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Maxima of Stationary Gaussian Processes

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Summary. Let $\{X_N, N = 0, \pm 1, \pm 2, \ldots\}$ be a stationary Gaussian stochastic process with means zero, variances one, and covariance sequence $\{r_N\}$. Let $Z_N = \max_{1 \leq k \leq N} X_k$. Limit properties are obtained for Z_N, as N approaches infinity. A double exponential limit law is known to hold if the random variables X_i are mutually independent, that is $r_N \equiv 0$, $N \neq 0$. BERMAN has shown that the same law holds in the case of dependence, provided r_N approaches zero "sufficiently fast". Specifically sufficient conditions are that either $\lim_{N\to\infty} r_N \log N = 0$, or $\sum_{N=1}^{\infty} r_N^2 < \infty$. In the present work, it is shown, however, that $\lim_{N\to\infty} r_N = 0$ is not sufficient. A corresponding law is obtained for a separable, measurable version of a continuous parameter process. Sufficient conditions are obtained for the "strong laws of large numbers",

$$Z_N - \sqrt{2 \log N} \to 0, \quad \text{a.s., and} \quad Z_N/\sqrt{2 \log N} \to 1, \quad \text{a.s.}$$

in both discrete, and continuous time.

Section 1

Let $\{X_N, N = 0, \pm 1, \pm 2, \ldots\}$ be a discrete parameter stationary Gaussian stochastic process, characterized by expectation, and covariance function, respectively:

$$EX_N \equiv 0,$$
$$EX_i X_{i+N} \equiv r_N, \quad r_0 \equiv 1.$$

A study is made of some of the limit properties of the r.v.s. (random variables)

$$Z_N = \max\{X_1, X_2, \ldots, X_N\} \tag{1.1}$$

as N becomes large. Corresponding laws are also considered for continuous parameter processes.

If the sequence $\{X_N, N = 1, 2, \ldots\}$ consists of independent r.v.s. having the d.f. (distribution function) $F(x)$, and if there exist sequences $\{a_N\}$ and $\{b_N\}$, $a_N > 0$, and a proper, non-degenerate d.f. $\Lambda(x)$, such that

$$\lim_{N\to\infty} P\{a_N^{-1}(Z_N - b_N) \leq x\} = \Lambda(x) \tag{1.2}$$

on all points in the continuity set of $\Lambda(x)$, we say that $\Lambda(x)$ is an extremal distribution, and that $F(x)$ lies "in its domain of attraction". GNEDENKO [*10*] has shown that $\Lambda(x)$ can have one of only three forms. These results are summarized and analyzed in the book by GUMBEL [*12*]. If X is normally distributed; that is if

$$P\{X \leq x\} \equiv \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

then, (1.2) is true, where

$$a_N \equiv (2\log N)^{-1/2}, \qquad b_N \equiv (2\log N)^{1/2} - \tfrac{1}{2}(2\log N)^{-1/2}(\log\log N + \log 4\pi), \tag{1.3}$$

and $\Lambda(x) \equiv \exp(-e^{-x})$,
as is shown in Cramér [5], pp. 374—75.

In the general case of variables having a stationary kind of dependence, a start was made by Watson [18]. He showed that the same law (1.2) holds for an M-dependent process, provided a mild additional restriction is satisfied. He showed that this condition holds for a Stationary Gaussian process. In other words (1.2) with (1.3), is true provided

$$r_N \equiv 0$$

for all but a finite number of integers N.

This result was extended by Berman [3], who showed that it is only necessary to assume that r_N approaches zero, as a limit "sufficiently fast"; specifically, that is, that either

$$\lim_{N\to\infty} r_N \log N = 0, \tag{1.4}$$

or

$$\sum_{N=1}^{\infty} r_N^2 < \infty. \tag{1.5}$$

An evident question is whether these conclusions can be weakened still further. In particular, might

$$\lim_{N\to\infty} r_N = 0 \tag{1.6}$$

be a sufficient condition. In Section 2, a class of processes is considered wherein (1.6) is satisfied but not (1.2), with (1.3). So the conditions (1.4) and (1.5) cannot be substantially improved.

In Section 3 sufficient conditions are found on the covariance function, and the spectrum, for stability and relative stability almost surely, (a.s.) or with probability one; respectively

$$Z_N - \sqrt{2\log N} \to 0, \quad \text{a.s.}, \tag{1.7}$$

and

$$Z_N/\sqrt{2\log N} \to 1, \quad \text{a.s.} \tag{1.8}$$

Theorems 3.2, and 3.3, show that (1.4) and (1.5) are sufficient. Some of the processes considered in Section 2, do not satisfy (1.7), and so (1.6) is not a sufficient condition for it. It is, however, as Theorem 3.4 shows, sufficient for (1.8).

Let $\{X(t), -\infty < t < \infty\}$ be a separable, measurable version of a continuous parameter stationary Gaussian process, with expectation and covariance, respectively

$$EX(t) \equiv 0,$$
$$EX(s)\,X(s+t) \equiv r(t), \quad r(0) \equiv 1.$$

In Section 4, the maximum

$$Z(t) \equiv \max_{0\leq s\leq t} X(s)$$

is considered. Theorems 4.4, and 4.5 give conditions, sufficient for a law of the form

$$\lim_{t\to\infty} P\{(A(t))^{-1}(Z(t)-B(t)) \leqq x\} = \Lambda(x).$$

The conditions are of two types. As in the case of discrete parameter processes, there are "mixing" conditions, which involve the behavior of $r(t)$, as t approaches infinity. But attention must also be given to "local" conditions, which concern the behavior of $r(t)$ in the limit, as t approaches zero.

In Section 5, both "mixing" and "local" conditions are found, in Theorems 5.1, through 5.5, sufficient for "stability", and "relative stability" almost surely, in the continuous parameter case.

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Section 2

The most important known results concerning the limiting distribution of Z_N, (1.1) in the Gaussian case, are summarized below. Then a class of processes is constructed which shows that these results cannot be significantly extended.

Theorem 2.1 (Berman [*3*]). *If, either*

$$\lim_{N\to\infty} r_N \log N = 0 \tag{2.1}$$

or

$$\sum_{N=1}^{\infty} r_N^2 < \infty, \tag{2.2}$$

then

$$\lim_{N\to\infty} P\{a_N^{-1}(Z_N - b_N) \leqq x\} = \Lambda(x), \tag{2.3}$$

where

$$a_N = (2\log N)^{-1/2},$$

and

$$b_N = (2\log N)^{1/2} - \tfrac{1}{2}(2\log N)^{-1/2}(\log\log N + \log 4\pi),$$

and

$$\Lambda(x) = \exp(-e^{-x}). \tag{2.4}$$

Theorem 2.1 clearly includes the case of independent r.v.s. ($r_N \equiv 0$, $N \neq 0$). The generalization employs Lemma 2.1, below, which makes possible a comparison between the results using the given measure, and those (known) using the independence measure. The measure of a Gaussian process, with means zero and variances one, is uniquely characterized by the covariance function r_{ij}.

Lemma 2.1 [*3*]. *Let P, and P^* be two normalized Gaussian measures, characterized by the covariance functions, respectively, $\{r_{ij}\}$, and $\{r_{ij}^*\}$. Then*

$$|P\{Z_N \leqq c\} - P^*\{Z_N \leqq c\}| \leqq D_N = \sum_{i,j=1}^{N-1} |r_{ij} - r_{ij}^*|\, \varphi(c, |r'_{ij}|), \tag{2.5}$$

where

$$\varphi(c, |r_{ij}|) = (1 - r_{ij}'^2)^{-1/2} \exp\{-c^2/(1 + |r_{ij}'|)\},$$

and

$$r_{ij}' = \max(r_{ij}, r_{ij}^*).$$

If the process is stationary,

$$D_N \leqq \sum_{j=1}^{N-1} (N - j) |r_j^* - r_j| \varphi(c, |r_j'|). \tag{2.5a}$$

Now it is shown that the condition

$$\lim_{N\to\infty} r_N = 0, \tag{2.6}$$

is not sufficient for (2.3). A class of processes is defined for which (2.6) holds, and such that,

$$\lim_{K\to\infty} P\{r_{N_K}^{-1}(Z_{N_K} - \sqrt{2(1 - r_{N_K}) \log N_K}) \leqq x\} = \Phi(x), \tag{2.7}$$

where

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2} dt, \tag{2.8}$$

and $\{N_K, K = 1, 2, \ldots\}$ is an infinite subsequence of the set of positive integers. By the essential uniqueness of such laws (2.7) clearly contradicts (2.3). This is so since in effect, if a limiting distribution exists for Z_N, in a process of this class, it must be normal.

Now it is useful to define a system of equally correlated r.v.s. Let $\{r_N\}$ be given by

$$r_0 = 1,$$
$$r_N = \varrho, \quad N \neq 0.$$

This measure will be designated by $P_\varrho(\cdot)$.

Lemma 2.2 (Berman [2]). *Let $\{X_N, N = 0, \pm 1, \pm 2, \ldots\}$ be a sequence of equally correlated jointly Gaussian random variables. Then the following representation is possible:*

$$X_N = U + Y_N, \tag{2.9}$$

where $\{Y_N, N = 0, \pm 1, \pm 2, \ldots\}$ are independent, and Gaussian r.v.s. with means zero, and variances $1 - \varrho$, and U is Gaussian with mean zero, and variance ϱ, and is independent of the Y_N.

This representation can be verified directly.

The general method of verifying the property (2.7), is using Lemma 2.1, to compare, for each N_K the behavior of Z_{N_K}, as given in (1.1), under the original measure $P(\cdot)$, with that under the measure $P_\varrho(\cdot)$,

where $$\varrho = r_{N_K}.$$

Note that the second measure is, itself, a function of K. The following lemma shows that if r_N approaches zero sufficiently slowly, the limiting distribution of

Z_N, under the equal correlation measures with $\varrho = r_N$, is Gaussian. The lemma does not actually require that r_N approach zero, but it includes this possibility. It is, in effect a generalization of BERMAN's result [2].

Lemma 2.3. *Let* $\{P_N(\cdot),\ N = 0, \pm 1, \pm 2, \ldots\}$ *be the sequence of measures:*

$$P_N(\cdot) \equiv P_{r_N}(\cdot)\,.$$

If

$$\lim_{N\to\infty} r_N \log N/(\log\log N)^2 = \infty\,, \tag{2.10}$$

then

$$\lim_{N\to\infty} P_N\{r_N^{-1/2}(Z_N - \sqrt{2(1-r_N)\log N}) \leqq x\} = \Phi(x)\,, \tag{2.11}$$

where $\Phi(x)$ *is given by* (2.8).

Proof. For a given N, consider the r.v.s. X_i, as given by

$$X_i = U^{(N)} + Y_i^{(N)}\,,$$

a representation, which is possible, by Lemma 2.2. Then

$$Z_N = U^{(N)} + W_N\,,$$

where

$$W_N \equiv \max\{Y_1^{(N)},\ Y_2^{(N)}, \ldots,\ Y_N^{(N)}\}\,.$$

The r.v. to be examined is

$$r_N^{-1/2}(Z_N - \sqrt{2(1-r_N)\log N}) = r_N^{-1/2}\,U_N + r_N^{-1/2}(W_N - \sqrt{2(1-r_N)\log N}). \tag{2.12}$$

Clearly $r_N^{-1/2}\,U_N$ is Gaussian, with mean zero, and variance one, for every N. To prove the lemma, then, it is sufficient to prove that the second term of the right side of (2.12), becomes concentrated at 0, with increasing N:

$$\begin{aligned}&\lim_{N\to\infty} P_N\{r_N^{-1/2}(W_N - \sqrt{2(1-r_N)\log N}) > \varepsilon\,(\leqq -\varepsilon)\}\\ &\quad= \lim_{N\to\infty} P_N\{\sqrt{2\log N}(W_N/\sqrt{1-r_N} - \sqrt{2\log N}) + \tfrac{1}{2}(\log\log N +\\ &\qquad + \log 4\pi) > (2r_N\log N)^{1/2}\,\varepsilon/\sqrt{1-r_N} + \tfrac{1}{2}(\log\log N + \log 4\pi)\\ &\quad(\leqq -(2r_N\log N)^{1/2}\,\varepsilon/\sqrt{1-r_N} + \tfrac{1}{2}(\log\log N + \log 4\pi))\}\,.\end{aligned} \tag{2.13}$$

But $W_N/\sqrt{1-r_N}$ is, by definition, the maximum of N independent, normalized Gaussian r.v.s. But by assumption (2.10)

$$\lim_{N\to\infty} (\varepsilon(2r_N\log N)^{1/2}/\sqrt{1-r_N} + (-)\tfrac{1}{2}\log\log N) = \infty\,, \quad \forall\,\varepsilon > 0\,. \tag{2.14}$$

By (2.13) and (2.14), and Theorem 2.1, then (2.11) follows, q.e.d.

Lemma 2.4. *Let* $P(\cdot)$ *refer to a stationary Gaussian process, with non-increasing covariance sequence* $\{r_N\}$. *Let* $\{N_K\}$ *be an increasing sequence of integers. Let*

$$P_K(\cdot) = P_\varrho(\cdot)\,,$$

where $\varrho = r_{N_K}$.

If

$$\lim_{N\to\infty} r_N = 0\,, \tag{2.15}$$

and $\exists \lambda > 0$, *such that*

$$\lim_{K\to\infty} N_K^{\lambda-1} \sum_{j=1}^{N_K} |r_j - r_{N_K}| = 0 , \tag{2.16}$$

then

$$\lim_{K\to\infty} | P\{A_K\} - P_K\{A_K\} | = 0 , \tag{2.17}$$

where $A_K = \{r_{N_K}^{-1/2} (Z_{N_K} - \sqrt{2(1 - r_{N_K}) \log N_K}) \leqq x\}$.

Proof. By Lemma 2.1,

$$| P\{A_K\} - P_K\{A_K\} | \leqq D_K \equiv D_{N_K}$$

where D_N is given by (2.5).

$$c^2 = (x r_N^{1/2} + \sqrt{2(1 - r_N) \log N})^2 = x^2 r_N + 2(1 - r_N) \log N + \\ + 2x \sqrt{2 r_N (1 - r_N) \log N} .$$

Let

$$\delta_N = \operatorname*{Sup}_{K \geqq N} r_K , \tag{2.18}$$

$$\delta_N^* = \operatorname*{Sup}_{K \geqq N} 2 r_K/(1 + r_K) . \tag{2.19}$$

By the conditions of the lemma, it is clear that,

$$\lim_{N\to\infty} \delta_N = 0 , \tag{2.20}$$

$$\lim_{N\to\infty} \delta_N^* = 0 . \tag{2.21}$$

So $\operatorname*{Sup}_{N \geqq 1} (1 - r_N^2)^{-1/2} = (1 - \delta_1^2)^{-1/2} < \infty$,

and in taking the limit that term can be replaced by a constant C_1. We replace D_N by the sum of D_N^1, and D_N^2, where the former involves summation from one to $[N^\gamma]$ and the latter from $[N^\gamma] + 1$, to $N - 1$. Both are shown to approach zero as a limit. Thus

$$D_N^1 \equiv C_1 N^{-1+2r_N} g_N \sum_{j=1}^{[N^\gamma]} |r_j| \exp 2(1 - r_N) \log N |r_j|/(1 + |r_j|) \leqq \\ \leqq C_1 N^{-1+2r_N+\gamma+(1-r_N)\delta_1^*} g_N \to 0, \quad \text{if} \quad \gamma < 1 - \delta_1^* ,$$

where $$g_N = \exp 2x \sqrt{2 r_N (1 - r_N) \log N}$$

approaches infinity more slowly than any power of N, and the inequality holds for sufficiently large N.

$$D_N^2 \leqq C_1 N^{-1+2r_N+(1-r_N)\delta_{[N^\gamma]}^*} \sum_{j=1}^{N} |r_j| \leqq C_1 N^{\lambda-1} \sum_{j=1}^{N} |r_j|$$

for sufficiently large N. By Lemma 2.1, (2.17) follows, q.e.d.

Theorem 2.2. *Let* r_N *be a sequence satisfying the conditions of Lemmas* 2.3, *and* 2.4. *Then* (2.7) *follows directly by combining the results.*

Now, a class of processes is constructed, which satisfies the conditions of Lemmas 2.3, and 2.4, and hence of Theorem 2.2. Thus it is established that (2.6) is not a sufficient condition for (2.3). Let

$$\{U_{k,i},\, k=1,2,\ldots,\; i=0,\pm 1,\pm 2,\ldots\}$$

be a doubly indexed sequence of mutually independent normalized Gaussian r.v.s. Let

$$X_N=\sum_{k=1}^{\infty} a_k \frac{1}{\sqrt{M_k}} \sum_{i=N+1}^{N+M_k} U_{k,i},$$

where $\sum_{k=1}^{\infty} a_k^2=1$.

Clearly $EX_N\equiv 0$,

$$r_N\equiv EX_k X_{k+N}\equiv\sum_{k=1}^{\infty} a_k^2\left(1-\frac{N}{M_k}\right)^+=\sum_{\{k:\,M_k>N\}} a_k^2\left(1-\frac{N}{M_k}\right).$$

Since $r_N\leqq \sum_{\{k:\,M_k>N\}} a_k^2$,
(2.6) is verified.

Let us choose the sequences $\{M_k\}$ and $\{N_k\}$, so that

$$M_k<N_k<M_{k+1},\quad k=1,2,\ldots$$

and furthermore

$$\lim_{k\to\infty} N_k^{\lambda}(M_k/N_k)=0\,, \tag{2.22}$$

and

$$\limsup_{k\to\infty} M_{k+1}^{\lambda}(N_k/M_{k+1})<\infty\,, \tag{2.23}$$

for some $\lambda>0$.

Let c be any real constant, $0<c<1$, and N sufficiently large. Then

$$r_N\geqq\sum_{\{k:\,M_{k-1}>N\}} a_k^2(1-M_{k-1}/M_k)\geqq c\sum_{\{k:\,M_{k-1}>N\}} a_k^2\,.$$

So r_N can be made to approach zero as slowly as desired, by making M_k approach infinity sufficiently fast. So (2.10) is verified.

It remains only to verify that (2.16) is satisfied. Note that

$$\frac{1}{N}\sum_{j=1}^{N}\left(1-\frac{j}{M}\right)^+-\left(1-\frac{N}{M}\right)^+=\frac{\min(N,M)-1}{2\max(N,M)},$$

$$H_N\equiv\frac{1}{N}\sum_{j=1}^{N}(r_j-r_N)=\sum_{k=1}^{\infty} a_k^2\,\frac{\min(N,M_k)-1}{2\max(N,M_k)},$$

$$N_k^{\lambda} H_{N_k}=N_k^{\lambda}\sum_{\{\nu:\,M_\nu\leqq N_k\}} a_\nu^2\,\frac{M_\nu-1}{2N_k}+N_k^{\lambda}\sum_{\{\nu:\,M_\nu>N_k\}} a_\nu^2\,\frac{N_k-1}{2M_\nu}\leqq$$

$$\leqq N_k^{\lambda}(M_k/2N_k)+M_{k+1}^{\lambda}(N_k/M_{k+1})\sum_{\nu=k+1}^{\infty} a_\nu^2\to 0\,,\quad\text{as}\quad k\to\infty\,.$$

So all of the conditions are verified. Specifically, if

$$M_k=2^{2^{2k}},\quad N_k=2^{2^{2k+1}} \tag{2.24}$$

then (2.22) and (2.23) hold, as it does for any $\{M_k\}$ or $\{N_k\}$ which approach ∞ more rapidly than (2.24).

It is interesting to note that these processes have a property even stronger than (2.6). Specifically they are purely non-deterministic. Let $\mathscr{F}_N$ be the sub-sigma field of events spanned by $X_N, X_{N-1}, \ldots$. Let

$$\mathscr{F}_{-\infty} \equiv \bigcap_{N=-\infty}^{\infty} \mathscr{F}_N .$$

Then $\mathscr{F}_{-\infty}$ is called the tail field. Clearly each of the variables $U_{k,i}$ is independent of $\mathscr{F}_{-\infty}$, and consequently, so is X_N for each N.

As is well known, using the spectral representation of a process, (see Grenander and Rosenblatt [11]),

$$r_N = \int_{-\pi}^{\pi} \cos N\,\omega\, dG(\omega) .$$

It is worthwhile to find conditions sufficient for the above results, in terms of the spectral d.f. $G(\omega)$. In Zygmund [19], page 13, it is shown that a sufficient condition for (2.2) is that

$$\int_{-\pi}^{\pi} g^2(\omega)\, d\omega < \infty , \tag{2.25}$$

where $G(\omega)$ is absolutely continuous, and

$$g(\omega) = dG(\omega)/d\omega .$$

From page 46, it can be concluded that a sufficient condition for (2.1) is that $g(\omega)$ satisfy a Lipschitz condition, of order α, for some α. From page 45, it is possible to conclude, from the absolute continuity of $G(\omega)$, that (2.6) holds. But the spectral distribution function of a purely non-deterministic process, is necessarily absolutely continuous. So absolute continuity is not sufficient for (2.3).

For general (not necessarily Gaussian) stationary processes, Chybisov [4] has found that the limiting distribution of Z_N, is, under certain circumstances, the same as in the case of independence, provided a "uniform mixing" condition holds. It is defined as follows. Let $\mathscr{F}_{ij}$ be the sub-sigma field generated by

$$X_i, X_{i+1}, X_{i+2}, \ldots, X_j .$$

If $D_k = \sup_{(A \in \mathscr{F}_{-\infty,-i},\, B \in \mathscr{F}_{i+k,\infty})} |P\{A\,B\} - P\{A\}\,P\{B\}| \to 0 \quad \text{as} \quad k \to \infty$

the process is said to be uniformly mixing. Kolmogorov and Rozanov [13] have proved that if a Gaussian process is uniformly mixing its spectral d.f. must be absolutely continuous, and furthermore $g(\omega)$ can have no discontinuities. Clearly then (2.25) holds. So uniform mixing is sufficient for (2.3).

Section 3

It follows from (2.3) that

$$Z_N - \sqrt{2 \log N} \to 0, \ \text{i.p.}$$

hat is that Z_N is "stable in probability". From this it follows that Z_N is "relatively table" in probability, i.e.

$$Z_N/\sqrt{2 \log N} \to 1, \quad \text{i.p.} \tag{3.1}$$

In addition, Berman has shown in [3] that (2.6) is sufficient for (3.1).

In this section, conditions are found wich are sufficient, respectively, for "stability" and "relative stability" almost surely (a.s), or, "with probability one". The problem is decomposed into an "upper" one (3.2), and a "lower" one (3.6).

Theorem 3.1. *If the variables X_N are normalized, and Gaussian,*

$$P\left\{\limsup_{N\to\infty}(Z_N-\sqrt{2\log N})\leqq 0\right\}=1. \tag{3.2}$$

Remark. It is important to note that this result does not require any kind of condition involving the dependence.

Proof of Theorem 3.1. It is sufficient to prove that for every $\varepsilon>0$,

$$Z_N>\sqrt{2\log N}+\varepsilon$$

only finitely many times, with probability one. For this it need only be shown that for every $\varepsilon>0$,

$$X_N>\sqrt{2\log N}+\varepsilon$$

only finitely many times, with probability one. But

$$\sum_{k=1}^{\infty}P\{X_k>\sqrt{2\log K}+\varepsilon\}\leqq\int_0^{\infty}g(x)\,d\Phi(x) \tag{3.3}$$

where

$$g(x)=\left(\max_k:\sqrt{2\log \mathrm{K}}+\varepsilon\leqq x\right)\leqq\exp(x-\varepsilon)^2/2.$$

So the integral (3.3) converges and the Theorem follows, by the Borel-Cantelli Theorem (Loève [14], page 228), q.e.d.

Now, the "lower" part of the problem (3.6) is considered.

Let $\varepsilon>0$ be arbitrary. Define

$$N_\varepsilon(M)=\min_K:\{\sqrt{2\log K}>M\varepsilon\}=[\exp M^2\varepsilon^2/2]+1, \tag{3.4}$$

where $[x]$ denotes the greatest integer less than or equal to x. Thus we have a subsequence "of very low density" of the sequence of integers, which will be useful in what follows. Define

$$A_\varepsilon(M)=\sqrt{2\log N_\varepsilon(M)}.$$

It can easily be seen that, for all $\varepsilon>0$,

$$M\varepsilon\leqq A_\varepsilon(M)\leqq M\varepsilon+M^{-1}\varepsilon^{-1}\exp(-M^2\varepsilon^2/2)(1+o(1))\quad\text{as}\quad M\to\infty. \tag{3.5}$$

Lemma 3.1. *If* $\forall\varepsilon>0$,

$$Z_{N_\varepsilon(M)}\leqq A_\varepsilon(M)-\varepsilon$$

only finitely many times, with probability one, then

$$P\left\{\liminf_{N\to\infty}(Z_N-\sqrt{2\log N})\geqq 0\right\}=1. \tag{3.6}$$

Proof. It is sufficient to show that for arbitrary $\varepsilon>0$,

$$Z_N\leqq\sqrt{2\log N}-\varepsilon \tag{3.7}$$

only finitely many times, with probability one. Because ε is arbitrary, eq. (3.7) could as well be written

$$Z_N \leqq \sqrt{2\log N} - 3\varepsilon. \tag{3.8}$$

Suppose (3.8) holds infinitely many times. Let N_0 be so chosen that for all $N_\varepsilon(M) > N_0$,

$$Z_{N_\varepsilon(M)} > A_\varepsilon(M) - \varepsilon.$$

Let N' be an integer, such that (3.8) holds, and

$$N_0 < N_\varepsilon(M) \leqq N' \leqq N_\varepsilon(M+1)$$

for some suitable M. Then

$$A_\varepsilon(M) - \varepsilon \leqq Z_{N_\varepsilon(M)} \leqq Z_{N'} \leqq \sqrt{2\log N'} - 3\varepsilon \leqq A_\varepsilon(M+1) - 3\varepsilon.$$

So

$$A_\varepsilon(M+1) - A_\varepsilon(M) > 3\varepsilon - \varepsilon = 2\varepsilon. \tag{3.9}$$

But (3.8) holds infinitely many times. So (3.9) must also. But this contradicts (3.5).

Lemma 3.2. *A sufficient condition for* (3.6) *is that*

$$\lim_{N\to\infty} (\log N)\, P\{Z_N \leqq \sqrt{2\log N} - \varepsilon\} = 0, \quad \forall \varepsilon > 0. \tag{3.10}$$

Proof. A sufficient condition for (3.6) is that,

$$\lim_{M\to\infty} M^2 P\{Z_{N_\varepsilon(M)} \leqq A_\varepsilon(M) - \varepsilon\} = 0, \quad \forall \varepsilon > 0 \tag{3.11}$$

since this clearly implies the summability on M of $P\{Z_{N_\varepsilon(M)} \leqq A_\varepsilon(M) - \varepsilon\}$. So by the Borel-Cantelli Theorem, and Lemma 3.1, (3.6) is true. But by the definition (3.4) (3.11) is equivalent to (3.10), q.e.d.

Taking the first term of the power series expansion of $-\log F(x)$ about $F(x) = 1$, it clearly follows that

$$-\log F(x) = (1 - F(x))(1 + o(1)), \quad \text{as} \quad F(x) \to 1. \tag{3.12}$$

Lemma 3.3 (Cramèr [5], page 374). *If X is a normalized Gaussian r.v.*

$$P\{X > x\} \leqq \phi(x),$$

and $\lim_{x\to\infty} P\{X > x\}/\phi(x) = 1$, *where*

$$\phi(x) = (2\pi)^{-1/2} x^{-1} \exp(-x^2/2) \tag{3.13}$$

and so by (3.12) *above*

$$\lim_{x\to\infty} (-\log P\{X \leqq x\})/\phi(x) = 1.$$

Lemma 3.4. *If $\{X_N, N = 1, 2, \ldots\}$ is a sequence of independent, normalized, Gaussian r.v.s., then*

$$\lim_{N\to\infty} (\log N)^k P\{Z_N \leqq \sqrt{2C\log N} - \varepsilon\} = 0, \quad \forall \varepsilon > 0, \quad \forall k > 0, \quad C > 0.$$

Proof. By Lemma 3.3, and e.g. for any c, $0 < c < 1$, and sufficiently large N,

$$-N \log \Phi\left(\sqrt{2C \log N} - \varepsilon\right) > cN\,\phi\left(\sqrt{2C\log N} - \varepsilon\right).$$

$$(\log N)^k \Phi^N\left(\sqrt{2C\log N} - \varepsilon\right) \leqq (\log N)^k \exp\left[-c\left(\sqrt{2C\log N} - \varepsilon\right)^{-1} \times \times \exp \varepsilon\left(\sqrt{2C\log N} - \varepsilon\right)\right] \to 0, \quad \text{as} \quad N \to \infty,$$

q.e.d.

Theorem 3.2. *If*

$$\sum_{N=1}^{\infty} r_N^2 < \infty, \qquad Z_N - \sqrt{2\log N} \to 0, \quad a.s. \tag{3.14}$$

Proof. By Lemmas 2.1, 3.2, and 3.4, it is sufficient to show that

$$\lim_{N\to\infty} (\log N)\, D_N = 0,$$

where D_N is given by (2.5a), and

$$c^2 = \left(\sqrt{2\log N} - \varepsilon\right)^2 = 2\log N - 2\varepsilon\sqrt{2\log N} + \varepsilon^2.$$

Then

$$(\log N)\, D_N = (\log N)\left(\exp 2\varepsilon \sqrt{2\log N}\right) N^{-1} \sum_{j=1}^{N} |r_j| \exp\left(2\log N\, |r_j|/(1 + |r_j|)\right). \tag{3.15}$$

Consider $D_N^{(1)}$, the expression on the right hand side of (3.15) above, but with summation from one to $[N^\gamma]$ where $0 < \gamma < 1$. Then

$$D_N^{(1)} \leqq \delta_1 \log N \left(\exp 2\varepsilon\sqrt{2\log N}\right) N^{-(1-\gamma)+\delta^*} \to 0, \quad \text{as} \quad N \to \infty$$

where $\delta_{N'}$, and δ_N^* are given respectively by (2.18) and (2.19). Now consider the remaining part of (3.15), $D_N^{(2)}$.

$$D_N^{(2)} \leqq \log N \left(\exp 2\varepsilon\sqrt{2\log N}\right) N^{\delta^*_{[N^\gamma]^{-1}}} \sum_{j=[N^\gamma]+1}^{N-1} |r_j|,$$

$$(D_N^{(2)})^2 \leqq (\log N)^2 \left(\exp 4\varepsilon\sqrt{2\log N}\right) N^{2\delta^*_{[N^\gamma]^{-1}}} \sum_{j=[N^\gamma]+1}^{N-1} r_j^2,$$

which approaches zero, since (3.14) implies (2.6), and hence (2.21), q.e.d.

Lemma 3.5 (BERMAN [3]). *Let P and P' be two normalized Gaussian measures characterized, respectively, by the covariance sequences $\{r_N\}$ and $\{r'_N\}$. If*

$$r_N \leqq r'_N \text{ for all } N,$$

then

$$P\{Z_N \leqq c\} \leqq P'\{Z_N \leqq c\}.$$

In other words $P\{Z_N \leqq c\}$ is a monotonic function of the covariances.

We define the function

$$L(N) \equiv \exp\left(\sqrt{\log N}/h_N\right) \tag{3.16}$$

where h_N is a non-decreasing sequence such that

$$\lim_{N\to\infty} h_N = \infty , \tag{3.17}$$

$$\forall \alpha > 0, \quad \lim_{N\to\infty} (\log N)^{-\alpha} h_N = 0 ,$$

and $\limsup_{N\to\infty} h_N/\sqrt{\delta_{L(N)} \log L(N)} < \infty$.

Let

$$A_N \equiv \sqrt{2(1-\delta_{L(N)}) \log [N/L(N)]} . \tag{3.18}$$

Lemma 3.6. *If*

$$\lim_{N\to\infty} r_N \log N = 0 ,$$

$$\lim_{N\to\infty} \delta_{L(N)} \sqrt{\log N} = 0 , \tag{3.19}$$

and

$$\lim_{N\to\infty} \left(A_N - \sqrt{2 \log N}\right) = 0 . \tag{3.20}$$

Proof. Let

$$h'_N = (\delta_N \log N)^{-1/2} ,$$

and

$$h_N = h'_{L(N)} .$$

Clearly h_N satisfies the conditions (3.17). By definition (3.16),

$$h_N \log L_{(N)} = \sqrt{\log N} ,$$

from which (3.19) follows. Eq. (3.20) follows directly.

Lemma 3.7. *A sufficient condition for* (3.6) *is that*

$$\forall \varepsilon > 0, \quad \lim_{N\to\infty} (\log N) P\{Z_N \leqq A_N - \varepsilon\} = 0 .$$

Proof. Note that by Lemma 3.6, for sufficiently large N,

$$P\{Z_N \leqq A_N - \varepsilon\} \leqq P\{Z_N \leqq \sqrt{2 \log N} - \varepsilon/2\} .$$

The result follows from Lemma 3.2, and the fact that ε is arbitrary, q.e.d.

Theorem 3.3. *If*

$$\lim_{N\to\infty} r_N \log N = 0 ,$$

$$Z_N - \sqrt{2 \log N} \to 0 , \quad a.s.$$

Proof. By Lemma 3.7, it is sufficient to prove

$$\forall \varepsilon > 0, \quad \lim_{N\to\infty} (\log N) P\{Z_N \leqq A_N - \varepsilon\} = 0 . \tag{3.21}$$

Let

$$Z_N^* = \max_{1 \leqq k \leqq [N/L(N)]} X_{kL(N)} . \tag{3.22}$$

Since, clearly $Z_N^* \leqq Z_{N'}$, it is sufficient to prove

$$\forall \varepsilon > 0, \quad \lim_{N\to\infty} (\log N) P\{Z_N^* \leqq A_N - \varepsilon\} = 0 .$$

But all correlations among the variables in (3.22) are $\leqq \delta_{L(N)}$. So, by Lemma 3.5, it is sufficient to prove that

$$\forall \varepsilon > 0, \lim_{N\to\infty} (\log N) P_N^*\{Z_N^* \leqq A_N - \varepsilon\} = 0, \tag{3.23}$$

where $P_N^*(\cdot)$ is the equal correlation measure with $\varrho = \delta_{L(N)}$. But using the representation (2.9), the expression (3.23) is dominated by

$$(\log N) P_N^*\{U \leqq A_N - \varepsilon/2\} + (\log N) P_N^*\{W_N \leqq A_N - \varepsilon/2\}. \tag{3.24}$$

Recalling the definitions (3.18) of A_N, (3.16) of $L(N)$, and Lemma 2.22 the expression or the left hand side of (3.24) is dominated by

$$\begin{aligned} &h_N \log L(N) P_N^*\{U/\sqrt{1-\delta_{L(N)}} \leqq \sqrt{2\log L(N)} - \varepsilon/2\} \leqq \\ &\leqq C' h_N \log L(N) \phi\left(\sqrt{2\log L(N)} - \varepsilon/2\right) \to 0, \quad \text{as} \quad N\to\infty, \end{aligned}$$

for any $C' > 1$, by Lemma 3.4, since $U/\sqrt{1-\delta_{L(N)}}$ is Gaussian, with mean zero, and variance one.

By the definition (3.16) of $L(N)$, Lemma 3.4, and the fact that $W_N/\sqrt{1-\delta_{L(N)}}$ is the maximum of $L(N)$ independent normalized Gaussian r.v.s., the expression on the right side of (3.24) converges to zero also, q.e.d.

Again it is natural to inquire whether

$$\lim_{N\to\infty} r_N = 0 \tag{3.25}$$

might be sufficient. To show that it is not, recall the class of processes considered in Section 2. From (2.7) it can be concluded that, on the subsequence $\{N_k\}$

$$\forall \varepsilon > 0, \lim_{k\to\infty} P\{Z_{N_k} - \sqrt{2(1-r_{N_k})\log N_k} > \varepsilon\} = 0. \tag{3.26}$$

This is consistent with (1.7) only if

$$\lim_{N\to\infty} r_N \sqrt{\log N} = 0, \tag{3.27}$$

but, it was noted that in this class, r_N may approach zero, as slowly as desired. Therefore (3.27), and hence (1.7) may be violated.

Theorem 3.4, below, will show, however, that (3.25) is sufficient for Relative Stability.

Lemma 3.8. *If for every* $\varepsilon > 0$, $a > 0$

$$Z_{N_\varepsilon(M)} \leqq a\sqrt{2\log N_\varepsilon(M)} \quad (1-\varepsilon) \tag{3.28}$$

only a finite number of times, then for every $\varepsilon > 0$, *and the same value* a,

$$Z_N \leqq a\sqrt{2\log N}(1-\varepsilon) \tag{3.29}$$

only a finite number of times.

Proof. It is sufficient to show that, under the hypothesis, for any $\varepsilon > 0$,

$$Z_N \leqq a\sqrt{2\log N}(1-2\varepsilon) \tag{3.30}$$

only a finite number of times. Let M_0 be an integer, so chosen that if $M \geqq M_0$, (3.28) does not hold. Let M', N' be integers such that

$$N_\varepsilon(M_0) \leqq N_\varepsilon(M') \leqq N' \leqq N_\varepsilon(M'+1),$$

and so that for N', (3.30) is true. Then

$$a\sqrt{2\log N_\varepsilon(M)}\,(1-\varepsilon) \leqq Z_{N_\varepsilon(M)} \leqq Z_{N'} \leqq a\sqrt{2\log N'}\,(1-\varepsilon) \leqq \\ \leqq a\sqrt{2\log N_\varepsilon(M+1)}\,(1-2\varepsilon)\,. \tag{3.31}$$

Therefore,

$$a(\varepsilon+o(1))\,(1-\varepsilon) - a\,\varepsilon\left(\sqrt{2\log N_\varepsilon(M)}+\varepsilon+o(1)\right) \geqq 0\,,$$

for infinitely many M, which plainly contradicts eq. (3.5), q.e.d.

Lemma 3.9. *If, for a fixed* $a > 0$

$$\forall\,\varepsilon > 0\,,\quad \lim_{N\to\infty}(\log N)\,P\{Z_N \leqq a\sqrt{2\log N}(1-\varepsilon)\} = 0 \tag{3.32}$$

then

$$P\{\limsup_{N\to\infty}(Z_N/\sqrt{2\log N}) \leqq a\} = 1\,. \tag{3.33}$$

The *proof* is identical to that of Lemma 3.2.

Theorem 3.4. *If*

$$\lim_{N\to\infty} r_N = 0$$

then $Z_N/\sqrt{2\log N} \to 1$, *a.s.*

Proof. From Lemma 3.9, it is clear that what is to be proved is (3.32) with $a = 1$. Let ϱ be arbitrary > 0. Let k be so chosen that $\delta_k < \varrho$, where δ_N is given by (2.18). Let

$$Z_N^{(k)} = \max_{1\leqq j\leqq N/k} X_{jk}\,.$$

First, it is proved that

$$P\left\{\liminf_{N\to\infty}(Z_N^{(k)}/\sqrt{2\log N}) \geqq \sqrt{1-\varrho}\right\} = 1\,. \tag{3.34}$$

Since

$$\lim_{N\to\infty}\frac{\log N/k}{\log N} = 1\,,$$

it is sufficient for (3.34) to prove that

$$P\left\{\liminf_{N\to\infty}(Z_N^{(k)}/\sqrt{2\log N/k}) \geqq \sqrt{1-\varrho}\right\} = 1\,.$$

Substituting N for N/k, by Lemma 3.9, it is sufficient that (3.32) hold with $a = \sqrt{1-\varrho}$. But using the representation (2.9), (3.32) is dominated by

$$(\log N)\,P\{U \leqq \sqrt{2(1-\varrho)\log N} - \varepsilon/2\} + \\ + (\log N)\,P\{W_N \leqq \sqrt{2(1-\varrho)\log N} - \varepsilon/2\}\,. \tag{3.35}$$

First consider the expression on the left side of (3.35). Since $U/\sqrt{\varrho}$ is normalized and Gaussian, by Lemma 3.3, and the form of $\phi(x)$ (3.13), it follows that this expression approaches zero, as N approaches infinity. Clearly $W_N/\sqrt{1-\varrho}$ is the maximum of N independent normalized Gaussian r.v.s. and so by Lemma 3.4, the right side also approaches zero. Since clearly

$$Z_N \geqq Z_N^{(k)},$$

$$P\left\{\liminf_{N\to\infty}(Z_N/\sqrt{2\log N}) \geqq \sqrt{1-\varrho}\right\} = 1,$$

but ϱ was arbitrary. So the result follows, q.e.d.

It is clear from the remarks at the end of Section 2, that uniform mixing is sufficient for stability, and that absolute continuity of the spectral density function is sufficient for relative stability.

Section 4

Let $\{X(t), -\infty < t < \infty\}$ be a separable, measurable version of a stationary Gaussian stochastic process with real valued parameter space. It is assumed without loss of generality that

$$EX(t) \equiv 0,$$
$$EX(s)X(s+t) \equiv r(t),$$

where $r(t)$ is, of course, the covariance function, which, by stationarity, does not depend on s. Let

$$Z(t) = \max_{0 \leqq s \leqq t} X(s). \tag{4.1}$$

The limiting behavior of $Z(t)$ is investigated.

Two theorems are stated concerning the sample functions of such processes, which are due to BELAYEV [1].

Theorem 4.1 (BELAYEV). *The sample functions of a S.G.S.P. are either* 1) *continious everywhere, with probability one, or* 2) *unbounded in every finite interval, with probability one.*

Theorem 4.2 (BELAYEV). *The sample functions of a S.G.S.P. are continuous everywhere with probability one, if*

$$\exists \beta > 1: \limsup_{t\to 0} |\log t|^{\beta}(1-r(t)) < \infty. \tag{4.2}$$

If the sample functions are of the type 2, above, it is apparent that

$$Z(t) \equiv \infty, \quad \forall t > 0.$$

So, of course, we are only interested in those processes, whose sample functions are of the type 1. The condition (4.2) is very mild and is easily verified in the cases considered, hereafter.

For discrete parameter processes, such as those considered in the previous sections, the conditions imposed consisted entirely of what are called "Mixing conditions"; that is those which concern the behavior of the covariance sequence $\{r_N\}$ as N becomes large. For continuous parameter processes, two types of conditions are involved. The "mixing conditions" concern the behavior of $r(t)$, as t becomes large. The "local conditions" involve the behavior, as t approaches zero, as for example (4.2).

To facilitate some of the proofs, which ensue, the following quantities are defined. Let ε and ω, be two real numbers, such that

$$0 \leqq \varepsilon < 1, \quad 0 < \omega .$$

Let the intervals $\{I_k,\ k = 1, 2, \ldots\}$ be defined

$$I_k \equiv \{(k-1+\varepsilon/2)\,\omega \leqq t \leqq (k-\varepsilon/2)\omega\}, \quad k = 1, 2, \ldots,$$
$$I \equiv \bigcup_{k=1}^{\infty} I_k$$

and let I^c be the complement of I. Let

$$Z(\varepsilon, \omega; t) \equiv \operatorname*{Sup}_{0 \leqq s \leqq t} X(s)\chi(\varepsilon, \omega; s),$$

where

$$\chi(\varepsilon, \omega; t) \equiv \begin{cases} 1, & t \in I, \\ -\infty, & t \in I^c \end{cases}$$

Note that

$$Z(0, \omega; t) \equiv Z(t).$$

Let $Z_N(t) \equiv \max\limits_{1 \leqq k \leqq N} X(kt/N)$,

and

$$Z_N(\varepsilon, \omega; t) \equiv \max_{1 \leqq k \leqq N} X(kt/N)\,\chi(\varepsilon, \omega; kt/N).$$

In addition to the original measure, we define the measure $P^*_\omega(\cdot)$ in the following way. Let

$$\mathscr{F}_{k,\omega} \equiv \mathscr{F}\{X(t), \quad (k-1)\,\omega \leqq t < k\omega\}.$$

Clearly $\mathscr{F} = \mathscr{F}\left(\bigcup\limits_{k=-\infty}^{\infty} \mathscr{F}_{k,\omega}\right)$,

where $\mathscr{F}$ is generated by the entire process. On each sub-sigma field $\mathscr{F}_{k,\omega}$ let

$$P^*_{\omega,k}(\cdot) = P(\cdot).$$

Then $P^*(\cdot)$, defined on $\mathscr{F}$, is the product measure. To indicate the specific meaning, let $\{X_k(t),\ k = 1, 2, \ldots\}$ be a sequence of processes, mutually independent and each having the measure $P(\cdot)$. Let

$$X^*(t) = X_{[t/\omega]}(t).$$

Then $X^*(t)$ has the measure $P^*_\omega(\cdot)$, which is clearly not stationary.

In addition, if λ is a positive real number, $P^{**}_{\omega,\lambda}(\cdot)$ is the measure defined for

$$X^{**}_\lambda(t) = Y_{[t/\omega]}(t),$$

where the $\{Y_k(t),\ k = 1, 2, \ldots\}$ are mutually independent processes, each having the covariance function, considered by SLEPIAN [17]:

$$r(t) = (1 - \lambda\,|t|)^+.$$

It is assumed that $\lambda\omega \leqq 1$. The lemmas which follow, support Theorem 4.3. Lemmas 4.1, and 4.2, relate $Z(\varepsilon, \omega; t)$ and $Z_N(\varepsilon, \omega; t)$ under the measures $P(\cdot)$ and $P^*_\omega(\cdot)$.

Lemma 4.1. *Let $P(\cdot)$ be a stationary measure conferring continuity. Let a, A be two numbers such that $0 \leqq a \leqq 1$, $A > 0$. If*

$$t/N \leqq \omega(1-\varepsilon)/2, \tag{4.3}$$

then

$$\begin{aligned} &P\{Z(\varepsilon,\omega;t) - Z_N(\varepsilon,\omega;t) > A\} \leqq \\ &\leqq N \sum_{k=0}^{\infty} 2^k \phi(A(1-a)a^k/\sigma(t/2^{k+1}N)) \end{aligned} \tag{4.4}$$

where $\phi(x)$ is given by (3.13), *and*

$$\sigma^2(t) = 2(1-r(t)). \tag{4.5}$$

The result remains true if $\varepsilon = 0$. The lemma is still true, if $P^*_\omega(\cdot)$ is used, instead of $P(\cdot)$.

Proof. Let the events B, B_k, $k = 0, 1, 2, \ldots$ be defined as follows:

$$\begin{aligned} B &\equiv \{Z(\varepsilon,\omega;t) - Z_N(\varepsilon,\omega;t) > A\}, \\ B_k &\equiv \{Z_{2^{k+1}N}(\varepsilon,\omega;t) - Z_{2^k N}(\varepsilon,\omega;t) > A(1-a)a^k\}. \end{aligned}$$

By continuity,

$$\begin{aligned} Z(\varepsilon,\omega;t) - Z_N(\varepsilon,\omega;t) &= \sum_{k=0}^{\infty} (Z_{2^{k+1}N}(\varepsilon,\omega;t) - Z_{2^k N}(\varepsilon,\omega;t)) \\ &= \lim_{k\to\infty} (Z_{2^k N}(\varepsilon,\omega;t) - Z_N(\varepsilon,\omega;t)) \quad \text{a.s.} \end{aligned}$$

Clearly

$$P\{B\} \leqq \sum_{k=0}^{\infty} P\{B_k\}. \tag{4.6}$$

Consider the following sets of points.

$$\text{Let} \quad t_j^{(k)} = jt/2^k N, \quad j = 1, 2, \ldots, 2^k N.$$

Evidently the point $t_{2j-1}^{(k+1)}$ is midway between the points $t_j^{(k)}$ and $t_{j-1}^{(k)}$. The condition (4.3) guarantees that if $t_{2j-1}^{(k+1)}$ belongs to I, so must either $t_{j-1}^{(k)}$ or $t_j^{(k)}$ or both. So if B_k is true, one of the variables $X(t_{2j-1}^{(k+1)})$ must excede its "neighbor" by an amount greater than or equal to $A(1-a)a^k$. If the "forward" neighbor does not belong to I, we use the "backward" one. We call this event E_j. Clearly, by Lemma 3.3,

$$P\{E_j\} \leqq P\{Y_j > A(1-a)a^k\} \leqq \phi(A(1-a)a^k/\sigma(t/2^{k+1}N)). \tag{4.7}$$

But

$$B_k \subset \bigcup_{j=1}^{2^k N} E_j. \tag{4.8}$$

Combining (4.6), (4.7), and (4.8), we get (4.4), q.e.d.

Lemma 4.2. *Let $X(t)$ be a S.G.S.P. with covariance function $r(t)$, such that*

$$\exists \alpha > 0: \limsup_{t\to 0} t^{-\alpha}(1-r(t)) < \infty. \tag{4.9}$$

Let $N(t)$ be an integer function of t, such that

$$\lim_{t\to\infty} N(t)/t(\log t)^{2/\alpha} = \infty\,, \tag{4.10}$$

$$\forall \beta > 1\,, \lim_{t\to\infty} N(t)/t^{\beta} = 0\,, \tag{4.11}$$

then

$$(2\log t)^{1/2}(Z(\varepsilon, \omega; t) - Z_{N(t)}(\varepsilon, \omega; t)) \to 0\,, \quad \text{i.p.}$$

under both measures $P(\cdot)$ and $P^*_\omega(\cdot)$.

Proof. Clearly the condition (4.9) implies (4.2), and so by Theorem 4.2, the sample functions are continuous with probability one. Let $\varepsilon > 0$ be arbitrarily chosen. It is sufficient to show that

$$\lim_{t\to\infty} P(t) = 0\,, \tag{4.12}$$

where

$$\begin{aligned} P(t) &\equiv P\{Z(\varepsilon, \omega; t) - Z_{N(t)}(\varepsilon, \omega; t) > \varepsilon/(2\log t)^{1/2}\} \leqq \\ &\leqq \sum_{k=0}^{\infty} 2^k N(t)\, \phi(\varepsilon(1-a)\, a^k/(2\log t)^{1/2}\, \sigma(t/2^{k+1} N(t))) \end{aligned} \tag{4.13}$$

by Lemma 4.1 where a is such that $0 < a < 1$, and will be chosen later, $\phi(x)$ is given by (3.13), and $\sigma^2(t)$ by (4.5). Let $\psi(t)$ be arbitrarily chosen, so that

$$\begin{aligned} &\lim_{t\to\infty} \psi(t) = \infty\,, \\ \forall \beta > 0,\ &\lim_{t\to\infty} \psi(t)/t^{\beta} = 0\,. \end{aligned} \tag{4.14}$$

Let $N(t) \equiv t(\log t)^{2/\alpha}\psi(t)$.

Then $N(t)$ satisfies the conditions (4.10) and (4.11). It is worthwhile to emphasize that $\psi(t)$ can approach infinity as slowly as desired.

By (4.9) there exist constants t_0, and C_0, such that, if $t \leqq t_0$,

$$1 - r(t) \leqq C_0 t^{\alpha}/2\,. \tag{4.15}$$

By (4.5) and (4.15),

$$\sigma^2(t/2^{k+1}N(t)) \leqq C_0\, 2^{-\alpha(k+1)}(\log t)^{-2}\psi^{-\alpha}(t)\,, \tag{4.16}$$

provided $t/2^{k+1}N(t)$ is less than t_0. We will say instead that $t > t_1$.

Let us define the quantities

$$\begin{aligned} y(t) &\equiv C_2\, \varepsilon(1-a)\, 2^{\alpha/2}(\log t)^{1/2}\psi^{\alpha/2}(t)\,, \\ C &= a \cdot 2^{\alpha/2}\,. \end{aligned} \tag{4.17}$$

It is clear that a can be chosen so that $C > 1$. Then by (4.13), (4.16), and (4.17)

$$P(t) \leqq N(t) \sum_{k=0}^{\infty} 2^k \phi(yC^k) = (N(t)/y(t)) \sum_{k=0}^{\infty} (2/C)^k \exp(-y^2 C^{2k}/2)\,. \tag{4.18}$$

Since C is fixed, k_0 can be so chosen that if $k > k_0$,

$$C^{2k} > k\,C^2\,.$$

The expression $P(t)$, (4.18) then is expressed as the sum of $P_1(t)$ and $P_2(t)$, where the former involves summation from one to k_0, the latter from k_0+1 to infinity. That is

$$P_2(t) \leqq (N(t)/\sqrt{2\pi}\,y(t)) \sum_{k=k_0+1}^{\infty} (2/C)^k \exp(-y^2 C^{2k})/2 =$$
$$= (N(t)/\sqrt{2\pi}\,y(t))\,(2/C)\exp(-k_0 C^2 y^2/2)/(1-(2/C)\exp(-C^2 y^2/2)).$$

Clearly then to prove the Lemma, it is only necessary to show that if $C_1 > 1$,

$$\lim_{t\to\infty} (N(t)/y(t)) \exp(-C_1^2 y^2(t))/2 = 0.$$

But this is equal to

$$\lim_{t\to\infty} (\psi(t)(\log t)^{2/\alpha}/y(t)) \exp\{\log t - C_1^2(\varepsilon(1-a)\,2^{\alpha/2}(\log t)^{1/2}\psi^{\alpha/2}(t))^2/2\} =$$
$$= \lim_{t\to\infty} (\psi(t)(\log t)^{2/\alpha}/y(t)) \exp\{\log t(1 - C_1^2\varepsilon^2(1-a)^2 2^\alpha \psi^\alpha(t)/2)\} = 0, \qquad \text{q.e.d.}$$

The following lemmas concern the upper tail of the distribution of the maximum for a particular process. The first is due to Slepian [*17*].

Lemma 4.3 (Slepian). *Let $X(t)$ be the S.G.S.P. having the covariance function*

$$r(t) = (1-|t|)^+. \tag{4.19}$$

Let $Q(x, t\colon u)\,dt$ be the conditional probability that $X(t)$ reached x for the first time, in the interval of time $t \leqq s \leqq t + dt \leqq 1$, given that $X(0) = u < x$.

Then

$$Q(x, t\colon u) = (2\pi)^{-1/2} t^{-3/2}(2-t)^{-1/2}|x-u| \exp\{-(x-u(1-t))^2/2t(2-t)\},$$
q.e.d. (4.20)

Lemma 4.4. *Let $X(t)$ be a S.G.S.P. defined on the interval $0 \leqq t \leqq 1$, with covariance function $(1-\lambda|t|)^+$. Let ε, and ω be arbitrarily chosen, so that $0 \leqq \varepsilon < \omega < 1$. Let*

$$Y = \max_{\{\omega\varepsilon/2 \leqq s \leqq \omega(1-\varepsilon/2)\}} X(s),$$
$$F(x) = P\{Y \leqq x\}.$$

Then

$$\lim_{x\to\infty} (1-F(x))/(2\pi)^{-1/2}\omega(1-\varepsilon)\,x\exp(-x^2/2) = 1. \tag{4.21}$$

Proof. Let $Q(x,t)$ be the unconditional version of the probability specified in Lemma 4.3, and let

$$J = \int_{\omega\varepsilon/2}^{\omega(1-\varepsilon/2)} Q(x,t)\,dt.$$

Then J is the joint probability that $X(0) \leqq x$, and that x is exceeded somewhere in the whole interval $\omega\varepsilon/2 \leqq s \leqq \omega(1-\varepsilon/2)$. Clearly

$$J \leqq 1 - F(x) \leqq J + P\{X(0) > x\}. \tag{4.22}$$

First $Q(x,t)$ is evaluated.

$$Q(x,t) = (2\pi)^{-1/2} t^{-1} I, \tag{4.23}$$

where

$$I = (2\pi)^{-1/2} t^{-1/2} (2-t)^{-1/2} \int_{-\infty}^{0} |x-u| \exp\{-(x-u(1-t))^2/2t(2-t)\} \times \\ \times \exp(-u^2/2)\, du\,.$$

First letting $u \equiv v + x$,

$$I = (2\pi t(2-t))^{-1/2} \int_{-\infty}^{0} |v| \exp\{-(v(1-t) - xt)^2/2t(2-t)\} \exp\{-(v+x)^2/2\}\, dv.$$

Now consider the expression in the brackets, multiplied by $2t(2-t)$

$$(v(1-t) - xt)^2 + t(2-t)(v+x)^2 = (v+xt)^2 + x^2(2t - t^2)\,.$$

Now let $w = -(v+xt)/\sqrt{t(2-t)}$. Then

$$I = e^{-x^2/2}(I_1 + I_2)\,, \tag{4.24}$$

where

$$\begin{aligned} I_1 &= (2\pi)^{-1/2} \sqrt{t(2-t)} \int_{-xt/\sqrt{t(2-t)}}^{\infty} w \exp(-w^2/2)\, dw \\ &= (2\pi)^{-1/2} \sqrt{t(2-t)} \int_{xt/\sqrt{t(2-t)}}^{\infty} w \exp(-w^2/2)\, dw = \\ &= (2\pi)^{-1/2} \sqrt{t(2-t)} \exp\{-x^2 t/t(2-t)\} \end{aligned} \tag{4.25}$$

since the integrand being odd, the integral from $-xt/\sqrt{t(2-t)}$ to $xt/\sqrt{t(2-t)}$ vanishes, and

$$I_2 = (2\pi)^{-1/2} xt \int_{-xt/\sqrt{t(2-t)}}^{\infty} \exp(-w^2/2)\, dw = xt\,\Phi(xt/\sqrt{t(2-t)})\,. \tag{4.26}$$

Combining (4.23), (4.24), (4.25), and (4.26),

$$Q(x,t) = (2\pi)^{-1/2} e^{-x^2/2} ((2\pi)^{-1/2} t^{-1/2} (2-t)^{1/2} \exp\{-(xt)^2/2t(2-t)\}) + \\ + (2\pi)^{-1/2} e^{-x^2/2} x\,\Phi(x\sqrt{t/(2-t)})\,.$$

Now, if $\omega\,\varepsilon/2 \leqq t \leqq \omega(1-\varepsilon/2)$, it is clear that

$$\lim_{x\to\infty} Q(x,t)/(2\pi)^{-1/2} x e^{-x^2/2} = 1, \text{ uniformly in } t.$$

By definition of J, and eq. (4.22)

$$(1 - F(x)) = J(1 + o(1))\,,$$

where

$$J = \omega(1-\varepsilon)\, x \exp(-x^2/2)\,(1 + o(1))\,, \quad \text{q.e.d.}$$

Lemma 4.5 shows that a d.f. $F(x)$ lies in the domain of attraction of the d.f. $\Lambda(x)$, if it has the upper tail equivalent (4.21).

Lemma 4.5. *Let $F(x)$ be a d.f. such that*

$$\lim_{x\to\infty} (1 - F(x))/(2\pi)^{-1/2} c x \exp\{-x^2/2\} = 1\,.$$

Then

$$\lim_{N\to\infty} F_N(a_N x + b_N) = \Lambda^c(x)\,, \tag{4.27}$$

where $a_N = (2\log N)^{-1/2}$

and $b_N = (2\log N)^{1/2} + \frac{1}{2}(2\log N)^{-1/2}(\log\log N - \log \pi)$

and $\Lambda(x)$ *is given by* (2.4).

Proof. By the asymptotic relationship (3.12), it is sufficient to prove that

$$\lim_{N\to\infty} N(1 - F(a_N x + b_N)) = c\,e^{-x}\,. \tag{4.28}$$

Clearly

$$a_N x + b_N = (2\log N)^{1/2} + (\tfrac{1}{2}\log\log N/(2\log N)^{1/2})\,(1 + o(1))\,,$$
$$(a_N x + b_N)^2/2 = x + \log\sqrt{\log N} + \log N - \log\sqrt{\pi} + o(1)\,,$$

from which (4.28) above follows, q.e.d.

Lemma 4.6. *Let* $X(t)$ *be characterized by the measure* $P^{**}_{\omega,\lambda}(\cdot)$. *That is assume it has the covariance function,*

$$r(s,t) \equiv 1 - \lambda\,|t-s|\,, \quad \text{if} \quad [s/\omega] = [t/\omega]\,,$$
$$\equiv 0\,.$$

Assume also that $\lambda\omega \leqq 1$. *Then*

$$\lim_{t\to\infty} P\{(A(t))^{-1}(Z(\varepsilon,\omega;t) - B(t)) \leqq x\} = \Lambda^{\lambda(1-\varepsilon)}(x) \tag{4.29}$$

where

$$A(t) = (2\log t)^{-1/2}\,, \tag{4.30}$$

$$B(t) = (2\log t)^{1/2} + \tfrac{1}{2}(2\log t)^{-1/2}(\log\log t - \log\pi)\,. \tag{4.31}$$

Proof. First the time axis is stretched. Let $\tau = \lambda t$. Then

$$Z(\varepsilon,\omega;t) = \max_{1\leqq k\leqq t/\omega} Y_k,\ Y^*\,,$$

where

$$Y_k \equiv \max_{\{(k-1+\varepsilon/2)\omega\leqq s\leqq(k-\varepsilon/2)\omega\}} X(s)\,,$$

and

$$Y^* \equiv \max_{\{t/\omega+\varepsilon/2\leqq s\leqq t\}} X(s)\,, \quad \text{if} \quad t \geqq (t/\omega + \varepsilon/2)\,\omega\,,$$
$$\equiv 0\,, \qquad \text{if} \quad t < (t/\omega + \varepsilon/2)\,\omega\,.$$

By stationarity,

$$P\{(A(t))^{-1}(Y^* - B(t)) \leqq x\} \leqq$$
$$\leqq P\left\{(A(t))^{-1}\left(\operatorname*{Sup}_{0\leqq s\leqq\omega} X(s) - B(t)\right) \leqq x\right\} \to 1\,, \quad \text{as} \quad t\to\infty\,.$$

So Y^* can be disregarded.

Replacing τ again by λt, by Lemma 4.4,

$$\lim_{x\to\infty} P\{Y_k > x\}/(2\pi)^{-1/2}\lambda\,\omega(1-\varepsilon)\,x\exp(-x^2/2) = 1\,.$$

So using the relationship (4.28) in Lemma 4.5,

$$\lim_{t\to\infty} [t/\omega]\,\lambda\,\omega(1-\varepsilon)\,P\{Y_k > A(t)\,x + B(t)\} = \lambda(1-\varepsilon)\,e^{-x}\,.$$

Since the variables Y_k are mutually independent, the result follows from Lemma 4.5, q.e.d.

Remark. It is important to note that the result of Lemma 4.6, is independent of ω.

Lemma 4.7. *Let $\varkappa(t) \leqq t$ be a function such that*

$$\limsup_{t\to\infty}\,(t-\varkappa(t)) = C < \infty\,. \tag{4.32}$$

Then

$$\lim_{t\to\infty} P\{(A(t))^{-1}(Z(\varkappa(t)) - B(t)) \leqq x\}$$

exists if and only if

$$\lim_{t\to\infty} P\{A(t))^{-1}(Z(t) - B(t)) \leqq x\}$$

does, and they are both equal, where $A(t)$ and $B(t)$ are given respectively by (4.30) *and* (4.31).

Proof. By stationarity,

$$P\{(A(t))^{-1}(Z(\varkappa(t)) - Z(t)) > \varepsilon\} \leqq P\{(A(t))^{-1}Z(c) > \varepsilon\} \to 0, \quad \text{as} \quad t\to\infty,$$

q.e.d.

Theorem 4.3. *Let $\{X(t),\ -\infty \leqq t \leqq \infty\}$ be a S.G.S.P. with covariance function $r(t)$, such that*

$$\lim_{t\to 0} t^{-1}(1-r(t)) = \lambda\,, \quad 0<\lambda<\infty\,, \tag{4.33}$$

$$\forall\,\varepsilon>0,\ \lim_{t\to\infty} D_\varepsilon(\varkappa(t)) = 0\,,$$

where

$$D_\varepsilon(t) = t^{-2}(\log t)^{-1}\sum_{J=1}^{N(t)}(N(t)-J)\,|r_\varepsilon(J\,t/N(t))|\exp\{r_\varepsilon(J\,t/(N(t))\,(2\log t + \\ + \log\log t)/(1+r_\varepsilon(J\,t/N(t)))\}\,, \tag{4.34}$$

and

$$r_\varepsilon(t) = \max\,(r(\varepsilon),\ r(t))\,. \tag{4.35}$$

Then

$$\lim_{t\to\infty} P\{(A(t))^{-1}(Z(t) - B(t)) \leqq x\} = (\Lambda(x))^\lambda = \Lambda(x-\log\lambda)\,, \tag{4.36}$$

where $A(t)$ and $B(t)$ are given by (4.30) *and* (4.31), *$\Lambda(x)$ by* (2.4), *$N(t)$ is such that* (4.10) *and* (4.11) *hold and $\varkappa(t)$ satisfies* (4.32).

Proof. Let ε and ω be arbitrarily chosen, as explained in the beginning of this section. Let

$$P(x,t) \equiv P\{(A(t))^{-1}(Z(t) - B(t)) \leqq x\}\,.$$

Let $P(x, \varepsilon, \omega; t)$, $P^*(x, \varepsilon, \omega; t)$, and $P_\lambda^{**}(x, \varepsilon, \omega; t)$ be similarly defined, using $Z(\varepsilon, \omega; t)$ instead of $Z(t)$ and the measures introduced at the beginning of this section. Let $P_{N(t)}(x, \varepsilon, \omega; t)$ and $P_{N(t)}^*(x, \varepsilon, \omega; t)$ be the corresponding functions using $Z_{N(t)}(\varepsilon, \omega; t)$.

Let λ^-, and λ^+ be arbitrarily chosen, but so that

$$1 - \lambda^- |t - s| \leqq r(|t - s|) \leqq 1 - \lambda^+ |t - s| \tag{4.37}$$

provided $|t - s| \leqq \omega$. Recall that, with the exception of $P(\cdot)$, these measures assign mutual independence on the various intervals I_k. Define the event

$$E \equiv \left\{ \sup_{s \in [0,t] \cap I^c} X(s) > A(t)\, x + B(t) \right\}.$$

It is the event that $A(t)\, x + B(t)$ is exceeded in the interval $[0, t]$ on the complement of I. Clearly, by Lemma 3.5,

$$|P(x, \varkappa(t)) - P(x, \varepsilon, \omega, \varkappa(t))| \leqq P\{E\} \leqq P_\omega^*\{E\} \leqq P_{\omega,\lambda^-}^{**}\{E\}.$$

Let

$$\psi(\varepsilon) \equiv \lim_{t\to\infty} P_{\omega,\lambda^-}\{E\} = 1 - (A(x))^{\lambda^- \varepsilon}.$$

Then

$$\limsup_{t\to\infty} \left(\liminf_{t\to\infty}\right) P(x, t) \leqq (\geqq) \limsup_{t\to\infty} \left(\liminf_{t\to\infty}\right) P(x, \varepsilon, \omega; t) + (-)\, \psi(\varepsilon),$$

and

$$\lim_{\varepsilon\to\infty} \psi(\varepsilon) = 0. \tag{4.38}$$

By Lemma 4.2, and by (4.30),

$$\begin{aligned}((Z(\varepsilon, \omega; t) - B(t))\,(A(t))^{-1} &- (Z_{N(t)}(\varepsilon, \omega; t) - B(t))\,(A(t))^{-1}) \\ &= (2\log t)^{1/2}\,(Z(\varepsilon, \omega; t) - Z_{N(t)}(\varepsilon, \omega; t)) \to 0, \quad \text{i.p.}\end{aligned}$$

with respect to both of the measures $P(\cdot)$ and $P_\omega^*(\cdot)$, and similarly with $\varkappa(t)$ instead of t. By the definitions this implies that

$$\limsup_{t\to\infty} \left(\liminf_{t\to\infty}\right) P(x, \varepsilon, \omega, \varkappa(t)) \leqq (\geqq)\, P_{N(t)}(x + (-)\, \varepsilon', \varepsilon, \omega, \varkappa(t))$$

for all $\varepsilon' \geqq 0$, and similarly for P^*. Now it is shown that

$$\lim_{t\to\infty} |P_{N(t)}(x, \varepsilon, \omega, \varkappa(t)) - P_{N(t)}^*(x, \varepsilon, \omega, \varkappa(t))| = 0. \tag{4.39}$$

By Lemma 2.1, this expression is dominated by $D_\varepsilon(t)$, (4.34). The term of the form $(1 - r^2)^{-1/2}$ can be replaced by a constant, since the use of the ε width intervals empowers the definition (4.35). The summation is assumed to take place only on those pairs of points such that $it/N(t)$, and $jt/N(t)$ are both on I. Also, in this case,

$$c^2 = 2\log t + \log\log t + O(1).$$

So (4.39) follows and hence,

$$\limsup_{t\to\infty} \left(\liminf_{t\to\infty}\right) P(x, \varepsilon, \omega, \varkappa(t)) \leqq (\geqq) \limsup_{t\to\infty} \left(\liminf_{t\to\infty}\right) P^*(x + (-)\, \varepsilon', \varepsilon, \omega, \varkappa(t))$$

for all ε', ε, and all x.

But by (4.37), and Lemma 3.5,

$$P^{**}_{\lambda^+}(x, \varepsilon, \omega; t) \leqq P^*(x, \varepsilon, \omega; t) \leqq P^{**}_{\lambda^-}(x, \varepsilon, \omega; t) .$$

By Lemmas 4.6, and 4.7,

$$\lim_{t\to\infty} P^{**}_{\lambda^{+(-)}}(x, \varepsilon, \omega, \varkappa(t)) = (\Lambda(x))^{\lambda^{+(-)}(1-\varepsilon)} .$$

Combining the above results,

$$\limsup_{t\to\infty} \left(\liminf_{t\to\infty}\right) P(x, \varkappa(t)) \leqq (\geqq) \left(\Lambda(x + (-)\, \varepsilon')\right)^{\lambda^{+(-)}(1-\varepsilon)} + (-)\, \psi(\varepsilon) .$$

But ε and ε' are arbitrary positive. The result does not depend on ω, which is also arbitrary. The local character (4.33) of the process makes it plain that λ^+ and λ^- can be chosen as close to λ as desired, by making ω sufficiently small. Recalling (4.38) it follows that

$$\lim_{t\to\infty} P(x, \varkappa(t)) = (\Lambda(x))^{\lambda} ,$$

and hence, by Lemma 4.7, the theorem follows, q.e.d.

The lemmas and theorems which follow simplify the condition (4.34).

Theorem 4.4. *If*

$$\lim_{t\to\infty} t^{-1}(1 - r(t)) = \lambda , \quad 0 < \lambda < \infty ,$$

and

$$\lim_{t\to\infty} r(t) (\log t)^3 = 0 \tag{4.40}$$

then

$$\lim_{t\to\infty} P\{(A(t))^{-1}(Z(t) - B(t)) \leqq x\} = \Lambda^{\lambda}(x) ,$$

where $A(t)$, $B(t)$, *and* $\Lambda(x)$ *are given by* (4.30), (4.31), *and* (2.4).

Proof. It is only necessary to show that the condition (4.34) of Theorem 4.3, is satisfied. Let $D(t)$ be expressed as the sum of $D_1(t)$ and $D_2(t)$, where the former contains the sum from one to $[t^{\gamma-1}N(t)]$, and the latter from $[t^{\gamma-1}N(t)] + 1$ to $N(t) - 1$, and γ, $0 < \gamma < 1$, will be specified latter. Let

$$\delta(t) = \operatorname*{Sup}_{t\leqq s} r(s) \tag{4.41}$$

and

$$\delta^*(t) = \operatorname*{Sup}_{t\leqq s} 2r(s)/(1 + r(s)) , \tag{4.42}$$

and note that (4.40) implies

$$\lim_{t\to\infty} r(t) = 0 ,$$

and hence $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \delta^*(t) = 0$.

So $\quad \exp\{r_\varepsilon(t)\,(2\log t + \log\log t)/(1 + r_\varepsilon(t))\} \leqq$

$$\leqq \exp\{\delta(\varepsilon)\,(\log t + \tfrac{1}{2}\log\log t)\} = t^{\delta(\varepsilon)}(\log t)^{\delta(\varepsilon)/2} .$$

Recalling the representation of $N(t)$ and the assumptions (4.10) and (4.11)

$$D_1(t) \leqq t^{-(1-\delta(\varepsilon))+\gamma} (\log t)^{3+\delta(\varepsilon)/2} \to 0, \quad \text{as} \quad t \to \infty$$

since by (4.14), $\psi(t)$ approaches infinity more slowly than any power of t. γ is chosen less than $1 - \delta(\varepsilon)$. This is clearly possible for every fixed $\varepsilon > 0$.

Now consider $D_2(t)$. By (4.40) above

$$\lim_{t\to\infty} \delta(t^\gamma)(2\log t + \log\log t) = 0$$

and

$$0 \leqq \exp\{r_\varepsilon(t)(2\log t + \log\log t)/(1 + r_\varepsilon(t))\} \leqq$$
$$\leqq \exp\{\delta(t^\gamma)(2\log t + \log\log t)\} \to 1, \quad \text{as} \quad t \to \infty.$$

So by Theorem 4.3, it remains to prove that

$$\lim_{t\to\infty} t^{-2}(\log t)^{-1} \sum_{j=[t^{\gamma-1}N(t)]+1}^{N(t)} (N(t) - j)\, r_\varepsilon(j\,t/N(t)) = 0.$$

Recalling the definition of $N(t)$, this expression is less than

$$\delta(t^\gamma)(\log t)^3 \psi^2(t) \to 0, \quad \text{as} \quad t \to \infty, \tag{4.43}$$

if $\psi^2(t)$ is set arbitrarily equal to

$$(\delta(t^\gamma))^{-1/2}(\log t)^{-3/2} \to \infty, \quad \text{as } t \to \infty.$$

If $\psi(t)$, so defined, does not satisfy (4.14), it can be replaced by a function which does, and (4.43) is still valid, q.e.d.

Lemma 4.8. *If*

$$\int_0^\infty r^2(t)\,dt < \infty, \tag{4.44}$$

then

$$\lim_{t\to\infty} r(t) = 0. \tag{4.45}$$

Proof. It will be shown that if (4.45) is not true, then (4.44) is not. In effect we assume there exists a real number $a > 0$, such that $|r(t)| > a$ infinitely many times. Consider a point t such that this is true. Let $\delta > 0$ be arbitrarily chosen. Let $t \in (t' - \delta, t' + \delta)$. Then, by the "Increments Inequality", (LOÈVE [14], p. 195)

$$|r(t) - r(t')| \leqq 2^{1/2}(1 - r(t - t'))^{1/2} \to 0, \quad \text{as} \quad t \to t'.$$

So δ may be chosen so that $t \in (t' - \delta, t' + \delta)$ implies $|r(t)| > a/2$.

$$\text{So} \int_{t'-\delta}^{t'+\delta} r^2(t)\,dt > \delta a^2/2 > 0.$$

But, by assumption, an infinite sequence of such points t_k can be found whose δ intervals are not overlapping. So

$$\int_0^\infty r^2(t)\,dt \geqq \sum_{k=0}^\infty \int_{t_k-\delta}^{t_k+\delta} r^2(t)\,dt = \infty.$$

Therefore (4.44) implies (4.45), q.e.d.

Theorem 4.5. *If*

$$\lim_{t\to 0} t^{-1}(1-r(t)) = \lambda\,, \quad 0<\lambda<\infty\,,$$

and

$$\int_0^\infty r^2(t)\,dt < \infty\,, \tag{4.46}$$

then

$$\lim_{t\to\infty} P\{(A(t))^{-1}(Z(t)-B(t)) \leqq x\} = \Lambda^\lambda(x)\,,$$

where $A(t)$ *and* $B(t)$ *are given by* (4.30) *and* (4.31), *and* $\Lambda(x)$ *by* (2.4).

Proof. Let $\varkappa(t)$ be a function such that

$$t N(t)/N(t) + 1 \leqq \varkappa(t) \leqq t\,,$$

where $N(t)$ is subject to the conditions (4.10) and (4.11).

Clearly, by Lemma 4.7 and Theorem 4.3 it is sufficient to show that

$$\forall\,\varepsilon > 0\,,\ \lim_{t\to 0} D_\varepsilon(\varkappa(t)) = 0\,.$$

As in the proof of Theorem 4.4, $D_\varepsilon(\varkappa(t))$ is expressed as the sum of $D_1(\varkappa(t))$ and $D_2(\varkappa(t))$. The former approaches zero as t approaches infinity. It is sufficient to show that

$$\forall\,\varepsilon > 0\,,\ \lim_{t\to\infty} D_2(\varkappa(t)) = 0\,.$$

We assume this is not so, and show that a contradiction develops. That is, we assume that there exists a constant $A > 0$, such that for infinitely many t,

$$D_2(\varkappa(t)) > A \tag{4.47}$$

for every possible value of $\varkappa(t)$. It is possible to bound $D_2(t)$

$$D_2(t) \leqq g(t)\,(N(t))^{-1} \sum_{j=[t^{\gamma-1}N(t)]+1}^{N(t)} |r(j\varkappa(t)/N(t))|\,,$$

where

$$g(t) \equiv (\varkappa(t))^{-2}\,(\log\varkappa(t))^{-1}\,(N(t))^2\,(\varkappa(t))^{2\delta(\varkappa(t))}\,\gamma\,(\log\varkappa(t))^{\delta(\varkappa(t))}$$

where $\delta(t)$ is given by (4.41). By (4.46) and Lemma 4.8,

$$\lim_{t\to\infty} \delta(\varkappa^\gamma(t)) = 0\,.$$

So, if $g(t)$ approaches infinity, it does so more slowly than any power of t.

If (4.47) holds for all possible functions $\varkappa(t)$, it follows, by integration, for a fixed t, that

$$g(t)\,(N(t))^{-1} \int_{tN(t)/N(t)+1}^{t} \sum_{j=[t^{\gamma-1}N(t)]+1}^{N(t)} |r(j\,u/N(t))|\,du \geqq$$
$$\geqq A\,(t - t\,N(t)/N(t) + 1) \geqq A\,t/N(t) + 1 = A/h(t)\,.$$

Clearly $h(t)$ approaches infinity more slowly than any power of t.

Therefore a sufficient condition is that

$$\lim_{t\to\infty} g(t)\,h(t)\,(N(t))^{-1} \sum_{j=[t^{\gamma-1}N(t)]+1}^{N(t)} |r(ju/N(t))|\,du = 0\,.$$

By the Cauchy-Schwarz inequality, a sufficient condition is that

$$\forall\varepsilon > 0\,,\quad \lim_{t\to\infty} D^*(t) = 0\,,$$

where

$$D^*(t) = g^2(t)\,h^2(t)\,(N(t))^{-1} \sum_{j=[t^{\gamma-1}N(t)]+1}^{N(t)} I_j\,,$$

and

$$I_j = \int_{tN(t)/N(t)+1}^{t} r^2(ju/N(t))\,du\,.$$

Using the transformation $s = ju/N(t)$,

$$D^*(t) \leqq g^2(t)\,h^2(t)\,(M(t))^{-1} \sum_{j=M(t)+1}^{N(t)} \int_{jt/(N(t)+1)}^{jt/N(t)} r^2(s)\,ds$$

where $M(t) = [t^{\gamma-1}N(t)]$.

Since $(j + 1/N + 1) - J/N \geqq 0$, there is no overlapping, in summing the integrals. Hence

$$D^*(t) \leqq g^2(t)\,h^2(t)\,(M(t))^{-1} \int_0^t r^2(s)\,ds$$

which approaches zero, as t approaches infinity, by the previously observed fact that $g(t)$ and $h(t)$ approach infinity more slowly than any power of t, and hence less rapidly than $M(t)$, q.e.d.

It seems probable, recalling Section 2, that

$$\lim_{t\to\infty} r(t) = 0$$

is not a sufficient "mixing condition" for (4.36). Though the author has not constructed a counterexample to establish this, it probably would not be impossible. Using the spectral representation

$$r(t) = \int_{-\infty}^{\infty} \cos\omega t\,dG(\omega)\,,$$

what condition on $G(\omega)$ is sufficient for (4.33)? It can be written, simply

$$\lim_{\omega\to\infty} (1 - G(\omega))/\omega = \lim_{\omega\to\infty} G(-\omega)/|\omega| = 1/\lambda\,.$$

If $G(\omega)$ is absolutely continuous, we can write

$$r(t) = \int_{-\infty}^{\infty} \cos\omega t\,g(\omega)\,d\omega$$

a sufficient condition for the "mixing condition" (4.40), is that $g(\omega)$ satisfy a Lipschitz condition of order α for some $\alpha > 0$. A sufficient condition for (4.46) is simply that

$$\int_{-\infty}^{\infty} g^2(\omega)\, d\omega < \infty. \tag{4.48}$$

Kolmogorov and Rozanov [13] have proven that if a continuous parameter stationary Gaussian process is "uniformly mixing", $g(\omega)$ must be well defined and continuous everywhere. Since, obviously,

$$r(0) = \int_{-\infty}^{\infty} g(\omega)\, d\omega < \infty,$$

(4.48) above, clearly follows. So the uniform mixing condition is sufficient for (4.36).

Cramèr [7] and [8], has obtained analogous results for a different class of stationary Gaussian processes. He considers processes whose realizations are everywhere differentiable with probability one. They satisfy the local condition:

$$r(t) = 1 - \omega_2 t^2/2 + o(t^2)$$

as $t \to 0$, and the mixing condition:

$$\exists \alpha > 0 : \lim_{t \to \infty} t^\alpha r(t) = 0.$$

The conclusion is the same as that of Theorem 4.3 except that

$$B(t) \equiv \sqrt{2 \log t} - \left(\log\left(2\pi/\sqrt{\omega_2}\right)/\sqrt{2 \log t}\right),$$

with $\lambda = 1$.

His method is based on the limiting probability distribution of the number of upcrossings of a high level. In the present case the realizations are not differentiable, and so the concept of an upcrossing is not meaningful.

The Ornstein-Uhlenbeck process is the only process, which is both a diffusion process, and a stationary Gaussian process. It is characterized by the covariance,

$$r(t) = e^{-\lambda t}.$$

Newell, in his study of diffusion processes [15] has shown that its maxima have the property (4.36). It clearly satisfies the conditions of both Theorems 4.4 and 4.5. So the result is again proved.

Section 5

In this section "local" and "mixing conditions" are found, which are sufficient for stability and relative stability, almost surely. As in Section 3, the problem is divided into the problems, respectively, of "upper", and "lower" stability. The former is considered first.

Lemma 5.1. *Let $\{X(t), -\infty < t < \infty\}$ be a S.G.S.P. having a covariance function $r(t)$ such that*

$$\exists \alpha > 0 : \limsup_{t \to 0} t^{-\alpha}(1 - r(t)) < \infty. \tag{5.1}$$

Let $1 - F_T(x) \equiv P\{Z(T) > x\}$
where T *is sufficiently small, so that*

$$r(t) \geqq 0, \quad \text{if} \quad 0 \leqq t \leqq T. \tag{5.2}$$

The condition (5.1) *clearly makes this possible.*

Let $M(x)$ *be a function, such that*

$$\forall \gamma > 0, \quad \lim_{x \to \infty} M(x + \gamma/x)/M(x) = 1, \tag{5.3}$$

and

$$\lim_{\to \infty} M(x)/x^{4/\alpha} = \infty. \tag{5.4}$$

Then

$$\limsup_{x \to \infty} (1 - F_T(x))/M(x)\,\phi(x) < 1, \tag{5.5}$$

where $Z(T)$ *is given by* (4.1) *and* $\phi(x)$ *by* (3.13).

Proof. Let $\gamma > 0$ be arbitrarily chosen. Note that

$$P\{Z(T) > x + \gamma/x\} \leqq P\{Z_{M(x)}(T) > x\} + P\{Z(T) - Z_{M(x)}(T) > \gamma/x\}.$$

Using Lemma 3.5, and the well known combinatorial formula

$$P\{Z_{M(x)}(T) > x\} \leqq P'\{Z_{M(x)}(T) > x\} \leqq M(x)\,P\{X > x\} \leqq M(x)\,\phi(x),$$

where $P'(\cdot)$ is the measure conferring independence on the component variables, and X is a normalized Gaussian r.v.

$$P(x) \equiv P\{Z(T) - Z_{M(x)}(T) > \gamma/x\} \leqq \sum_{k=0}^{\infty} P\{D_k\}, \tag{5.6}$$

where

$$D_k \equiv \{Z_{2^{k+1}M(x)}(T) - Z_{2^k M(x)}(T) > \gamma a^k (1-a)/x\},$$

and a is a positive number less than one.

Let

$$E_j^{(k)} \equiv \{X((2j-1)\,T/2^{k+1}) - X(j\,T/2^k) > \gamma a^k (1-a)/x\}.$$

By Lemma 3.3,

$$P\{E_j^{(k)}\} \leqq \phi(z_k),$$

where $z_k = \gamma a^k (1-a)/x\sigma$,

$$\sigma^2 = 2(1 - r(T/2^{k+1} M(x))) \leqq C_1 2^{-\alpha k} M(x)^{-\alpha},$$

for some constant C_1, and sufficiently large x (that is for sufficiently small $(T/M(x))$), by the condition (5.1), uniformly for all k. But

$$z_k \geqq C_2 a^k 2^{\alpha k/2} (M(x))^{\alpha/2}/x = C_2\,\omega^k (M(x))^{\alpha/2}/x,$$

where $\omega = a \cdot 2^{\alpha/2}$.

Recalling that a can be chosen arbitrarily less than one, it is chosen so that

$$\omega > 1.$$

$$P\{D_k\} \leqq \sum_{j=1}^{2^k M(x)} P\{E_j\} \leqq 2^k M(x)\,\phi(z_k).$$

By (5.6),

$$P(x) \leqq g(y(x), \omega)/\phi(x) = (\phi(x))^{-1} \sum_{k=0}^{\infty} 2^k \phi(y(x)\omega^k), \tag{5.7}$$

where $y(x) = C_2(M(x))^{\alpha/2}$. Since ω is fixed, there exists an integer k_0, so that if $k > k_0$,

$$\omega^k > k\omega.$$

The function $g(y, \omega)$ can be expressed as the sum of two functions $g_1(y, \omega)$, and $g_2(y, \omega)$, where the former is the sum from one to k_0 in eq. (5.7), and the latter from $k_0 + 1$ to infinity.

$$g_2(y, \omega) \leqq (2\pi)^{-1/2} y^{-1}(2/\omega) \exp(-\omega y^2/2)/(1 - \exp(-\omega y^2/2)).$$

To prove

$$\limsup_{x\to\infty} P\{Z(T) > x + \gamma/x\}/M(x)\phi(x) < 1, \tag{5.8}$$

it is sufficient to prove that the right side of (5.7) approaches zero, as x approaches infinity. Hence it is sufficient to prove $g_2(y(x), \omega)$ approaches zero. It need only be shown then, that

$$\lim_{x\to\infty} (\exp(-C_3 y^2(x))/2)/y(x)\phi(x) = 0.$$

But this equals

$$(\exp(-C_3 C_2^2(M(x))^{\alpha}/2x^2 + x^2/2))(2\pi)^{1/2} x/y(x).$$

But

$$(C_3 C_2^2(M(x))^{\alpha}/2x^2 - x^2/2) = x^2/2\,(C_3 C_2^2(M(x))^{\alpha}/x^4 - 1) \to \infty$$

as $x \to \infty$. Note that

$$\lim_{x\to\infty} M(x + \gamma/x)\phi(x + \gamma/x)/M(x)\phi(x) = e^{-\gamma}.$$

Let x' be chosen for each x, so that

$$x = x' + \gamma/x'.$$

Then

$$\limsup_{x\to\infty} P\{Z(T) > x\}/M(x)\phi(x) = \limsup_{x\to\infty} (P\{Z(T) > x' + \gamma/x'\}/M(x') \times \\ \times \phi(x'))\, M(x')\phi(x')/M(x)\phi(x) \leqq e^{\gamma}.$$

Since γ was arbitrary, (5.5) follows.

Corollary 5.1. *If*

$$\exists \alpha > 0: \limsup_{t\to 0} t^{-\alpha}(1 - r(t)) < \infty,$$

then $\forall \lambda > (4/\alpha) - 1, \quad \limsup_{x\to\infty} (1 - F_T(x))/x^{\lambda} \exp(-x^2/2) < 1.$

The *proof* follows from Lemma 5.1, and its conditions 5.3, and 5.4, which are easily seen to be satisfied by

$$M(x) = x^{\lambda+1}.$$

Theorem 5.1. *If*

$$\exists \alpha > 0: \limsup_{t \to 0} t^{-\alpha}(1 - r(t)) < \infty .$$

Then $\limsup_{t \to \infty} \left(Z(t) - \sqrt{2 \log t}\right) \leqq 0$ *a.s.*

That is to say that upper stability holds. As in the discrete parameter case, no "mixing condition" is involved here.

Proof of Theorem 5.1. First note that for any real $C > 0$, it is sufficient to prove that

$$\limsup_{t \to \infty} \left(Z(t) - \sqrt{2 \log C t}\right) \leqq 0 , \quad \text{a.s.}$$

Note that

$$Z(t) - \sqrt{2 \log N t} \leqq Z^*_{N+1} - \sqrt{2 \log N T} ,$$

where $N = t/T$,

$$Z^*_N = \max_{1 \leqq k \leqq N} X^*_k ,$$

$$X^*_N = \operatorname*{Sup}_{\{(N-1)T \leqq s \leqq NT\}} X(s)$$

and T satisfies (5.2). So it is sufficient to establish that

$$\limsup_{N \to \infty} \left(Z^*_N - \sqrt{2 \log N}\right) \leqq 0 , \quad \text{a.s.}$$

So, it must be demonstrated that $\forall \varepsilon > 0$,

$$X^*_N > \sqrt{2 \log N} + \varepsilon$$

only finitely many times, with probability one. But, clearly

$$\sum_{N=1}^{\infty} (2 \log N + \varepsilon)^{\lambda} \exp\left(-\sqrt{2 \log N} + \varepsilon\right)^2/2 < \infty , \quad \forall \varepsilon > 0 .$$

So by Corollary 5.1, and the Borel-Cantelli Theorem, the theorem holds, q.e.d.

Now consider the problem of "lower stability".

Theorem 5.2. *If*

$$\lim_{t \to \infty} r(t) \log t = 0 ,$$

then $\liminf_{t \to \infty} \left(Z(t) - \sqrt{2 \log t}\right) \geqq 0$ a.s.

Proof. Clearly, for all $t \geqq 0$

$$Z(t) - \sqrt{2 \log t} \geqq Z_t - \sqrt{2 \log [t] + 1} ,$$

where

$$Z_N = \max_{1 \leqq k \leqq N} X(k) . \tag{5.9}$$

The result then follows immediately from Theorem 3.3, q.e.d.

Lemma 5.2. *Let*

$$t_\varepsilon(M) = \left(t: \sqrt{2 \log t} = \varepsilon M\right) = \exp(M^2 \varepsilon^2/2) .$$

If

$$Z(t_\varepsilon(M)) < (M-1)\,\varepsilon \tag{5.10}$$

only finitely many times $\forall\,\varepsilon > 0$, *with probability one, then*

$$\liminf_{t\to\infty}\left(Z(t) - \sqrt{2\log t}\right) \geqq 0\,, \quad a.s.$$

The *proof* follows, with minor modifications from that of Lemma 3.1.

Lemma 5.3. *A sufficient condition for "lower stability" is*

$$\lim_{t\to\infty}(\log t)\,P\{Z(t) \leqq \sqrt{2\log t} - \varepsilon\} = 0\,, \quad \forall\,\varepsilon > 0\,. \tag{5.11}$$

As in the proof of Lemma 3.2, the sufficiency of this condition follows from the fact that it implies the summability on M, of the probability of the event in (5.10). See the proof of Lemma 3.2.

Theorem 5.3. *If*

$$\int_0^\infty r^2(t)\,dt < \infty\,,$$

then $\liminf_{t\to\infty}\left(Z(t) - \sqrt{2\log t}\right) \geqq 0$, *a.s.*

Proof. Clearly, it is sufficient that for some function $\varkappa(t)$ satisfying (4.32)

$$\liminf_{t\to\infty}\left(Z(\varkappa(t)) - \sqrt{2\log t}\right) \geqq 0\,, \quad \text{a.s.}$$

But

$$Z(\varkappa(t)) - \sqrt{2\log t} \leqq Z_{[t]} - \sqrt{2\log([t]+1)}$$

where $Z_{[t]} = \max\limits_{1\leqq k\leqq t} X_{k\varkappa(t)/t}$.

The remainder of the proof is a direct result of Lemma 4.7, Theorem 3.2, and the reasoning of Theorem 4.5, q.e.d.

Theorems 5.1, 5.2, and 5.3, combine to give

Theorem 5.4. *If*

$$\exists\,\alpha > 0\colon \limsup_{t\to\infty} t^{-\alpha}(1 - r(t)) < \infty\,,$$

and either $\lim\limits_{t\to\infty} r(t)\log t = 0$, *or* $\int_0^\infty r^2(t)\,dt < \infty$, *then* $Z(t) - \sqrt{2\log t} \to 0$, *a.s.*

Theorem 5.5. *If*

$$\exists\,\alpha > 0\colon \limsup_{t\to\infty} t^{-\alpha}(1 - r(t)) < \infty\,,$$

and

$$\lim_{t\to\infty} r(t) = 0\,, \tag{5.12}$$

then $Z(t)/\sqrt{2\log t} \to 1$, *a.s.*

Proof. It is sufficient to establish "lower relative stability". But

$$Z(t)/\sqrt{2\log t} \geqq Z_{[t]}/\sqrt{2\log([t]+1)}\,,$$

where Z_N is given by (5.9). The result follows from Theorem 3.4, q.e.d.

As remarked at the end of Section 4, a counterexample could probably be constructed which would show that (5.12) was not sufficient for stability.

The remarks concerning the spectrum and the strong mixing condition are also valid.

Clearly the absolute continuity of the spectral density function is sufficient for relative stability.

CRAMÈR has shown [6] that

$$\lim_{t\to\infty} P\left[\left|Z(t)-\sqrt{2\log t}\right| \geqq \log\log t/\sqrt{\log t}\right]=0,$$

provided the spectral density is of bounded variation, and is such that

$$\exists a>1: \int_0^\infty \omega^2(\log(1+\omega))^a g(\omega)\,d\omega<\infty.$$

It has been shown by SHUR [16] that, under the same conditions, for any $\varepsilon>0$,

$$Z(t)-\sqrt{2\log t}<(1+\varepsilon)\log\log t/\sqrt{2\log t}$$

for all sufficiently large t, with probability 1. This implies the present results for the class of processes satisfying the above conditions on the spectrum. Those processes satisfy the local condition:

$$r(t)=1-\omega_2 t^2/2+o(t^2),$$

and the mixing condition

$$\limsup_{t\to\infty} t\,|r(t)|<\infty.$$

It would be worthwhile to obtain similar results for a wider class of processes.

References

[1] BELAYEV, YU. K.: Continuity and holders conditions for sample functions of stationary Gaussian processes. Proc. fourth Berkeley Symposium math. Statistics, Probability, Vol. II, 23—33 (1961).

[2] BERMAN, S. M.: Equally correlated random variables. Sankhya A. **24**, 155—156 (1962).

[3] — Limit theorems for the maximum term in stationary sequences. Ann. math. Statistics **35**, 502—516 (1964).

[4] CHYBISOV, D. M.: Remarks on the work of N. V. SMIRNOV „Limiting distribution laws for the terms of a variational series". Theor. Probab. Appl. **7**, 357—358 (1962).

[5] CRAMÈR, H.: Mathematical methods of statistics. Princeton: University Press 1951.

[6] — On the maximum of a normal stationary stochastic process. Bull. Amer. math. Soc. **68**, 512—516 (1962).

[7] — A limit theorem for the maximum values of certain stochastic processes. Theor. Probab. Appl. **10**, 126—128 (1965).

[8] — On the intersections between the trajectories of a normal stationary process and a high level. Ark. Mat. **6**, 337—349 (1966).

[9] GEFFROY, J.: Contribution a la theorie des valeurs extremes. Thesis, University of Paris, 1958.

[10] GNEDENKO, B. V.: Sur la distribution du terme maximum d'une serie aleatoire. Ann. of Math., II. Ser. **44**, 423—453 (1943).

[11] GRENANDER, U., and M. ROSENBLATT: Statistical analysis of stationary time series. New York: John Wiley and Sons 1957.
[12] GUMBEL, E. J.: Statistics of Extremes. Columbia: University Press 1958.
[13] KOLMOGOROV, A. N., and YU. A. ROZANOV: On strong mixing conditions for stationary Gaussian processes. Theor. Probab. Appl. **5**, 204–207 (1960).
[14] LOÈVE, M.: Probability theory. Princeton: D. Van Nostrand Co. Inc. 1955.
[15] NEWELL, G. F.: Asymptotic extreme value distribution for one-dimensional diffusion processes. J. Math. Mech. **11**, 481–496 (1962).
[16] SHUR, M. G.: On the maximum of a Gaussian stationary process. Theor. Probab. Appl. **10**, 354–357 (1965).
[17] SLEPIAN, D.: First passage time for a particular Gaussian process. Ann. math. Statistics **32**, 610–612 (1961).
[18] WATSON, G. S.: Extreme values from M-dependent stationary Gaussian process. Ann. math. Statistics **25**, 798–800 (1954).
[19] ZYGMUND, A.: Trigonometrical series. New York: Dover Publ. 1955.

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