(u)S_m(u)$ and $p(u)T_m(u)$ instead of μ and ν, respectively. Choosing $\varepsilon = t^{-(s+p_1+p_2)/2}$ yields that the second term of the r.h.s. in (11.26) in [2], p. 98, is of order $O(t^{-(s+p_1+p_2)/2})$ (notice that $A \subset \mathbb{R}^{p_2}$ is convex). With formula (2.20) of [9] we now compute the characteristic function of $p(u)(S_m(u)-T_m(u))$, say $k_m(u,v)$:

$$\begin{aligned} k_m(u,v) &= p(u)\int \exp(iv^T x)(S_m(u)-T_m(u))(dx) \\ &= p(u)(2\pi)^{-p_1} m^{-p_1/2} \int_K \exp(-iu^T w)(h_m(w,v)-f_m(w,v))\,dw \end{aligned}$$

where $K=\{w \in \mathbb{R}^{p_1}: -\pi m^{1/2} \leqq w_j < \pi m^{1/2}, j=1,\dots,p_1\}$. The inversion formula (21.29) in [2], p. 230, implies that $k_m(u,v)$ is a linear combination of terms of the following kind:

$$m^{-p_1/2}\int_K \exp(-iu^T w)[D^\beta(h_m(w,v)-f_m(w,v))]\,dw$$

where $\beta=(\beta_1,\dots,\beta_p)$ is a nonnegative integral vector with $\beta_1+\dots+\beta_p$ not exceeding the degree of p. Using Theorem 9.12 in [2], p. 83, and the Cramér type condition for X, we obtain that for nonnegative integral p-vectors β

$$\int |D^\beta k_m(u,v)|\, 1_{\{\|v\| \leqq t^{(s+p_1+p_2)/2}\}}\,dv$$

is of order $O(t^{-(s-2)/2})$. This together with Lemma (4.7) proves (4.6).

Let $X, X_1, X_2, \dots$ be a sequence of independent identically distributed nonnegative random variables with positive mean μ, variance σ^2, satisfying

$$EX^s < \infty$$

for some integer $s \geqq 3$. Write U for the renewal function of X, i.e. for $t>0$ let

$$U(t) = 1 + \sum_{n=1}^{\infty} P\{X_1+\dots+X_n < t\}.$$

(4.8) **Lemma.** a) *If* X *satisfies Cramér's condition*

$$\limsup_{t\to\infty} |E\exp(itX)| < 1$$

then

(4.9)
$$U(t) = \frac{t}{\mu} + \frac{\sigma^2+\mu^2}{2\mu^2} + o(t^{-(s-2)/2}).$$

b) *If X has a nonlattice distribution, i.e. for all $t \neq 0$ $|E\exp(itX)| \neq 1$, then* (4.9) *holds for* $s=2$.

c) *If X has a lattice distribution with minimal lattice $\mathbb{Z}$, then* (4.9) *is true for* $t \in \mathbb{Z}$.

Better higher order approximations for $U(t)$ are given by van der Genugten (1969) for the case that X has a moment generating function.

Proof. Part b) is relation (4.5) in Feller (1966), p. 357. Let $Y_1, Y_2, \dots$ be independent normally distributed random variables with mean μ, and variance σ^2, and $U'(t) = \sum_{n=0}^{\infty} P\{Y_1 + \dots + Y_n < t\}$. Then

$$U(t) - U'(t) = \sum{}'(P\{X_1 + \dots + X_n < t\} - P\{Y_1 + \dots + Y_n < t\}) + o(t^{-(s-2)/2})$$

where the dashed sum extends over $n \geqq 1$ satisfying $|n - t/\mu| \leqq \varepsilon t$, $\varepsilon > 0$ small. Let $\Psi_{n,s}(t)$ be the formal Edgeworth expansion for $P\{n^{-1/2}(X_1 + \dots + X_n - n\mu) < t\}$, which has derivative

$$\varphi_\sigma(x) \sum_{j=0}^{s-2} n^{-j/2} p_j(x).$$

Here $p_0, \dots, p_{s-1}$ are polynomials and φ_σ is the density of the normal law with mean zero and variance σ^2.

a) We have uniformly for n in the range of summation

$$P\{X_1 + \dots + X_n < t\} = \Psi_{n,s}(n^{-1/2}(t - n\mu)) + o(t^{-(s-2)/2})(1 + |n^{-1/2}(t - n\mu)|^s)^{-1}$$

(see [2], p. 215, Corollary 20.5). Write

$$a = (t/\mu)^{-1/2}(n - t\mu).$$

Then

$$n\mu/t = 1 + (t/\mu)^{-1/2} a.$$

Replace $n^{-1/2}$ and $n^{-j/2}$ in $\Psi_{n,s}$ by suitable expressions in a and $(t/\mu)^{-1/2}$. Then

$$U(t) - U'(t) = \sum{}' \Psi^*_{t,s}(t^{-1/2}(t - n\mu)) + o(t^{-(s-2)/2})$$

where $\Psi^*_{t,s}$ has derivative

$$\varphi_\sigma(x) \sum_{j=1}^{s-2} t^{-j/2} p_j^*(x)$$

and p_j^*, $j = 1, \dots, s-2$, are polynomials. The Euler-Maclaurin formula yields that for $j = 1, \dots, s-2$

$$t^{-j/2} \sum{}'(\varphi_\sigma p_j^*)(t^{-1/2}(t - n\mu)) = t^{-(j-1)/2} \int \varphi_\sigma(x) p_j^*(x)\,dx + o(t^{-(s-2)/2}).$$

This implies that there exist constants $a_0, \ldots, a_{s-3}$ with

(4.10) $$U(t) = U'(t) + \sum_{j=0}^{s-3} t^{-j/2} a_j + o(t^{-(s-2)/2})$$

For $U'(t)$, Theorem 3 in [7], p. 332/333 yields the approximation

$$U'(t) = t/\mu + \tfrac{1}{2}(\sigma^2 + \mu^2)/\mu^2 + o(t^{-(s-2)/2})$$

i.e.

(4.11) $$U(t) = t/\mu + \tfrac{1}{2}(\sigma^2 + \mu^2)/\mu^2 + \sum_{j=0}^{s-3} a_j t^{-j/2} + o(t^{-(s-2)/2}).$$

Consider a random variable Y with $EX^k = EY^k$, $k = 1, \ldots, s$, such that the distribution of Y has a Lebesgue-density and a moment generating function. If we repeat the above proof for Y instead of X we arrive at (4.11) with the same constants a_j. Comparing (4.11) with (1.12) in [7] we see that $a_j = 0$, $j = 0, \ldots, s-3$. This proves part a) of the lemma.

c) Corollary 22.3 in [2], p. 237, implies that uniformly for $z \in \mathbb{Z}$ and n in the range of summation

$$|P\{X_1 + \ldots + X_n = z\} - \psi_{n,s}(n^{-1/1}(z - n\mu))|(1 + |n^{-1/2}(z - n\mu)|^s) = o(t^{-(s-1)/2})$$

where $\psi_{n,s}$ is the derivative of $\Psi_{n,s}$. Then uniformly for $z \in \mathbb{Z}$ and n in the range of summation

$$|P\{X_1 + \ldots + X_n < z\} - G_{n,s}(n^{-1/2}(z - n\mu))|(1 + |n^{-1/2}(z - n\mu)|^{s/2}) = o(t^{-(s-)/2})$$

where $G_{n,s}(x)$ is the approximation for $F_n(x)$ in (23.3), [2], p. 238. Since z is an integer, the argument in the functions S_α in (23.3) are zero. Hence we can repeat the replacement procedure of $t^{-1/2}$ for $n^{-1/2}$. We thus obtain that uniformly for $z \in \mathbb{Z}$ and n in the range of summation

$$|P\{X_1 + \ldots + X_n < z\} - G^*_{t,s}(t^{-1/2}(z - n\mu))|(1 + |t^{-1/2}(z - n\mu)|^{s/2}) = o(t^{-(s-2)/2})$$

where $G^*_{t,s}$ has derivative

$$\varphi_\sigma(x) \sum_{j=0}^{s-2} t^{-j/2} p^*_j(x)$$

and p^*_j are polynomials. Now the proof is completed as in case a). The random variable Y is here chosen such that Y has a lattice distribution with minimal lattice $\mathbb{Z}$.

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