Z. Wahrscheinlichkeitstheorie verw. Gebiete 32, 209–244 (1975)

Diffusion Processes Associated with Lévy Generators

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Diffusions associated with a Lévy-type generator $\mathscr{L}$ are discussed from the point of view of solving the martingale problem for $\mathscr{L}$. Existence of solutions is demonstrated under the assumptions of continuity. As for uniqueness, a localization procedure is developed to show that uniqueness for the whole process follows from uniqueness up until the first time has a jump of size greater than ε. This fact is then applied to prove uniqueness for various special classes of $\mathscr{L}$'s.

0. Introduction

In [7], the martingale problem associated with a time dependent, second order elliptic differential operator

$$L_t = \frac{1}{2}\sum_1^d a^{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_1^d b^i(t,x)\frac{\partial}{\partial x_i}$$

was introduced and discussed. That is, we said that a probability measure P on $C([0,\infty), R^d)$ solves the martingale problem for L_t starting from (s,x) if $P(x(s)=x)=1$ and $f(x(t)) - \int_s^t L_u f(x(u))\,du$ is a P-martingale for all $f \in C_0^\infty(R^d)$. It was shown there that if a is bounded, positive definite valued, and continuous, and if b is bounded and measurable, then the martingale problem for L_t is *well-posed*, in the sense that for each (s,x) there is exactly one solution to the martingale problem for L_t. The point of formulating the theory of diffusion processes in this way was to make precise what is the connection between diffusions and differential operators. The idea was that the martingale condition is the minimal property connecting a process with an operator. What is interesting is that in some cases it turned out to completely determine the process.

In the present paper, we will carry out a similar program for the class of *Lévy generators* $\mathscr{L}_t = L_t + K_t$, where L_t is as before and

$$K_t f(x) = \int \left(f(x+y) - f(x) - \frac{\langle y, \nabla f(x)\rangle}{1+|y|^2} \right) M(t,x;dy).$$

Here $M(t,x;\cdot)$ is a *Lévy jump measure* for each (t,x). That is, $M(t,x;.)$ is a σ-finite measure on $R^d \setminus \{0\}$ such that

$$\int \frac{|y|^2}{1+|y|^2} M(t,x;dy)$$

is finite. The martingale problem for $\mathscr{L}_t$ is that of finding for each (s,x) a probability measure P on $D([0,\infty), R^d)$ (the space of right-continuous functions

* The author was partially supported by N. S. F. grant GP-40383 while doing this research.

having left limits) such that $P(x(s)=x)=1$ and $f(x(t)-\int_s^t \mathscr{L}_u f(x(u))\,du$ is a martingale for all $f\in C_0^\infty(R^d)$ and proving that there is at most one such P.

This program is carried out as follows. In Section (1) we develop the stochastic calculus associated with such processes. Section (2) is devoted to the proof of existence. The proof of uniqueness is broken into three parts. Section (3) develops a localization procedure which enables us to use perturbation theory. The actual proofs of uniqueness are done in Sections (4) and (5).

Aside from the desire to extend the class of processes to which the martingale procedure applies, the motivation for this work comes from the study of diffusions with boundary conditions (cf. [8] and the thesis of Anderson [1]). In [8], it was shown how the problem of uniqueness for a diffusion satisfying boundary conditions can be split into proving uniqueness of the process up until it first hits the boundary and then proving uniqueness of the so called boundary process. The boundary process turns out to be governed by a Lévy generator. Unfortunately, the results in this paper cannot be used to handle the boundary processes which arise in [8]. However, they are just what is needed in Anderson's work.

The techniques developed in this paper can be adapted to prove existence and uniqueness for the martingale problem associated with operators

$$L_t = a(t,x)\,S^{(\alpha)} + K_t,$$

where $a(t,x)$ is a bounded, continuous positive function,

$$S^{(\alpha)} f(x) = \int \left(f(x+y)\cdot f(x) - \frac{(y, f(x))}{1+|y|^2} \right) \frac{dy}{|y|^{d+\alpha}} \qquad (1 \leqq \alpha < 2)$$

and

$$K_t f(x) = \int \left(f(x+y) - f(x) - \frac{(y, f(x))}{1+|y|^2} \right) M(t,x;dy)$$

where $M(t,x;\cdot)$ is a Lévy jump satisfying

$$\lim_{\delta\searrow 0}\, \sup_{t,x} \int_{|y|\leqq\delta} |y|^\alpha M(t,x;dy) = 0.$$

It has recently come to the author's attention that an article by Komatsu [10] on this subject has appeared. Komatsu's approach is quite different from the one taken here in that it carries out the perturbation and piecing arguments in a more analytic way. His results and the ones here do not imply one another, but they do have a large region of intersection.

Finally, it is a pleasure to acknowledge the contributions of S.R.S. Varadhan, E. Fabes, and N. Riviére to this paper. The basic ideas of most of what follows were hashed out, over a period of years, in conversations with Varadhan; and the L^p-estimates which appear in the appendix were obtained with the help of Fabes and Riviére.

1. Stochastic Calculus

Let $\Omega = D([0,\infty), R^d)$ be the space of right continuous functions ω on $[0,\infty)$ into R^d having left limits. Given $\omega\in\Omega$, let $x(t,\omega)$ denote the position of ω at time t. For $0\leqq s\leqq t$, set $\mathscr{M}_t^s = \mathscr{B}[x(u)\colon s\leqq u\leqq t]$, and take $\mathscr{M}^s = \sigma\left(\bigcup_{t\geqq s} \mathscr{M}_t^s\right)$.

Throughout this section we will be dealing with the following quantities. The function $a: [s, \infty)\times\Omega\to S_d$ is bounded and s-non-anticipating (i.e., a is $\mathscr{B}_{[s,\infty)}\times\mathscr{M}^s$-measurable and $a(t)$ is $\mathscr{M}_t^s$-measurable for each $t\geqq s$). (Here, and later on, S_d is used to stand for the class of symmetric, real, non-negative definite $d\times d$-matrices.) The function $b: [s,\infty)\times\Omega\to R^d$ is bounded and s-non-anticipating. From a and b, we form the operator

$$L_t=\tfrac{1}{2}\sum_1^d a^{ij}(t)\frac{\partial^2}{\partial x_i\,\partial x_j}+\sum_1^d b^i(t)\frac{\partial}{\partial x_i}.$$

Finally, $M: [s,\infty)\times\Omega\times\mathscr{B}_{R^d\setminus\{0\}}\to[0,\infty)$ is function with the following properties:

(i) for each $t\geqq s$ and $\omega\in\Omega$, $M(t,\omega;.)$ is a σ-finite measure on $R^d\setminus\{0\}$ satisfying the condition

$$\sup_{\substack{t\geqq s\\ \omega\in\Omega}}\int\frac{|y|^2}{1+|y|^2}M(t,\omega;dy)<\infty,$$

(ii) for each $\Gamma\in\mathscr{B}_{R^d\setminus\{0\}}$,

$$\int_\Gamma\frac{|y|^2}{1+|y|^2}M(t,\omega;dy)$$

is an s-non-anticipating function.

With M we associate the operator

$$K_t f(x)=\int\left(f(x+y)-f(x)-\frac{\langle y,\nabla f(x)\rangle}{1+|y|^2}\right)M(t;dy)$$

on $C_b^2(R^d)$, the space of bounded functions on R^d having two bounded, continuous derivatives.

Let $\alpha: [s,\infty)\times\Omega\to R^d$ be an s-non-anticipating function which is right continuous and has left limits. The following Theorem is proved in exactly the same way as Theorem (2.1) in [8].

Theorem (1.1). *Let $\mathscr{L}_t=L_t+K_t$. Suppose P is a probability measure on $\langle\Omega,\mathscr{M}^s\rangle$. Then the following are equivalent:*

(1) *$f(\alpha(t))-\int_s^t\mathscr{L}_u f(\alpha(u))\,du$ is a P-martingale for all $f\in C_0^\infty(R^d)$* (i.e., *$\langle f(\alpha(t))-\int_s^t\mathscr{L}_u f(\alpha(u))\,du,\mathscr{M}_t^s,P\rangle$ is a martingale*) (*$C_0^\infty(R^d)$ is the used of C^∞ functions having compact support*),

(2) *$f(t,\alpha(t))-\int_s^t\left(\frac{\partial}{\partial u}+\mathscr{L}_u\right)f(u,\alpha(u))\,du$ is a P-martingale for all $f\in C_b^{1,2}([s,\infty)\times R^d)$ (the space of bounded functions on $[s,\infty)\times R^d$ having one bounded continuous derivate in t and two bounded continuous derivatives in x),*

(3) *for all uniformly positive $f\in C_b^{1,2}([s,\infty)\times R^d)$,*

$$f(t,\alpha(t))\exp\left[-\int_s^t\frac{\left(\frac{\partial}{\partial_u}+\mathscr{L}_u\right)f(u,\alpha(u))}{f(u,\alpha(u))}\,du\right]$$

is a P-martingale,

(4) *for all* $\theta \in R^d$,

$$\exp\left[i\left\langle\theta, \alpha(t)-\alpha(s)-\int_s^t b(u)\,du\right\rangle+\tfrac{1}{2}\int_s^t \langle\theta, a(u)\theta\rangle\,du\right.$$
$$\left.-\int_s^t du\int\left(e^{i\langle\theta,y\rangle}-1-\frac{i\langle\theta,y\rangle}{1+|y|^2}\right)M(u,dy)\right]$$

is a P-martingale.

Let P be a probability measure on $\langle\Omega, \mathcal{M}^s\rangle$ and let τ be a finite s-stopping time (i.e., $\tau\colon \Omega\to[s,\infty)$ satisfies $\{\tau\leqq t\}\in\mathcal{M}_t^s$ for all $t\geqq s$). Define

$$\mathcal{M}_\tau^s=\{A\in\mathcal{M}^s\colon A\cap\{\tau\leqq t\}\in\mathcal{M}_t^s \text{ for all } t\geqq s\}.$$

It is not hard to see that $\mathcal{M}_\tau^s=\mathcal{B}[x(t\wedge\tau)\colon t\geqq s]$. In particular $\mathcal{M}_\tau^s$ is countably generated. Thus there is a mapping $\omega\to P_\omega$ such that

(i) for each ω, P_ω is a probability measure on $\langle\Omega,\mathcal{M}^s\rangle$,

$$P_\omega(x(t\wedge\tau)=x(t\wedge\tau(\omega),\omega), t\geqq s)=1,$$

(ii) for $A\in\mathcal{M}^s$, $\omega\to P_\omega(A)$ is $\mathcal{M}_\tau^s$-measurable and $P(A)=P(A\,|\,\mathcal{M}_\tau^s)$ (a.s., P).

Such a map is called a regular conditional probability distribution of P given $\mathcal{M}_\tau^s$ (abbr. r.c.p.d. of $P\,|\,\mathcal{M}_\tau^s$). Its existence was discussed in [7] in a slightly different situation. For a more up to date account of these matters, see [4]. The next Theorem is proved in the same way as Theorem (3.1) of [7].

Theorem (1.2). *Let P be a probability measure on $\langle\Omega,\mathcal{M}^s\rangle$ such that $f(\alpha(t))-\int_s^t\mathcal{L}_u f(\alpha(u))\,du$ is a P-martingale for all $f\in C_0^\infty(R^d)$. Given a finite s-stopping time τ, let P_ω be a r.c.p.d. of $P\,|\,\mathcal{M}_\tau^s$. Then there is an $N\in\mathcal{M}_\tau^s$ such that $P(N)=0$ and when $\omega\notin N$*

$$f(\alpha(t\vee\tau(\omega)))-\int_{\tau(\omega)}^{t\vee\tau(\omega)}\mathcal{L}_u f(\alpha(u))\,du$$

is a P_ω-martingale for all $f\in C_0^\infty(R^d)$.

For $t\geqq s$ and $\Gamma\in\mathcal{B}_{R^d\setminus B(0,\delta)}$, where $\delta>0$ and $B(0,\delta)\equiv\{x\colon |x|<\delta\}$, define $\eta(t,\Gamma)=\sum_{u\leqq t}\mathcal{X}_\Gamma(\alpha(u)-\alpha(u-))$, the number of jumps of $\alpha(u)$, $s\leqq u\leqq t$, such that $\alpha(u)-\alpha(u-)\in\Gamma$. It is easy to check that $\eta(t,\Gamma)$ is a finite s-non-anticipating function.

Theorem (1.3). *Let P be a probability measure on $\langle\Omega,\mathcal{M}^s\rangle$ such that $f(\alpha(t))-\int_s^t\mathcal{L}_u f(\alpha(u))\,du$ is a P-martingale for all $f\in C_0^\infty(R^d)$. Let g be a bounded measurable function on R^d which vanishes in a neighborhood of the origin. Then for all $\theta\in R^d$:*

$$\exp\left[i\left\langle\theta,\alpha(t)-\alpha(s)-\int_s^t b(u)\,du\right\rangle+\int g(y)\,\eta(t,dy)+\tfrac{1}{2}\int_s^t\langle\theta,a(u)\,\theta\rangle\,du\right.$$
$$\left.-\int_s^t du\int e^{i\langle\theta,y\rangle+g(y)}-1-\frac{i\langle\theta,y\rangle}{1+|y|^2}M(u;dy)\right]$$

is a P-martingale.

Proof. Clearly it is enough to prove the assertion when $s=0$ and $g\in C_0^\infty(R^d\setminus\{0\})$.

Let $t_0\geqq 0$ be given and suppose P_ω is a r.c.p.d. of $P|\mathscr{M}_{t_0}^0$. Then P-almost surely,

$$E^{P_\omega}\left[f\left(\alpha(t)-\alpha\left(\frac{k}{n}\right)\right)\exp\left[-\int_{k/n}^{t}\frac{\mathscr{L}_u f\left(\alpha(u)-\alpha\left(\frac{k}{n}\right)\right)}{f\left(\alpha(u)-\alpha\left(\frac{k}{n}\right)\right)}du\right]\right]$$

$$=f\left(\alpha(t_0)-\alpha\left(\frac{k}{n}\right)\right)\exp\left[-\int_{k/n}^{t_0}\frac{\mathscr{L}_u\left(\alpha(u)-\alpha\left(\frac{k}{n}\right)\right)}{f\left(\alpha(u)-\alpha\left(\frac{k}{n}\right)\right)}du\right]$$

for $\frac{k}{n}\leqq t_0\leqq t$ and $f\in C_b^2(R^d)$ which are uniformly positive. Hence if $f\in C_b^2(R^d)$ is uniformly positive, then for each $n\geqq 1$:

$$\left(\prod_0^{[nt]} f\left(\alpha\left(\frac{k+1}{n}\wedge t\right)-\alpha\left(\frac{k}{n}\wedge t\right)\right)\right)\exp\left[-\sum_0^{[nt]}\int_{\frac{k}{n}\wedge t}^{\frac{k+1}{n}\wedge t}\frac{\mathscr{L}_u f\left(\alpha(u)-\alpha\left(\frac{k}{n}\right)\right)}{f\left(\alpha(u)-\alpha\left(\frac{k}{n}\right)\right)}du\right]$$

is a P-martingale.

Let $g\in C_0^\infty(R^d\setminus\{0\})$ and $\theta\in R^d$ be given. Clearly the preceding applies to $f(x)=e^{i\langle\theta,x\rangle+g(x)}$. Hence

$$\begin{aligned}X_n(t)=\exp\Bigg[&i\left\langle\theta,\alpha(t)-\alpha(0)-\int_0^t b(u)\,du\right\rangle\\&+\sum_0^\infty g\left(\alpha\left(\frac{k+1}{n}\wedge t\right)-\alpha\left(\frac{k}{n}\wedge t\right)\right)+\frac{1}{2}\int_0^t\langle\theta,a(u),\theta\rangle\,du\\&-\int_0^t du\int\left(e^{i\langle\theta,y\rangle+g(\Delta_n(u)+y)-g(\Delta_n(u))}-1-\frac{\langle y,i\theta+\nabla g(\Delta_n(u))\rangle}{1+|y|^2}\right)M(u,dy)\\&-2i\int_0^t\langle\theta,a(u)\nabla g(\Delta_n(u))du\rangle-\int_0^t L_u g(\Delta_n(u))\,du\Bigg]\end{aligned}$$

is a P-martingale, where $\Delta_n(u)=\alpha(u)-\alpha\left(\frac{[nu]}{n}\right)$. Note that

$$\left|\int\left(e^{i\langle\theta,y\rangle+g(\Delta_n(u)+y)-g(\Delta_n(u))}-1-\frac{\langle y,i\theta+\nabla g(\Delta_n(u))\rangle}{1+|y|^2}\right)M(u,dy)\right|$$

is bounded independent of u, ω and n. Hence $E^P[|X_n(t)|^2]\leqq Ae^{Bt}$, and so for each t, $\{X_n(t)\}_1^\infty$ is uniformly P-integrable. Since $\Delta_n(u)\to 0$ for all but a countable number of u and $\sum_0^\infty g\left(\alpha\left(\frac{k+1}{n}\right)\wedge t\right)-\alpha\left(\frac{k}{n}\wedge t\right)\to\int g(y)\eta(t,dy)$, we see that $X_n(t)$ tends in $L^1(P)$ to the asserted martingale.

Corollary (1.3.1). *Define* $\tilde{\eta}(t,\Gamma)=\eta(t,\Gamma)-\int_0^t M(u;\Gamma)du$. *Then for measurable* g *on* $R^d\setminus\{0\}$ *satisfying the condition* $|g(y)|^2\leqq C\frac{|y|^2}{1+|y|^2}$, $\int_{|y|\geqq\varepsilon} g(y)\tilde{\eta}(t,dy)$ *P-almost surely converges uniformly in t as* $\varepsilon\downarrow 0$ *to a P-martingale* $\int g(y)\tilde{\eta}(t,dy)$ *which is right continuous and has left limits. In fact,*

$$\exp\left[i\left\langle\theta,\alpha(t)-\alpha(s)-\int_s^t b(u)\,du\right\rangle+\int g(y)\tilde{\eta}(t,dy)+\frac{1}{2}\int_s^t\langle\theta,a(u))\theta\rangle\,du\right.$$
$$\left.-\int_s^t du\int\left(e^{i\langle\theta,y\rangle+g(y)}-1-\frac{i\langle\theta,y\rangle}{1+|y|^2}-g(y)\right)M(u;dy)\right]$$

is a P-martingale for all $\theta\in R^d$.

Proof. Given $\varepsilon>0$, set $g_\varepsilon(y)=\mathscr{X}_{[\varepsilon,\infty)}(|y|)g(y)$. Then

$$X_\varepsilon(t)\equiv\exp\left[i\left\langle\theta,\alpha(t)-\alpha(s)-\int_s^t b(u)\,du\right\rangle+\int g_\varepsilon(y)\tilde{\eta}(t,dy)+\frac{1}{2}\int_s^t\langle\theta,a(u)\theta\rangle\,du\right.$$
$$\left.-\int_s^t du\int\left(e^{i\langle\theta,y\rangle+g_\varepsilon(y)}-1-\frac{i\langle\theta,y\rangle}{1+|y|^2}-g_\varepsilon(y)\right)M(u;dy)\right]$$

is a P-martingale. Moreover, it is easy to check that $E^P[|X_\varepsilon(t)|^2]\leqq Ae^{Bt}$, where A and B are independent of $\varepsilon>0$. Hence it is enough to prove the asserted convergence of $\int g_\varepsilon(y)\tilde{\eta}(t,dy)$. To this end, take $\theta=0$ and replace g by λg in the definition of $X_\varepsilon(t)$. Differentiating once and then twice with respect to λ, one sees, after setting $\lambda=0$, that

$$\int g_\varepsilon(y)\tilde{\eta}(t,dy)$$

and

$$\left(\int g_\varepsilon(y)\tilde{\eta}(t,dy)\right)^2-\int_s^t du\int g_\varepsilon^2(y)M(u;dy)$$

are P-martingales. Hence, if $0<\varepsilon_1<\varepsilon_2$,

$$E^P\left[\left(\int g_{\varepsilon_2}(y)\tilde{\eta}(t,dy)-\int g_{\varepsilon_1}(y)\tilde{\eta}(t,dy)\right)\right]=E^P\left[\int_s^t du\int_{\varepsilon_1<|y|<\varepsilon_2} g^2(y)M(u;dy)\right]$$
$$\to 0$$

as $\varepsilon_2,\varepsilon_1\downarrow 0$. Applying Doob's martingale inequality, we conclude that $\int g_\varepsilon(y)\tilde{\eta}(t,dy)$ P-almost surely converges uniformly with respect to t in compacts.

Corollary (1.3.2). *Given* $\delta>0$, *define*

$$\gamma_\delta(t)=\alpha(t)-\int_{|y|<\delta} y\tilde{\eta}(t,dy)-\int_{|y|\geqq\delta} y\eta(t,dy)$$

and

$$c_\delta(t)=b(t)+\int_{|y|<\delta}\frac{y|y|^2}{1+|y|^2}M(t;dy)-\int_{|y|\geqq\delta}\frac{y}{1+|y|^2}M(t,dy).$$

Then $\gamma(t)=\gamma_\delta(t)-\int_s^t c_\delta(u)du$ is independent of $\delta>0$. Moreover, for all θ,θ', and $\theta''\in R^d$,

$$X_\theta(t)\,Y_{\theta'}(t)\,Z_{\theta''}(t)$$

is a P-martingale, where:

$$X_\theta(t)=\exp\left[i\langle\theta,\gamma(t)-\gamma(s)\rangle+\frac{1}{2}\int_s^t\langle\theta,a(u)\theta\rangle du\right],$$

$$Y_{\theta'}(t)=\exp\left[i\langle\theta',\int_{|y|<\delta} y\tilde{\eta}(t,dy)\rangle-\int_s^t du\int_{|y|<\delta}(e^{i\langle\theta',y\rangle}-1-i\langle\theta',y\rangle)M(u;dy)\right]$$

$$Z_{\theta''}(t)=\exp\left[i\langle\theta'',\int_{|y|\geqq\delta} y\tilde{\eta}(t,dy)\rangle-\int_s^t du\int_{|y|\geqq\delta}(e^{i\langle\theta'',y\rangle}-1)M(u;dy)\right].$$

Proof. For $R>\delta$, take $g_R(y)=i\langle\theta'-\theta,y\rangle\mathscr{X}_{[0,\delta)}(|y|)+i\langle\theta''-\theta,y\rangle\mathscr{X}_{[\delta,R)}(|y|)$ in Corollary (1.3.1). One then sees that:

$$H^{(R)}(t)=\exp\left[i\left\langle\theta,\alpha(t)-\alpha(s)-\int_{|y|<\delta} y\tilde{\eta}(t,dy)-\int_{\delta\leqq|y|\leqq R} y\eta(t,dy)-\int_s^t c_\delta(u)du\right\rangle\right.$$
$$\left.+\frac{1}{2}\int_s^t\langle\theta,a(u)\theta\rangle du\right]\times Y_{\theta'}(t)\times\exp\left[i\langle\theta'',\int_{\delta\leqq|y|\leqq R} y\eta(t,dy)\rangle\right.$$
$$\left.-\int_s^t du\int_{|y|\geqq\delta}(e^{i\langle\theta''\mathscr{X}_{[\delta,R]}(|y|)+\theta\mathscr{X}_{(R,\infty)}(|y|),y\rangle}-1)M(u,dy)\right]$$

is a P-martingale. Since $E^P[|(H^{(R)}(t))|^2]\leqq Ae^{Bt}$ independent of $R>0$, it follows that $X_\theta(t)Y_{\theta'}(t)Z_{\theta''}(t)$ is also a P-martingale.

Corollary (1.3.3). *Given $\delta>0$, let*

$$\alpha_\delta(t)=\alpha(t)-\int_{|y|\geqq\delta} y\eta(t,dy),$$

$$b_\delta(t)=b(t)-\int_{|y|\geqq\delta}\frac{y}{1+|y|^2}M(t,dy).$$

Then for any bounded measurable function h: $R^d\to R$, we have:

$$\exp\left[i\left\langle\theta,\alpha_\delta(t)-\alpha_\delta(s)-\int_s^t b_\delta(u)du\right\rangle\right.$$
$$\left.+\tfrac{1}{2}\int_s^t\langle\theta,a(u)\theta\rangle-\int_s^t du\int_{|y|<\delta}\left(e^{i\langle\theta,y\rangle}-1-\frac{i\langle\theta,y\rangle}{1+|y|^2}\right)M(u,dy)\right]$$
$$\times\exp\left[\int_{|y|\geqq\delta} h(y)\eta(t,dy)-\int_s^t du\int(e^{h(y)}-1)M(u;dy)\right]$$

is a P-martingale.

Corollary (1.3.4). *Define $\gamma(t)$ as in Corollary* (1.3.2). *Then $\gamma(t)$ admits a stochastic calculus like the one developed in* [7]. *In particular, if θ: $[s,\infty)\times\Omega\to C^d$ is bounded and s-non-anticipating, then $\int_s^t\langle\theta(u),d\gamma(u)\rangle$ is a continuous P-martingale, and, in fact,*

$$\exp\left[\int_s^t\langle\theta(u),d\gamma(u)\rangle-\tfrac{1}{2}\int_s^t\langle\theta(u),a(u)\theta(u)\rangle du\right]\times Y_{\theta'}(t)\times Z_{\theta''}(t)$$

is a P-martingale for all $\theta', \theta'' \in R^d$. *Thus, if* $c\colon [s,\infty)\times\Omega \to R^d$ *is s-non-anticipating and* P' *is defined on* $\langle\Omega, \mathcal{M}^s\rangle$ *so that for all* $t>s$

$$\frac{dP'|_{\mathcal{M}_t^s}}{dP|_{\mathcal{M}_t^s}} = \exp\left[\int_s^t \langle c(u), d\gamma(u)\rangle - \tfrac{1}{2}\int_s^t \langle c(u), a(u)\, c(u)\rangle\, du\right],$$

then

$$f(\alpha(t)) - \int_s^t (\mathcal{L}_u + (a(u)\, c(u))\cdot\nabla)\, f(\alpha(u))\, du$$

is a P'*-martingale.*

Proof. The only assertion that isn't immediate from [7] is the last. However, it is easily proved by putting $\theta(u) = i\theta + c(u)$, taking $\theta' = \theta'' = \theta$, and applying Theorem (1.1).

2. Existence

The purpose of this section is to show how to construct a solution to the martingale problem for a given Lévy generator. Our basic result in this direction is the following perturbation theorem.

Theorem (2.1). *Let* $\mathcal{L}_t = L_t + K_t$ *be a Lévy generator. Assume that there exists a family of probability measures* $Q_{s,x}$, $(s,x)\in[0,\infty)\times R^d$, *on* $\langle\Omega,\mathcal{M}^s\rangle$ *such that* $Q_{s,x}$ *is a solution to the martingale problem for* $\mathcal{L}_t$ *starting at* (s,x) *and* $(s,x)\to Q_{s,x}(A)$ *is* $\mathcal{B}_{[0,t]}\times\mathcal{B}_{R^d}$*-measurable for* $A\in\mathcal{M}^t$. *Let* $M'\colon [0,\infty)\times R^d\times\mathcal{B}_{R^d\setminus\{0\}}\to[0,\infty)$ *be a uniformly bounded jump measure and define*

$$K'_t f(x) = \int (f(x+y) - f(x))\, M'(t, x; dy).$$

Then there is a family of probability measures $P_{s,x}$, $(s,x)\in[0,\infty)\times R^d$, *on* $\langle\Omega,\mathcal{M}^s\rangle$ *such* $P_{s,x}$ *solves the martingale problem for* $\mathcal{L}_t + K_t$ *starting at* (s,x) *and* $(s,x)\to P_{s,x}(A)$ *is* $\mathcal{B}_{[0,t]}\times\mathcal{B}_{R^d}$*-measurable for* $A\in\mathcal{M}^t$. *Moreover, if M is the jump measure associated with* K_t *and* $\Gamma_0\in\mathcal{B}_{R^d\setminus\{0\}}$ *is a set such that* $M'(t,x;R^d\setminus\Gamma_0) = M(t,x;\Gamma_0)\equiv 0$, *then* $P_{s,x}$ *can be chosen so that:*

$$P_{s,x}(x(\tau)-x(\tau-)\in\Gamma \mid \tau \text{ and } x(t) \text{ for } t<\tau) = \frac{M'(\tau, x(\tau-); \Gamma\cap\Gamma_0)}{M'(\tau, x(\tau-); \Gamma_0)} \quad (\text{a.s., } P_{s,x})$$

on the set where $\tau = \inf\{t\geq s\colon x(t)-x(t-)\in\Gamma_0\}$ *is finite.*

Proof. Let $\tilde\Omega = \Omega\times[0,\infty)^N$, where $N=\{0,1,\ldots,n,\ldots\}$. Define $x(t,\tilde\omega) = x(t,\omega)$ and $\tau_n(\tilde\omega) = \alpha_n$, $n\geq 0$, where $\tilde\omega = (\omega;\alpha_0,\ldots,\alpha_n,\ldots)$. Put $\tilde{\mathcal{M}}^s_{t,n} = \mathcal{B}[x(u), \tau_k\colon s\leq u\leq t$ and $0\leq k\leq n\}$ and $\tilde{\mathcal{M}}^s_{\cdot,n} = \sigma(\bigcup_{t\geq s}\tilde{\mathcal{M}}^s_{t,n})$. Given $n\geq 0$ and $\tilde\omega\in\tilde\Omega$, let $\mu_{n,\tilde\omega}$ be defined on $\mathcal{B}[\tau_n]$ so that

$$\mu_{n+1,\tilde\omega}([t,\infty)) = \exp\left[-\int_{\tau_n(\tilde\omega)}^t M'(u, x(u,\tilde\omega))\, du\right]$$

where $M'(t,x) = M'(t,x;R^d\setminus\{0\})$. Define for $n\geq 1$

$$\tilde Q^{(n)}_{\tilde\omega} = \left[\int \delta_\omega \mathop{\otimes}_{\tau_n}^{s} Q_{\tau_n(\tilde\omega),\, x(\tau_n(\tilde\omega)-,\,\omega)+y}\, \bar M'(\tau_n(\tilde\omega), x(\tau_n(\tilde\omega)-,\tilde\omega); dy)\right] \times \delta_{\{\tau_0(\tilde\omega)\}}\times\cdots\times\delta_{\{\tau_n(\tilde\omega)\}}\times\mu_{n+1,\tilde\omega}$$

if $\tau_n(\tilde{\omega})<\infty$, and

$$\tilde{Q}^{(n)}_{\tilde{\omega}}=\delta_\omega\times\delta_{\tau_0(\tilde{\omega})}\times\cdots\times\delta_{\tau_n(\tilde{\omega})}\times\mu_{n+1,\tilde{\omega}}$$

otherwise. Here

$$\bar{M}'(t,x;\Gamma)\equiv\frac{M'(t,x;\Gamma)}{M'(t,x;R^d\setminus\{0\}}$$

and $\delta_\omega\overset{s}{\underset{t}{\otimes}}Q_{t,x(x-,\omega)}$ is the measure on $\langle\Omega,\mathcal{M}^s\rangle$ satisfying

$$\delta_\omega\overset{s}{\underset{t}{\otimes}}Q_{t,x(t-,\omega)}(A\cap B)=\mathcal{X}_A(\omega)\,Q_{t,x(t-,\omega)}(B)$$

for $A\in\sigma(\bigcup_{s\leqq u<t}\mathcal{M}^s_u)$ and $B\in\mathcal{M}^t$. Given $n\geqq 0$, define $P^{(n)}_{s,x}$ on $\langle\tilde{\Omega},\tilde{\mathcal{M}}^s_{,n}\rangle$ so that $P^{(0)}_{s,x}=Q_{s,x}\times\delta_{\{s\}}$ and

$$P^{(n+1)}_{s,x}(\tilde{A})=E^{P^{(n)}_{s,x}}[Q^{(n+1)}_{\tilde{\omega}}(\tilde{A})].$$

It is clear that $(s,x)\to P^{(n)}_{s,x}(\tilde{A})$ is $\mathcal{B}_{[0,t]}\times\mathcal{B}_{R^d}$-measurable for $\tilde{A}\in\tilde{\mathcal{M}}^t_{,n}$ and that $P^{(n)}_{s,x}(x(s)=x)=1$. We will show that if $s\leqq t_1<t_2$, $A\in\mathcal{M}^s_{t_1}$, and $f\in C_0^\infty(R^d)$, then

$$\begin{aligned}&E^{P^{(n)}_{s,x}}\left[\left(f(x(t_2))-\int_s^{t_2}(\mathcal{L}_u+\mathcal{X}_{\tau_n>u}K'_u)\,f(x(u))\,du\right)\mathcal{X}_A\right]\\&=E^{P^{(n)}_{s,x}}\left(f(x(t_1))-\int_s^{t_1}(\mathcal{L}_u+\mathcal{X}_{\tau_n>u}K'_u)\,f(x(u))\,du\right)\mathcal{X}_A\Bigg].\end{aligned}\tag{2.1}$$

The first step in the proof of (2.1) is to show by induction that for $n\geqq 1$:

$$\begin{aligned}E^{P^{(n)}_{s,x}}[f(x(t_2))\mathcal{X}_{A\cap\{\tau_n>t_2\}}]&=E^{P^{(n)}_{s,x}}[f(x(t_1))\mathcal{X}_{A\cap\{\tau_n>t_1\}}]\\&+E^{P^{(n)}_{s,x}}\left[\mathcal{X}_A\int_{t_1}^{t_2}(\mathcal{X}_{\tau_n>u}\mathcal{L}_u+\mathcal{X}_{\tau_{n-1}>u}K'_u)\,f(x(u))\,du\right]\\&-E^{P^{(n)}_{s,x}}\left[\mathcal{X}_A\int_{t_1}^{t_2}\mathcal{X}_{\tau_{n-1}\leqq u<\tau_n}M'(u,x(u))\,f(x(u))\,du\right].\end{aligned}$$

First note that

$$\begin{aligned}E^{P^{(1)}_{s,x}}[\mathcal{X}_{A\cap\{\tau_1>t_2\}}f(x(t_2))]&=E^{Q_{s,x}}\left[\mathcal{X}_A\,f(x(t_2))\,e^{-\int_s^{t_2}M'(u,x(u))\,du}\right]\\&=E^{Q_{s,x}}\left[\mathcal{X}_A f(x(t_1))\,e^{-\int_s^{t_1}M'(u,x(u))\,du}\right]\\&\quad+E^{Q_{s,x}}\left[\mathcal{X}_A\int_{t_1}^{t_2}(\mathcal{L}_u-M'(u,x(u)))\,f(x(u))\,e^{-\int_s^u M(\sigma,x(\sigma))\,d\sigma}du\right]\\&=E^{P^{(1)}_{s,x}}[\mathcal{X}_A f(x(t_1))\mathcal{X}_{\tau_1>t_1}]\\&\quad+E^{P^{(1)}_{s,x}}\left[\mathcal{X}_A\int_{t_1}^{t_2}\mathcal{X}_{\tau_1>u}(\mathcal{L}_u-M'(u,x(u)))\,f(x(u))\,du\right]\\&=E^{P^{(1)}_{s,x}}[\mathcal{X}_A f(x(t_1))\mathcal{X}_{\tau_1>t_1}]\\&\quad+E^{P^{(1)}_{s,x}}\left[\mathcal{X}_A\int_{t_1}^{t_2}(\mathcal{X}_{\tau_1>u}\mathcal{L}_u+\mathcal{X}_{\tau_0>u}K'_u)\,f(x(u))\,du\right]\\&\quad-E^{P^{(1)}_{s,x}}\left[\mathcal{X}_A\int_{t_1}^{t_2}\mathcal{X}_{\tau_0\leqq u<\tau_1}M'(u,x(u))\,f(x(u))\,du\right].\end{aligned}$$

We have used here the martingale version of the Feynman-Kac formula (cf. Lemma (2.1) in [8]). Next assume (2.2) holds. Then

$$\begin{aligned} &E^{P^{(n+1)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_{n+1}>t_2\}} f(x(t_2))\right] \\ &\quad = E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n>t_2\}} f(x(t_2))\right] + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n\leqq t_2<\tau_{n+1}\}} f(x(t_2))\right] \end{aligned}$$

and

$$\begin{aligned} &E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n\leqq t_2<\tau_{n+1}\}} f(x(t_2))\right] \\ &\quad = E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n\leqq t_2\}} f(x(t_2)) e^{-\int_{\tau_n}^{t_2} M'(u,x(u))\,du}\right] \\ &\quad = E^{P^{(n-1)}_{s,x}}\left[\mathscr{X}_{\tau_n(\bar\omega)\leqq t_1} E^{Q^{(n)}_{\bar\omega}}\left[\mathscr{X}_A f(x(t_2)) e^{-\int_{\tau_n(\bar\omega)}^{t_2} M'(u,x(u))\,du}\right]\right] \\ &\qquad + E^{P^{(n-1)}_{s,x}}\left[\mathscr{X}_{A\cap\{t_1<\tau_n(\bar\omega)\leqq t_2\}} E^{Q^{(n)}_{\bar\omega}}\left[f(x(t_2)) e^{-\int_{\tau_n(\bar\omega)}^{t_2} M'(u,x(u))\,du}\right]\right] \\ &\quad = E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n\leqq t_1<\tau_{n+1}\}} f(x(t_1))\right] \\ &\qquad + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n\leqq t_1\}} \int_{t_1}^{t_2} \mathscr{X}_{\tau_{n+1}>u}(\mathscr{L}_u - M'(u,x(u))) f(x(u))\,du\right] \\ &\qquad + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{t_1<\tau_n\leqq t_2\}} \int f(x(\tau_n-)+y) \bar M'(\tau_n, x(\tau_n-); dy)\right] \\ &\qquad + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{t_1<\tau_n\leqq t_2\}} \int_{\tau_n}^{t_2} \mathscr{X}_{\tau_{n+1}>u}(\mathscr{L}_u - M'(u,x(u))) f(x(u))\,du\right] \\ &\quad = E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n\leqq t_1<\tau_{n+1}\}} f(x(t_1))\right] \\ &\qquad + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n\leqq t_1\}} \int_{t_1}^{t_2} \mathscr{X}_{\tau_{n+1}>u}(\mathscr{L}_u - M'(u,x(u))) f(x(u))\,du\right] \\ &\qquad + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_A \int_{t_1}^{t_2} \mathscr{X}_{\tau_{n-1}\leqq u<\tau_n}\,du \int f(x(u)+y) M'(u,x(u);dy)\right] \\ &\qquad + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n>t_1\}} \int_{t_1}^{t_2} \mathscr{X}_{\tau_n\leqq u<\tau_{n+1}}(\mathscr{L}_u - M'(u,x(u))) f(x(u))\,du\right] \\ &\quad = E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n\leqq t_1\leqq\tau_{n+1}\}} f(x(t_1))\right] \\ &\qquad + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_A \int_{t_1}^{t_2} \mathscr{X}_{\tau_n\leqq u<\tau_{n+1}}(\mathscr{L}_u - M'(u,x(u))) f(x(u))\,du\right] \\ &\qquad + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_A \int_{t_1}^{t_2} \mathscr{X}_{\tau_{n-1}\leqq u<\tau_n}\,du \int f(x(u)+y) M'(u,x(u);dy)\right]. \end{aligned}$$

Using the induction hypothesis, we now conclude that (2.2) holds for $n+1$.

One next has to check that

$$\begin{aligned} &E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{t_1<\tau_n\leqq t_n\}} f(x(t_2))\right] \\ &\quad = E^{P^{(n)}_{s,x}}\left[\mathscr{X}_A \int_{t_1}^{t_2} \mathscr{X}_{\tau_{n-1}\leqq u<\tau_n}\,du \int f(x(u)+y)\, M'(u,x(u);dy)\right] \\ &\qquad + E^{P^{(n)}_{s,x}}\left[\mathscr{X}_{A\cap\{\tau_n>t_1\}} \int_{t_1}^{t_2} \mathscr{X}_{\tau_n\leqq u}\, \mathscr{L}_u f(x(u))\,du\right] \end{aligned} \tag{2.3}$$

and

$$E^{P_{s,x}^{(n)}}[\mathscr{X}_{A\cap\{\tau_n\leqq t_1\}} f(x(t_2))] = E^{P_{s,x}^{(n)}}[\mathscr{X}_{A\cap\{\tau_n\leqq t_1\}} f(x(t_1))] + E^{P_{s,x}^{(n)}}\left[\mathscr{X}_{A\cap\{\tau_n\leqq t_1\}} \int_{t_1}^{t_2} \mathscr{L}_u f(x(u))\,du\right]. \tag{2.4}$$

These relations are proved in much the same manner as was (2.2), only they are much easier and do not involve induction.

Combining (2.2), (2.3), and (2.4), we arrive at (2.1).

Finally, set $\tilde{\mathscr{B}}_n = \mathscr{B}[x(u)\colon s \leqq u < \tau_n]$. Since

$$P_{s,x}^{(n)}(\tau_n \leqq t+s) \leqq 1 - e^{-\lambda t} \sum_{0}^{n-1} \frac{(\lambda t)^k}{k!},$$

where $\lambda = \sup_{t,x} M'(t,x)$, it is easy to see (for instance by Tulcea's extension theorem) that there is a unique probability measure $P_{s,x}^{(\infty)}$ on $\langle \tilde{\Omega}, \tilde{\mathscr{M}}^s_0\rangle$ such that $P_{s,x}^{(\infty)} = P_{s,x}^{(n)}$ on $\tilde{\mathscr{B}}_n$, $n \geqq 1$. Let $P_{s,x}$ be the measure induced on $\langle \Omega, \mathscr{M}^s\rangle$ by $P_{s,x}^{(\infty)}$. Clearly $P_{s,x}$ has all the asserted properties.

Theorem (2.2). *Let $\mathscr{L}_t = L_t + K_t$ be a Lévy generator. Assume that:*

(i) *the diffusion coefficients a of L_t are bounded and continuous,*

(ii) *the drift coefficients b of L_t admit the decomposition: $b = ac_1 + c_2$, where c_1 is bounded and measurable and c_2 is bounded and continuous,*

(iii) *the jump measure M of K_t has the property that for all $\varphi \in C_b(R^d)$, $\int \frac{|y|^2}{1+|y|^2}\varphi(y)\,M(t,x;dy)$ is bounded and continuous*

Then for each $(s,x) \in [0,\infty) \times R^d$ there is a solution $P_{s,x}$ to the martingale problem for $\mathscr{L}_t$.

Proof. In view of Corollary (1.3.4), we may assume that $c_1 \equiv 0$. Let $\psi \in C^\infty$ be chosen so that $0 \leqq \psi \leqq 1$, $\psi(y) = 0$ for $|y| \leqq \frac{1}{2}$, and $\psi(y) = 1$ for $|y| \geqq 1$. Given $\delta > 0$, define M^δ so that $\frac{dM^\delta(t,x;\cdot)}{dM(t,x;\cdot)} = \psi_\delta$, where $\psi_\delta(y) = \psi(y/\delta)$. Set $c_\delta(t,x) = \int \frac{y}{1+|y|^2} M^\delta(t,x;dy)$. Then c_δ is bounded and continuous. Let $L_t^\delta = L_t - c_\delta \cdot \nabla$. By Theorem (4.1) in [7], there is for each (s,x) a solution $Q_{s,x}^\delta$ to the martingale problem for L_t^δ starting from (s,x). By standard selection theorems (see, for example, Kuratowski and Ryll-Nardzewski [5]), we can choose $Q_{s,x}^\delta$ so that $Q_{s,x}^\delta(A)$ is $\mathscr{B}_{[0,t]} \times \mathscr{B}_{R^d}$-measurable for $A \in \mathscr{M}^t$. Hence, by Theorem (2.1), we can construct a $P_{s,x}^\delta$ which solves the martingale problem for $L_t + K_t^\delta$, where

$$K_t^\delta f(x) = \int \left(f(x+y) - f(x) - \frac{\langle y, \nabla f(x)\rangle}{1+|y|^2}\right) M^\delta(t,x;dy).$$

By Theorem (A.1) in the appendix, $\{P_{s,x}^\delta\colon \delta > 0\}$ is relatively weakly compact. Moreover, $K_t^\delta f(x) \to K_t f(x)$ uniformly on compacts for $f \in C_0^\infty(R^d)$. Hence, every limit point of $P_{s,x}^\delta$, as $\delta \downarrow 0$, is a solution to the martingale problem for $\mathscr{L}_t$ starting from (s,x).

3. Uniqueness, General Considerations

The point of the present section is to give a procedure for localizing the problem of proving uniqueness of solutions to the martingale problem. Throughout, we will be dealing with the following situation.

$\mathscr{L}_t = L_t + K_t$ and $\mathscr{L}'_t = L'_t + K'_t$ are Lévy generators. There exist (s_0, x_0), $\varepsilon > 0$, $T > s_0$, and an open set $G \ni x_0$ such that

$$\begin{aligned}
&\text{(i) } a'(s,x) = a(s,x) \text{ in } [s_0, T] \times G,\\
&\text{(ii) } b'(s,x) = b(s,x) - \int_{|y| \geqq \varepsilon} \frac{y}{1+|y|^2} M(s,x;dy) \text{ in } [s_0, T] \times G,\\
&\text{(iii) } M'(s,x;\Gamma) = M(s,x;\Gamma \cap B(0,\varepsilon)),\ (s,x) \in [s_0, T] \times G \text{ and}\\
&\qquad\qquad \Gamma \in \mathscr{B}_{R^d \setminus \{0\}}.
\end{aligned} \tag{3.1}$$

Here, of course, a, b, and M and a', b', and M' stand for the diffusion, drift, and jump parts of $\mathscr{L}_t$ and $\mathscr{L}'_t$, respectively. We will also be assuming that there is a measurable family $\{P'_{s,x}: (s,x) \in [0,\infty) \times R^d\}$ of solutions to the martingale problem for $\mathscr{L}'_t$. Our aim is to show how uniqueness of solutions for the $\mathscr{L}'_t$-martingale problem implies uniqueness for the $\mathscr{L}_t$-problem.

In accordance with the notation used in Section (1), set

$$x_\varepsilon(t) = x(t) - \int_{|y| \geqq \varepsilon} y\,\eta(t, dy).$$

By Corollary (1.3.3), if P_{s_0, x_0} is a solution of the martingale problem for $\mathscr{L}_t$ starting from (s_0, x_0) and $f \in C_b^2(R^d)$, then

$$f(x_\varepsilon(t)) - \int_{s_0}^{t} \mathscr{L}_u^{(\varepsilon)} f(x_\varepsilon(u))\,du$$

is a P_{s_0, x_0}-martingale, where

$$\begin{aligned}
a^{(\varepsilon)}(u) &= a(u, x(u)),\\
b^{(\varepsilon)}(u) &= b(u, x(u)) - \int_{|y| \geqq \varepsilon} \frac{y}{1+|y|^2} M(u, x(u); dy),\\
M^{(\varepsilon)}(u; \Gamma) &= M(u, x(u); \Gamma \cap B(0, \varepsilon)),
\end{aligned} \tag{3.2}$$

are the diffusion, drift, and jump parts of $\mathscr{L}_u^{(\varepsilon)}$.

Theorem (3.1). *Let $\sigma = \inf\{t \geqq s_0: x(t) \notin G\} \wedge T$ and $\tau = \inf\{t \geqq s_0: |x(t) - x(t-)| \geqq \varepsilon\}$. Put $\zeta = \sigma \wedge \tau$. Given a probability measure P on $\langle \Omega, \mathscr{M}^{s_0} \rangle$ such that $P(x(s_0) = x_0) = 1$ and*

$$f(x_\varepsilon(t)) - \int_{s_0}^{t} \mathscr{L}_u^{(\varepsilon)} f(x_\varepsilon(u))\,du$$

is a P-martingale for all $f \in C_b^2(R^d)$, define

$$Q_\omega = \delta_\omega \underset{\zeta(\omega)}{\overset{s_0}{\otimes}} P'_{\zeta(\omega), x_\varepsilon(\zeta(\omega), \omega)}$$

and

$$\tilde{P}(A) = E^P[Q_\omega(A)], \qquad A \in \mathscr{M}^{s_0}.$$

Then $\tilde{P}$ solves the martingale problem for $\mathscr{L}_t'$ starting from (s_0, x_0). In particular, one can take $P = P_{s_0, x_0}$.

Proof. Let $s_0 \leqq t_1 < t_2$ and $A \in \mathscr{M}_{t_1}^{s_0}$. Then

$$E^{\tilde{P}}[\mathscr{X}_A f(x(t_2))] = E^P[\mathscr{X}_{A \cap \{\zeta > t_2\}} f(x_\varepsilon(t_2))] + E^P[\mathscr{X}_{A \cap \{t_1 < \zeta \leqq t_2\}} E^{P_{\zeta(\omega), x_\varepsilon(\zeta)}}[f(x(t_2))]] + E^P[\mathscr{X}_{\{\zeta \leqq t_1\}} E^{Q_\omega}[\mathscr{X}_A f(x(t_2))]] = I_1 + I_2 + I_3.$$

Note that

$$I_2 = E^P[\mathscr{X}_{A \cap \{t_1 < \zeta \leqq t_2\}} f(x_\varepsilon(\zeta))] + E^{\tilde{P}}\left[\mathscr{X}_{A \cap \{t_1 < \zeta \leqq t_2\}} \int_\zeta^{t_2} \mathscr{L}_u' f(x(u))\, du\right]$$

and so

$$I_1 + I_2 = E^P[\mathscr{X}_{A \cap \{\zeta > t_1\}} f(x_\varepsilon(\zeta \wedge t_2))] + E^{\tilde{P}}\left[\mathscr{X}_{A \cap \{t_1 < \zeta \leqq t_2\}} \int_\zeta^{t_2} \mathscr{L}_u' f(x(u))\, du\right]$$

$$= E^P[\mathscr{X}_{A \cap \{\zeta > t_1\}} f(x_\varepsilon(t_1))] + E^P\left[\mathscr{X}_{A \cap \{\zeta > t_1\}} \int_{t_1}^{\zeta \wedge t_2} \mathscr{L}_u^{(\varepsilon)} f(x_\varepsilon(u))\, du\right]$$

$$+ E^{\tilde{P}}\left[\mathscr{X}_{A \cap \{\zeta > t_1\}} \int_{\zeta \wedge t_2}^{t_2} \mathscr{L}_u' f(x(u))\, du\right]$$

$$= E^{\tilde{P}}[\mathscr{X}_{A \cap \{\zeta > t_1\}} f(x(t_1))] + E^{\tilde{P}}\left[\mathscr{X}_{A \cap \{\zeta > t_1\}} \int_{t_1}^{t_2} \mathscr{L}_u' f(x(u))\, du\right],$$

since $\mathscr{L}_u^{(\varepsilon)} f(x_\varepsilon(u)) = \mathscr{L}_u' f(x(u))$ for $u < \zeta$. Also,

$$I_3 = E^{\tilde{P}}[\mathscr{X}_{A \cap \{\zeta \leqq t_1\}} f(x(t_1))] + E^{\tilde{P}}\left[\mathscr{X}_{A \cap \{\zeta \leqq t_1\}} \int_{t_1}^{t_2} \mathscr{L}_u' f(x(u))\, du\right].$$

Thus,

$$E^{\tilde{P}}[\mathscr{X}_A f(x(t_2))] = E^{\tilde{P}}[\mathscr{X}_A f(x(t_1))] + E^{\tilde{P}}\left[\mathscr{X}_A \int_{t_1}^{t_2} \mathscr{L}_u' f(x(u))\, du\right].$$

Corollary (3.1.1). *Let P, $\tilde{P}$, and ζ be as in Theorem (3.1) and define $\mathscr{M}_{\zeta-}^{s_0} = \mathscr{B}[x_\varepsilon(t \wedge \zeta): t \geqq s_0]$. If there is only one solution P_{s_0, x_0}' to the martingale problem for $\mathscr{L}_t'$ starting from (s_0, x_0), then $\tilde{P} = P_{s_0, x_0}'$ on $\mathscr{M}_{\zeta-}^{s_0}$.*

We now see that uniqueness for $\mathscr{L}_t'$ implies uniqueness for P_{s_0, x_0} on $\mathscr{M}_{\zeta-}^{s_0}$. Our next step is to show that uniqueness of P_{s_0, x_0}' implies uniqueness of P_{s_0, x_0} on $\mathscr{M}_\zeta^{s_0}$. To do this it suffices to prove that the distribution of $E_{s_0, x_0}[\eta(\zeta; \Gamma) | \mathscr{M}_{\zeta-}^{s_0}]$ is uniquely determined for $\Gamma \in \mathscr{B}_{R^d \setminus B(0, \varepsilon)}$. What we are going to find is that there is an $\mathscr{M}_{\zeta-}^{s_0}$-measurable function σ' such that for $\Gamma \in \mathscr{B}_{R^d \setminus B(0, \varepsilon)}$:

$$E_{s_0, x_0}[\eta(\zeta; \Gamma) | \mathscr{M}_{\zeta-}^{s_0}] = \int_{s_0}^{\sigma'} e^{-\int_{s_0}^t M^{(\varepsilon)}(u, x_\varepsilon(u \wedge \zeta))\, du} M(t, x_\varepsilon(t \wedge \zeta); \Gamma)\, dt, \tag{3.3}$$

where $M^{(\varepsilon)}(s, x) = M(s, x; R^d \setminus B(0, \varepsilon))$. Eq. (3.3) will be proved under the assumption that P_{s_0, x_0}' is unique.

We will use the following lemma; its proof is elementary.

Lemma (3.1). *Let $\langle M, \mathscr{B}, P\rangle$ be a probability space and $\mathscr{B}_t$, $t \geqq 0$, a non-decreasing family of sub σ-algebras. Suppose X, Y: $[0, \infty) \times M \to C$ are locally* *bounded, right-continuous, non-anticipating functions such that $X(t)$, $Y(t)$, and $X(t)\,Y(t)$ are P-martingales. If τ is a bounded stopping time and $\frac{dQ}{dP}=Y(\tau)$, then $X(t)$ is again a Q-martingale.*

Theorem (3.2). *Let I be an index set and for each $\alpha\in I$ let X_α: $[s,\infty)\times\Omega\to C$ be a locally bounded, right continuous, s-non-anticipating function. Suppose $\mathcal{M}'$ is a sub σ-algebra of $\mathcal{M}^s$ and P' is a probability measure on $\langle\Omega,\mathcal{M}'\rangle$ such that for any probability measure P on $\langle\Omega,\mathcal{M}^s\rangle$ satisfying*

(i) *$P=P'$ on $\mathcal{M}_s^s\cap\mathcal{M}'$,*

(ii) *$X_\alpha(t)$ is a P-martingale for all $\alpha\in I$,*

$P=P'$ on $\mathcal{M}'$. Let Y: $[0,\infty)\times\Omega\to[0,\infty)$ be a locally bounded right continuous, s-non-anticipating function such that $Y(s)\equiv 1$. If P is a probability measure on $\langle\Omega,\mathcal{M}^s\rangle$ satisfying (i), (ii), *and:*

(iii) *$Y(t)$ and $Y(t)\,X_\alpha(t)$ are P-martingales for all $\alpha\in I$,*

then for any bounded s-stopping time σ, $E^P[Y(\sigma)|\mathcal{M}']=1$ (a.s., P).

Proof. Define Q on $\langle\Omega,\mathcal{M}^s\rangle$ by $\frac{dQ}{dP}=Y(\sigma)$. Given $s\leqq t_1<t_2$ and $A\in\mathcal{M}_{t_1}^s$,

$$E^Q[X_\alpha(t_2)\,\mathcal{X}_A]=E^P[X_\alpha(t_2)\,Y(\sigma)\,\mathcal{X}_A]$$
$$=E^P[X_\alpha(t_1)\,Y(\sigma)\,\mathcal{X}_A]$$

where we have used Lemma (3.1). Hence $X_\alpha(t)$ is Q-martingale for all $\alpha\in I$. Also, if $A\in\mathcal{M}_s^s$, then $Q(A)=E^P[Y(\sigma)\,\mathcal{X}_A]=E^P[Y(s)\,\mathcal{X}_A]=P(A)$. Hence $Q=P'$ on $\mathcal{M}_s^s\cap\mathcal{M}'$. This shows that $Q=P$ on $\mathcal{M}'$, and so for $A\in\mathcal{M}'$:

$$E^P[\mathcal{X}_A\,E[Y(\sigma)|\mathcal{M}']]=E^P[\mathcal{X}_A\,Y(\sigma)]=Q(A)=P(A).$$

Theorem (3.3). *If there is only one solution P'_{s_0,x_0} to the martingale problem for $\mathcal{L}'_t$ starting from (s_0,x_0), then there is an $\mathcal{M}_{\zeta-}^{s_0}$-measurable function σ': $\Omega\to[s_0,T]$ such that Eq.* (3.2) *holds for any solution P_{s_0,x_0} to the martingale problem for $\mathcal{L}_t$ starting from (s_0,x_0). In particular, P_{s_0,x_0} is uniquely determined on $\mathcal{M}_\zeta^{s_0}$.*

Proof. Let

$$X_\theta(t)=\exp\left[i\left\langle\theta,x_\varepsilon(t)-x_\varepsilon(s_0)-\int_{s_0}^t b^{(\varepsilon)}(u)\,du\right\rangle+\tfrac{1}{2}\int_{s_0}^t\langle\theta,a^{(\varepsilon)}(u)\,\theta\rangle\,du\right.$$
$$\left.-\int_{s_0}^t du\int\left(e^{i\langle\theta,y\rangle}-1-\frac{i\langle\theta,y\rangle}{1+|y|^2}\right)M^{(\varepsilon)}(u;dy)\right]$$

and

$$Y_\lambda(t)=\exp\left[\lambda\eta(t,\Gamma)-\int_{s_0}^t du\int_\Gamma(e^{\lambda y}-1)\,M(u,x(u);dy)\right],$$

where $a^{(\varepsilon)}$, $b^{(\varepsilon)}$, and $M^{(\varepsilon)}$ are given in (3.1) and $\lambda\in R$ and $\theta\in R^d$. By Corollary (1.3.3), $X_\theta(t)\,Y_\lambda(t)$ is a P_{s_0,x_0}-martingale for all $\theta\in R^d$. Hence, by Theorems (3.1) and (3.2),

$$E^{P_{s_0,x_0}}[Y_\lambda(t\wedge\zeta)|\mathcal{M}_{\zeta-}^{s_0}]=1\qquad(\text{a.s., }P_{s_0,x_0}).\tag{3.4}$$

Since σ is a s_0-stopping time and $\sigma\leqq T$, we can find a measurable function f: $(R^d)^N\to[s_0,T]$ and $s_0\leqq t_0<\cdots<t_n<\cdots\leqq T$ such that

$$\sigma=f(x(t_0),\ldots,x(t_n),\ldots).$$

Define
$$\sigma' = f\big(x_\varepsilon(t_0 \wedge \zeta), \dots, x_\varepsilon(t_n \wedge \zeta), \dots\big).$$

It is easily checked that $\sigma = \sigma'$ if $\tau > \sigma$.

We now compute $P_{s_0,x_0}(\zeta \leqq t \mid \mathcal{M}^{s_0}_{\zeta-})$. Note that

$$\begin{aligned} P_{s_0,x_0}(\zeta \leqq t \mid \mathcal{M}^{s_0}_{\zeta-}) &= \mathcal{X}_{[0,t]}(\sigma')\, P_{s_0,x_0}(\tau > \sigma \mid \mathcal{M}^{s_0}_{\zeta-}) + P_{s_0,x_0}(\tau \leqq \sigma \wedge t \mid \mathcal{M}^{s_0}_{\zeta-}) \\ &= \mathcal{X}_{[0,t]}(\sigma')\, E^{P_{s_0,x_0}}\big[1 - \eta\big(\zeta; R^d \setminus (B(0,\varepsilon))\big) \big| \mathcal{M}^{s_0}_{\zeta-}\big] \\ &\quad + E^{P_{s_0,x_0}}\big[\eta\big(t \wedge \zeta; R^d \setminus B(0,\varepsilon)\big)\big| \mathcal{M}^{s_0}_{\zeta-}\big]. \end{aligned}$$

According to (3.4),

$$\begin{aligned} E^{P_{s_0,x_0}}[\eta(t \wedge \zeta; \Gamma) \mid \mathcal{M}^{s}_{\zeta-}] &= E^{P_{s_0,x_0}}\left[\int_{s_0}^{t \wedge \zeta} M(u, x(u); \Gamma)\, du \,\middle|\, \mathcal{M}^{s_0}_{\zeta-}\right] \\ &= \int_{s_0}^{t} P_{s_0,x_0}(\zeta > u \mid \mathcal{M}^{s_0}_{\zeta-})\, M(u, x_\varepsilon(u \wedge \zeta); \Gamma)\, du \end{aligned} \tag{3.5}$$

for any $\Gamma \in \mathcal{B}_{R^d \setminus B(0,\varepsilon)}$. In particular, we get

$$\begin{aligned} &P_{s_0,x_0}(\zeta \leqq t \mid \mathcal{M}^{s_0}_{\zeta-}) \\ &\quad = \mathcal{X}_{[0,t]}(\sigma') \left(1 - \int_0^t P_{s_0,x_0}(\zeta > u \mid \mathcal{M}^{s_0}_{\zeta-})\, M^{(\varepsilon)}(u, x_\varepsilon(u \wedge \zeta))\, du\right) \\ &\qquad + \int_{s_0}^{t} P_{s_0,x_0}(\zeta > u \mid \mathcal{M}^{s_0}_{\zeta-})\, M^{(\varepsilon)}(u, x_\varepsilon(u \wedge \zeta))\, du, \end{aligned}$$

and so

$$P_{s_0,x_0}(\zeta > t \mid \mathcal{M}^{s_0}_{\zeta-}) = \mathcal{X}_{(t,\infty)}(\sigma')\, e^{-\int_{s_0}^{t \wedge s_0} M^{(\varepsilon)}(u, x_\varepsilon(u \wedge \zeta))\, du}.$$

Plugging this back into (3.5) and setting $t = T$, we obtain Eq. (3.2).

Finally, since $x(\zeta) = x_\varepsilon(\zeta) + \int_{|y| \geqq \varepsilon} y\, \eta(\zeta; dy)$, we now see that the distribution of $x(\zeta)$ under P_{s_0,x_0} given $\mathcal{M}^{s_0}_{\zeta-}$ is uniquely determined, and, therefore by Corollary (3.1.1), P_{s_0,x_0} is uniquely determined on $\mathcal{M}^{s_0}_{\zeta}$.

Corollary (3.3.1). *Let $\mathcal{L}_t$ be a Lévy generator. For each $(s_0, x_0) \in [0,\infty) \times R^d$ suppose there exists a Lévy generator $\mathcal{L}'_t$, for which the martingale problem is well-posed, and an $\varepsilon = \varepsilon(s_0, x_0)$, for which (3.1) holds with $T = s_0 + \varepsilon$ and $G = B(x_0, \varepsilon)$. If $\varepsilon(s_0, x_0)$ is uniformly positive, then the martingale problem for $\mathcal{L}_t$ starting from any point has at most one solution.*

Proof. Let P_{s_0,x_0} solve the martingale problem for $\mathcal{L}_t$ starting from (s_0, x_0). Define $\zeta_0 \equiv s_0$ and

$$\zeta_{n+1} = \big(\inf\{t \geqq \zeta_n \colon |x(t) - x(t-)| \geqq \varepsilon \text{ or } |x(t) - x(\zeta_n)| \geqq \varepsilon\}\big) \wedge (\zeta_n + \varepsilon),$$

where $\varepsilon > 0$ is chosen so that $\varepsilon(s, x_0) \geqq \varepsilon$ for all (s, x). Using Theorem (3.3) together with Theorem (1.2), one can prove by induction that P_{s_0,x_0} is uniquely determined on $\mathcal{M}^{s_0}_{\zeta_n}$ for all n. Hence it is enough to check that $P_{s_0,x_0}(\zeta_n > t) \to 1$ as $n \to \infty$ for $t > s_0$.

First note that by the proof of Theorem (A.1) in the appendix,

$$P_{s,x}\Big(\sup_{s \leqq u \leqq t} |x(u) - x(s)| \geqq \varepsilon/2\Big) \leqq C(t-s),$$

where C is independent of s, x, and the particular solution $P_{s,x}$ of the martingale problem for $\mathscr{L}_t$ starting from (s, x). Hence we can choose $0<\delta<\varepsilon$ so that

$$P_{s,x}(\zeta^s \leqq s+\delta) \leqq \tfrac{1}{2},$$

where $\zeta^s = (\inf\{t \geqq s: |x(t)-x(s)| \geqq \varepsilon \text{ or } |x(t)-x(t-)| \geqq \varepsilon\}) \wedge (s+\varepsilon)$. Set $\zeta^s_{n+1} = \zeta^{\zeta^s_n}$. Then

$$\begin{aligned} P_{s,x}(\zeta^s_{n+1} \leqq s+\delta) &= P_{s,x}(\zeta^s_n \leqq s+\delta, \zeta^s_{n+1} \leqq s+\delta) \\ &= E^{P_{s,x}}\left[\mathscr{X}_{\zeta^s_n \leqq s+\delta} P_\omega(\zeta^s_{n+1} \leqq s+\delta)\right] \\ &\leqq E^{P_{s_0,x_0}}\left[\mathscr{X}_{\zeta^{s_0}_n \leqq s_0+\delta} P_\omega\left(\zeta^{\zeta^{s_0}_n(\omega)} \leqq \zeta^{s_0}_n(\omega)+\delta\right)\right] \\ &\leqq \tfrac{1}{2} P_{s,x}(\zeta^s_n \leqq s+\delta), \end{aligned}$$

where P_ω is a r.c.p.d. of $P_{s,x} | \mathscr{M}^s_{\zeta^s_n}$. We have used here the fact that P_ω is P_{s,x_0} – almost surely a solution of the martingale problem for $\mathscr{L}_t$ starting from $(\zeta^s_n(\omega), x(\zeta^s_n(\omega), \omega))$. Thus

$$P_{s,x}(\zeta^s_n \leqq s+\delta) \leqq 2^{-n}.$$

We next prove by induction on $m \geqq 1$ that

$$\sup_{s,x} P_{s,x}(\zeta^s_n \leqq s+m\delta) \to 0 \quad \text{as } n \to \infty.$$

Indeed,

$$P_{s,x}(\zeta^s_{2n} \leqq s+(m+1)\delta) \leqq P_{s,x}(\zeta^s_n \leqq s+m\delta) + P_{s,x}(\zeta^s_n \geqq s+m\delta, \zeta^s_{2n} \leqq s+(m+1)\delta)$$

and

$$P_{s,x}(\zeta^s_n \geqq s+m\delta, \zeta^s_{2n} \leqq s+(m+1)\delta) \leqq E^{P_{s,x}}\left[P_\omega\left(\zeta^{\zeta^s_n(\omega)} \leqq \zeta^s_n(\omega)+\delta\right)\right]$$

where P_ω is as before. Hence

$$\sup_{s,x} P_{s,x}(\zeta^s_{2n} \leqq s+(m+1)\delta) \leqq 2 \sup_{s,x} P_{s,x}(\zeta^s_n \leqq s+m\delta).$$

4. Uniqueness, the Elliptic Case

In this section we will prove uniqueness for the martingale problem associated with Lévy operators $\mathscr{L}_t = L_t + K_t$ whose coefficients satisfy:

(i) $a: [0,\infty) \times R^d \to S_d$ is bounded and continuous and each $a(s,x)$ is positive definite,

(ii) $b: [0,\infty) \times R^d \to R^d$ is bounded and measurable, (4.1)

(iii) $\int_\Gamma \frac{y}{1+|y|^2} M(s,x;dy)$ is bounded and continuous for all $\Gamma \in \mathscr{B}_{R^d \setminus \{0\}}$.

Actually, the proof will be first carried out under the more stringent conditions:

(i) $a: [0,\infty) \times R^d \to S_d$ is bounded, uniformly continuous, and uniformly continuous,

(ii) $b \equiv 0$, (4.2)

(iii) $\int \frac{|y|^2}{1+|y|^2} M(s,x;dy)$ is bounded and continuous for all

$$\Gamma \in \mathscr{B}_{R^d \setminus \{0\}} \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \sup_{(s,x)} \int_{|y|<\varepsilon} \frac{|y|^2}{1+|y|^2} M(s,x;dy) = 0.$$

Once this has been done, it will be easy to relax these assumptions to those in (4.1).

The idea behind our proof will be to take advantage of the localization procedure developed in the preceding section. Under the conditions in (4.2), one can find for each $\varepsilon>0$ a $\delta>0$ such that

$$\begin{aligned}&\text{(i)}\ \|a(t,y)-a(s,x)\|^2\equiv\sum_{i,j=1}^{d}|a^{ij}(t,y)-a^{ij}(s,x)|^2<\varepsilon^2\\&\qquad\text{for } (t,y)\in[s,s+\delta)\times B(x,\delta),\\&\text{(ii)}\ \int_{B(0,\delta)}\frac{|y|^2}{1+|y|^2}M(s,x;dy)<\varepsilon \text{ for all } (s,x).\end{aligned}\tag{4.3}$$

Hence, by Corollary (3.3.1), it suffices for us to show that if $A\in S_d$ is positive definite, then there is an $\varepsilon>0$ such that

$$\begin{aligned}&\text{(i)}\ \|a(s,x)-A\|<\varepsilon \text{ for all } (s,x),\\&\text{(ii)}\ \int\frac{|y|^2}{1+|y|^2}M(s,x;dy)<\varepsilon \text{ and } M(s,x;R^d\setminus B(0,1))=0 \text{ for all } (s,x)\end{aligned}\tag{4.4}$$

implies uniqueness holds. The critical step in this proof is to obtain the estimate contained in Theorem (4.1) below.

Lemma (4.1). *Let a: $[s,\infty)\times\Omega\to S_d$ and M: $[s,\infty)\times\Omega\times\mathscr{B}_{R^d\setminus\{0\}}\to[0,\infty)$ be s-non-anticipating functions. Assume that*

(i) $\mu|\theta|^2\leqq\langle\theta,a(t)\theta\rangle\leqq\dfrac{|\theta|^2}{\mu}$, *$t\in[s,\infty)$ and $\theta\in R^d$, for some $\mu>0$,*

(ii) $a(t)=a\left(s+\dfrac{[N(t-s)]}{N}\right)$, *$t\in[s,\infty)$, for some $N\geqq1$,*

(iii) $M(t,R^d\setminus\{0\})\leqq B$, *$t\in[s,\infty)$, for some $B<\infty$.*

Let α: $[s,\infty)\times\Omega\to R^d$ be an s-non-anticipating, right-continuous function having left limits and suppose P is a probability measure on $\langle\Omega,\mathscr{M}^s\rangle$ for which:

$$\exp\left[i\langle\theta,\alpha(t)-\alpha(s)\rangle+\tfrac{1}{2}\int_s^t\langle\theta,a(u)\theta\rangle\,du-\int_s^t du\int(e^{i\langle\theta,y\rangle}-1)\,M(u,dy)\right]$$

is a P-martingale for all $\theta\in R^d$. Then if $p>\dfrac{d+2}{2}$ and $T>s$:

$$\left|E^P\left[\int_s^T f(t,\alpha(t))\,dt\right]\right|\leqq C_p\|f\|_{L^p([s,T]\times R^d)},\qquad f\in C_0([s,T]\times R^d).$$

Proof. First note that since $M(t,R^d\setminus\{0\})$ is bounded, we can let $\delta=0$ in Corollary (1.3.2) and obtain:

$$\alpha(t)=\gamma(t)+\int y\,\eta(t,dy).$$

Next set

$$\beta(t)=\int_s^t a^{-\frac{1}{2}}(u)\,d\gamma(u).$$

Then $\beta(\cdot)$ is a P-Brownian motion and

$$\gamma(t)-\gamma(s)=\int_s^t a^{\frac{1}{2}}(u)\,d\beta(u).$$

Define $\tau_0 \equiv s$ and

$$\tau_{n+1} = (\inf\{t \geqq \tau_n \colon \eta(t, R^d) > \eta(\tau_n, R^d)\}) \wedge \left(s + \frac{[N(\tau_n - s)] + 1}{N}\right) \wedge T.$$

Since $M(t; R^d \setminus \{0\})$ is bounded, $\eta(\cdot, R^d)$ has only a finite number of jumps in $[s, T]$. Hence $\tau_n = T$ for all but a finite number of n's. Let $P_\omega^{(n)}$ denote the r.c.p.d. of $P \mid \mathscr{M}_{\tau_n}^s$.

Given $f \in C_0([s, T] \times R^d)$ which is $\geqq 0$, we now have:

$$\begin{aligned} E^P\left[\int_s^T f(t, \alpha(t))\, dt\right] &= \sum_0^\infty E^P\left[\int_{\tau_n}^{\tau_{n+1}} f(t, a(t))\, dt\right] \\ &\leqq \sum_0^\infty E^P\left[\int_s^T \mathscr{X}_{[0,t)}(\tau_n(\omega))\, E^{P_\omega^{(n)}} f(t, \alpha(\tau_n(\omega)) + a^{\frac{1}{2}}(\tau_n(\omega))(\beta(t) - \beta(\tau_n(\omega)))\, dt\right. \\ &\leqq \sum_0^\infty E^P\left[\mathscr{X}_{[0,T)}(\tau_n) \int_{\tau_n}^T dt \int g_{a(\tau_n)}(t - \tau_n, y - \alpha(\tau_n))\, f(t, y)\, dy\right] \\ &\leqq A \|f\|_{L^p([s,T] \times R^d)} \sum_0^\infty E^P[\mathscr{X}_{[0,T)}(\tau_n)] \end{aligned}$$

where $p > \frac{d+2}{2}$,
$$g_c(t, y) = \frac{1}{(2\pi t)^{d/c} (\det c)^{\frac{1}{2}}}\, e^{-\langle y, c^{-1} y\rangle/2t}$$

for positive definite $c \in S_d$, and

$$A = \sup_{\mu I \leqq c \leqq \frac{1}{\mu} I} \left(\int_s^t dt \int |g_c(t, y)|^q dy\right)^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Finally, note that $\sum_0^\infty \mathscr{X}_{[0,T)}(\tau_n)$ is the number of n's such that $\tau_n < T$. Hence $\sum_0^\infty \mathscr{X}_{[0,T)}(\tau_n) \leqq 1 + N(T-s)\eta(T, R^d)$, and so

$$\sum_0^\infty E^P[\mathscr{X}_{[0,T)}(\tau_n)] \leqq 1 + NB(T-s)^2. \tag{4.5}$$

Lemma (4.2). *Let* $a\colon [s, \infty) \times \Omega \to S_d$, $b\colon [s, \infty) \times \Omega \to R^d$ *and* $M\colon [s, \infty) \times \Omega \to \mathscr{B}_{R^d \setminus \{0\}} \to [0, \infty)$ *be s-non-anticipating functions. Assume that* a, b, *and* $\int \frac{|y|^2}{1+|y|^2} M(t, dy)$ *are bounded. Let* $\alpha\colon [s, \infty) \times \Omega \to R^d$ *be an s-non-anticipating, right continuous function having left limits. Suppose that there exist* $T > s$, $1 < p_1 < p_2 < \infty$ *and* $C_p < \infty$, $p_1 < p < p_2$, *such that*

$$\left|E^Q\left[\int_s^T f(t, \alpha(t))\, dt\right]\right| \leqq C_p \|f\|_{L^p([s,T] \times R^d)}, \qquad f \in C_0([s, T] \times R_d),$$

whenever Q *is a probability measure on* $\langle \Omega, \mathscr{M}^s \rangle$ *for which*

$$\exp\left[i\left\langle \theta, \alpha(t) - \alpha(s) - \int_s^t b(u)\, du\right\rangle + \frac{1}{2}\int_s^t \langle \theta, a(u)\theta\rangle\, du \right. \\ \left. - \int_s^t du \int \left(e^{i\langle \theta, y\rangle} - 1 - \frac{i\langle \theta, y\rangle}{1+|y|^2}\right) M(u, dy)\right]$$

is a Q-martingale for all $\theta \in R^d$. Then for $p_1 < p < p_2$:

$$\left| E^P\left[\int_s^T f(t, \alpha(t))\,dt\right]\right| \leqq C_{\frac{p+p_1}{2}} e^{B_p(T-s)} \|f\|_{L^p([s,T]\times R^d)}, \qquad f \in C_0([s,T]\times R^d),$$

whenever P is a probability measure on $\langle \Omega, \mathscr{M}^s\rangle$ such that

$$\exp\left[i\left\langle \theta, \alpha(t)-\alpha(s)-\int_s^t b'(u)\,du\right\rangle + \frac{1}{2}\int_s^t \langle \theta, a(u)\theta\rangle\,du \right.$$
$$\left. - \int_s^t du \int \left(e^{i\langle\theta,y\rangle} - 1 - \frac{i\langle\theta,y\rangle}{1+|y|^2}\right) M(u,dy)\right]$$

is a P-martingale for all $\theta \in R^d$, where $b' = b + ac$ and c: $[s,\infty)\times\Omega \to R^d$ is a bounded, s-non-anticipating function. The constant B_p depends only on the interval (p_1, p_2), and the bounds on a and c.

Proof. Let P be given, and define $\gamma(\cdot)$ accordingly as in Corollary (1.3.2). Define

$$X(t) = \exp\left[-\int_s^t c(u)\,d\gamma(u) - \frac{1}{2}\int_s^t \langle c(u), a(u)c(u)\rangle\,du\right].$$

Then $X(t)$ is a P-martingale and so we can define Q on $\langle\Omega,\mathscr{M}^s\rangle$ by $\frac{dQ}{dP} = X(t)$ on $\mathscr{M}_t^s$, $t \geqq s$. By Corollary (1.3.4),

$$\exp\left[i\left\langle \theta, \alpha(t)-\alpha(s)-\int_s^t b(u)\,du\right\rangle + \frac{1}{2}\int_s^t \langle \theta, a(u)\theta\rangle\,du \right.$$
$$\left. - \int_s^t du \int \left(e^{i\langle\theta,y\rangle} - 1 - \frac{i\langle\theta,y\rangle}{1+|y|^2}\right) M(u,dy)\right]$$

is a Q-martingale for all $\theta \in R^d$. Hence

$$\left| E^P\left[\int_s^T f(t,\alpha(t))\,dt\right]\right| = \left| E^Q\left[\left(\int_s^T f(t,\alpha(t))\,dt\right) X^{-1}(T)\right]\right|$$
$$\leqq \left(E^Q\left[\int_s^T |f(t,\alpha(t))|^u\,dt\right]\right)^{1/u} (T-s)^{1/v} (E^Q[X^{-v}(T)])^{1/v}$$

for $1 < u < \infty$ and $\frac{1}{u} + \frac{1}{v} = 1$. Given $p_1 < p < p_2$, choose $1 < u < \infty$ so that $p/u = (p+p_1)/2$. Then

$$\left| E^Q\left[\int_s^T |f(t,\alpha(t))|^u\,dt\right]\right| \leqq C_{\frac{p+p_1}{2}} \|f\|_{L^p([s,T]\times R^d)}.$$

Moreover,

$$E^Q[X^{-v}(T)] = E^P[X^{1-v}(T)]$$
$$= E^P \exp\left[(1-v)\int_s^T c(t)\,d\gamma(t) - \frac{(1-v)^2}{2}\int_s^T \langle c(t), a(t)\,c(t)\rangle\,dt\right]$$
$$\times \exp\left[\frac{v(v-1)}{2}\int_s^T \langle c(t), a(t)\,c(t)\rangle\,dt\right],$$

and therefore:

$$\left(E^Q[X^{-v}(T)]\right)^{\frac{1}{v}} \leqq e^{\frac{v-1}{2}B(T-s)}$$

where $B = \sup_{t \geqq s} \langle c(t), a(t)\, c(t)\rangle$.

Theorem (4.1). *Let* $a: [s, \infty) \times \Omega \to S_d$, $b: [s, \infty) \times \Omega \to R^d$, *and*

$$M: [s, \infty) \times \Omega \times \mathscr{B}_{R^d \setminus \{0\}} \to [0, \infty)$$

be s-non-anticipating functions, and let $\alpha: [s, \infty) \times \Omega \to R^d$ *be an s-non-anticipating, right continuous function having left limits. Given a positive definite* $A \in S_d$ *and* $\frac{d+2}{2} < p_1 < p_2 < \infty$, *there exists an* $\varepsilon > 0$ *such that*

(i) $\sup_{t \geqq s} \|a(t) - A\| < \varepsilon$

and

(ii) $\sup_{t \geqq s} \int \frac{|y|^2}{1+|y|^2} M(t, dy) < \varepsilon$

imply

$$\left| E^P \left[\int_s^T f(t, \alpha(t))\, dt \right] \right| \leqq C_p e^{B_p(T-s)} \|f\|_{L^p([s,T] \times R^d)}, \qquad f \in C_0([s, T] \times R^d),$$

for $p_1 < p < p_2$ *and* $T > s_1$ *whenever P is a probability measure on* $\langle \Omega, \mathscr{M}^s \rangle$ *for which*

$$\exp\left[i\left\langle \theta, \alpha(t) - \alpha(s) - \int_s^t b(u)\, du \right\rangle + \frac{1}{2} \int_s^t \langle \theta, \alpha(u)\, \theta \rangle\, du \right.$$
$$\left. - \int_s^t du \int \left(e^{i\langle \theta, y \rangle} - 1 - \frac{i\langle \theta, y \rangle}{1+|y|^2} \right) M(u, dy) \right]$$

is a P-martingale for all $\theta \in R^d$. *The* ε *dependends only on the interval* (p_1, p_2) *and the largest number* $\mu > 0$ *such that* $\mu |\theta|^2 \leqq \langle \theta, A\theta \rangle \leqq \frac{1}{\mu} |\theta|^2$. *The constants* C_p *and* B_p *depend on* ε, (p_1, p_2), μ *and the bound on b.*

Proof. In view of Lemma (4.2), it suffices to prove this estimate when P makes

$$\exp\left[i\langle \theta, \alpha(t) - \alpha(s) \rangle + \frac{1}{2} \int_s^t \langle \theta, a(u)\, \theta \rangle\, du \right.$$
$$\left. - \int_s^t du \int \left(e^{i\langle \theta, y \rangle} - 1 - i\langle \theta, y \rangle \mathscr{X}_{B(0,1)}(y) \right) M(u, dy) \right] \tag{4.6}$$

into a martingale for all $\theta \in R^d$. By Lemma (4.1) and (4.2), we know that if a and M satisfy the conditions of Lemma (4.1), then for such P:

$$\left| E^P \left[\int_s^T f(t, \alpha(t))\, dt \right] \right| \leqq C \|f\|_{L^p([s,T] \times R^d)}$$

for $p > \frac{d+2}{2}$. What we must show is that by a proper choice of ε, this constant C can be made independent of the N and B in Lemma (4.1). To this end, define

$$G^T \phi(t, x) = \int_{t \wedge T}^T du \int g_A(u-t, y-x)\, \phi(u, y)\, dy.$$

Then, if $f=G^T\phi$ and $\phi\in C_0^\infty([s,T)\times R^d)$:

$$-E^P[f(s,\alpha(s))]=-E^P\left[\int_s^T \phi(u,\alpha(u))\,du\right]$$
$$+\frac{1}{2}E^P\left[\int_s^T\sum_1^d (a^{ij}(u)-A^{ij})\frac{\partial^2 f}{\partial x_i\partial x_j}(u,\alpha(u))\,du\right]$$
$$+E^P\left[\int_s^T du\int (f(u,\alpha(u)+y)-f(u,\alpha(u)))-\mathscr{X}_{B(0,1)}(y)\langle y,\nabla_x f(u,\alpha(u))\rangle M(u,dy)\right].$$

Define v on $\mathscr{B}_{[s,T]\times R^d}$ by

$$v(\psi)=E^P\left[\int_s^T \psi(u,\alpha(u))\,du\right],\qquad \psi\in C_b([s,T]\times R^d).$$

Then

$$|v(\phi)|\leqq|E^P[G^T\phi(s,\alpha(s))]|+\frac{1}{2}\varepsilon v\left(\left(\sum^d\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)^2\right)^{\frac{1}{2}}\right)$$
$$+2\varepsilon v\left(\sup_{|y|\leqq 1}\frac{|f(t,x+y)-f(t,x)-\langle y,\nabla_x f(t,x)\rangle|}{|y|^2}\right)$$
$$+4\varepsilon(T-s)\|f\|_{L^\infty([s,T]\times R^d)}.$$

Assuming that a and M satisfy the conditions of Lemma (4.1) and therefore that $v\in L^q([s,T]\times R^d)$ for $q<\frac{d+2}{d}$, we now find that

$$\|v\|_{L^q}\|\phi\|_{L^p}\leqq A_p(T-s)^{\frac{1}{q}-\frac{d}{2p}}\|\phi\|_{L^p}+\frac{\varepsilon A'_p}{2}\|v\|_{L^q}\|\phi\|_{L^p}$$
$$+2\varepsilon A''_p\|v\|_{L^q}\|\phi\|_{L^p}+4\varepsilon A_p(T-s)^{\frac{1}{q}-\frac{d}{2p}+1}\|\phi\|_{L^p}.$$

Hence if

$$\varepsilon\left(\frac{A'_p}{2}+2A''_p\right)\leqq\frac{1}{2},\tag{4.7}$$

then

$$\|v\|_{L^q}\leqq 2A_p\left((T-s)^{\frac{1}{q}-\frac{d}{2p}}+4\varepsilon(T-s)^{\frac{1}{q}-\frac{d}{2p}+1}\right).\tag{4.8}$$

The constants A_p, A'_p, and A''_p are derived in the appendix. Clearly, ε can be chosen so that (4.7) is satisfied for all $p_1<p<p_2$ simultaneously.

Now suppose that a and M satisfy the conditions of theorem and that P makes (4.6) into a martingale for all $\theta\in R^d$. Assume that ε is small enough that (4.7) holds for all $p_1<p<p_2$. For $\delta>0$, define

$$\alpha^\delta(t)=\alpha(t)-\int_{|y|<\delta} y\tilde{\eta}(t,dy).$$

Then, by Corollary (1.3.1),

$$\exp\left[i\langle\theta,\alpha^\delta(t)-\alpha^\delta(s)\rangle+\frac{1}{2}\int_s^t\langle\theta,a(u)\theta\rangle\,du\right.$$
$$\left.-\int_s^t du\int_{|y|\geqq\delta}\left(e^{i\langle\theta,y\rangle}-1-i\langle\theta,y\rangle\mathscr{X}_{B(0,1)}(y)\right)M(u,dy)\right]$$

is a P-martingale for all $\theta \in R^d$. Next set

$$\beta(t) = \int_s^t a^{-\frac{1}{2}}(u)\, d\gamma(u).$$

Then $\beta(\cdot)$ is a P-Brownian motion and

$$\gamma(t) - \gamma(s) = \int_s^t a^{\frac{1}{2}}(u)\, d\beta(u).$$

Choose $a_n\colon [s, \infty) \times \Omega \to S_d$ so that each a_n satisfies the condition (ii) of Lemma (4.1) and condition (i) of this theorem and $E^P \int_s^t \|a_n(u) - a(u)\|^2 du \to 0$ as $n \to \infty$ for each $t > s$. Put

$$\alpha_n^\delta(t) = \alpha^\delta(t) - \gamma(t) + \int_s^t a_n^{\frac{1}{2}}(u)\, d\beta(u).$$

Then

$$E^P[|\alpha_n^\delta(t) - \alpha_n(t)|^2] \to 0 \quad \text{as } n \to \infty \text{ and } \delta \downarrow 0$$

for each $t \geqq s$. Defining

$$v_n^\delta(\psi) = E^P\left[\int_s^t f(t, \alpha_n^\delta(t))\, dt\right],$$

we now see that $v_n^\delta \to v$ weakly. Finally,

$$\exp\left[i\langle\theta, \alpha_n^\delta(t) - \alpha_n^\delta(s)\rangle + \frac{1}{2}\int_s^t \langle\theta, a_n(u)\theta\rangle\, du - \int_s^t du \int_{|y| \geqq \delta} (e^{i\langle\theta, y\rangle} - 1 - i\langle\theta, y\rangle \mathscr{X}_{B(0,1)}(y))\, M(u, dy)\right]$$

is a P-martingale for all $\theta \in R^d$, and so

$$|v_n^\delta(\phi)| \leqq D_p \|\phi\|_{L^p([s,T] \times R^d)}, \quad f \in C_0([s, T] \times R^d),$$

for $p_1 < p < p_2$ and D_p the constant in (4.8). Thus v satisfies the same inequality.

Lemma (4.3). *Let $\mu > 0$ and $\frac{d+2}{2} < p_1 < p_2 < \infty$ be given. Suppose $A \in S_d$ satisfies $\mu|\theta|^2 \leqq \langle\theta, A\theta\rangle \leqq \frac{|\theta|^2}{\mu}$, $\theta \in R^d$, and let $a\colon [0, \infty) \times R^d \to S_d$ and $M\colon [0, \infty) \times R^d \times \mathscr{B}_{R^d \setminus \{0\}} \to [0, \infty)$ satisfy the conditions in (4.4) with ε chosen so that (4.7) holds for $p \in (p_1, p_2)$. Define*

$$\mathscr{L}_t^{(0)} f(x) = \frac{1}{2}\sum_1^d a^{ij}(t, x)\frac{\partial^2 f}{\partial x_i \partial x_j} + \int (f(x+y) - f(x) - \langle y, \nabla_x f(x)\rangle)\, M(t, x; dy).$$

Then for $0 \leqq s < T$ and $p \in (p_1, p_2)$:

$$\|(\mathscr{L}_t^{(0)} - \tfrac{1}{2}\Delta_A) \circ G^T \phi\|_{L^p([s,T] \times R^d)} \leqq \tfrac{1}{2}\|\phi\|_{L^p([s,T] \times R^d)},$$

where: $\Delta_A = \sum_1^d A^{ij}\frac{\partial^2}{\partial x_i \partial x_j}$ and G^T is given by:

$$G^T \phi(t, x) = \int_s^t du \int g_A(u - t, y - x)\phi(u, y)\, dy.$$

In particular, $H_s^T = (I - (\mathscr{L}_t^{(0)} - \frac{1}{2}\Delta_A) \circ G^T)^{-1}$ *exists as a bounded operator on* $L^p([s, T] \times R^d)$ *onto itself and is consistently defined for* $p \in (p_1, p_2)$.

Proof. The proof of everything except the consistency has been essentially carried out in the proof of Theorem (4.1). The proof of consistency is an immediate consequence of the representation of H_s^T in terms of a Neumann series.

Theorem (4.2). *Let* $\mu > 0$, (p_1, p_2), A, ε, a *and* M *be given as in Lemma* (4.3) *and define* G^T *and* H_s^T *accordingly. Suppose* P *is a probability measure on* $\langle \Omega, \mathscr{M}^s \rangle$ *such that* $P(x(s) = x) = 1$ *and*

$$\exp\left[i\langle \theta, x(t) - x(s)\rangle + \frac{1}{2}\int_s^t \langle \theta, a(u, x(u))\theta\rangle du - \int_s^t du \int \left(e^{i\langle\theta, y\rangle} - 1 - i\langle\theta, y\rangle \mathscr{X}_{B(0,1)}(y)\right) M(u, x(u); dy)\right]$$

is a P-martingale for all $\theta \in R^d$. *Then for* $0 \leqq s < T$:

$$E^P\left[\int_s^t \phi(t, x(t))\, dt\right] = G^T \circ H_s^T \phi(s, x), \qquad \phi \in C_0([s, T] \times R^d). \tag{4.9}$$

Proof. Let $\Phi = \{\phi \in C_b([s, T] \times R^d) \colon G^T \circ H_s^T \phi \in C_0^{1,2}([s, T] \times R^d)\}$. Then Φ is dense in $L^p([s, T] \times R^d)$. Thus, since both sides of (4.9) are L^p-continuous, we need only prove (4.9) for $\phi \in \Phi$. But $\phi \in \Phi$ implies

$$f = G^T \circ H_s^T \phi(T, \cdot) = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial t} + \mathscr{L}_t^{(0)}\right) f = -\phi, \quad s \leqq t < T,$$

where $\mathscr{L}_t^{(0)}$ is given as in Lemma (4.3). Hence

$$G^T \circ H_s^T \phi(t \wedge T, x(t \wedge T)) + \int_s^{t \wedge T} \phi(u, x(u))\, du$$

is a P-martingale, and so (4.9) holds.

Corollary (4.2.1). *Under the conditions stated in Theorem* (4.2), *the martingale problem for* $\mathscr{L}_t^{(0)}$ *is well-posed, where* $\mathscr{L}_t^{(0)}$ *is the operator described in Lemma* (4.3).

Proof. Existence has already been established. To prove uniqueness, note that for each (s, x) and $t > s$, there is a probability $F(s, x; t, \Gamma)$ on R^d such that

$$P(x(t) \in \Gamma) = F(s, x; t, \Gamma), \qquad \Gamma \in \mathscr{B}_{R^d},$$

for any solution P starting from (s, x). In fact,

$$\int \phi(y) F(s, x; t, dy) = \lim_{h \downarrow 0} \frac{G^{t+h} \circ H_s^{t+h}(s, x) - G^t \circ H_s^t \phi(s, x)}{h}$$

for $\phi \in C_0(R^d)$. In particular $F(s, x; t, \Gamma)$ is measurable in (s, x) for all $t > s$ and $\Gamma \in \mathscr{B}_{R^d}$.

Now suppose that P and P' are solutions starting at (s, x). We will show that

$$P(x(t_1) \in \Gamma_1, \ldots, x(t_n) \in \Gamma_n) = P'(x(t_1) \in \Gamma_1, \ldots, x(t_n) \in \Gamma_n)$$

for all $n \geqq 1$, $s < t_1 < \cdots < t_n$, and $\Gamma_1, \ldots, \Gamma_n \in \mathscr{B}_{R^d}$. This is obvious from the preceding paragraph when $n = 1$. Assume it for n, and let P_ω and P'_ω be the r.c.p.d.'s of $P | \mathscr{M}_{t_n}^s$

and $P'|\mathcal{M}^s_{t_n}$, respectively. Then P_ω and P'_ω are each a.s. solutions starting from $(t_n, x(t_n, \omega))$, and so

$$P_\omega(x(t_{n+1})\in\Gamma_{n+1})=F(t_n, x(t_n,\omega); t_{n+1}, \Gamma_{n+1}) \quad (\text{a.s.}, P),$$

$$P'_\omega(x(t_{n+1})\in\Gamma_{n+1})=F(t_n, x(t_n,\omega); t_{n+1}, \Gamma_{n+1}) \quad (\text{a.s.}, P).$$

This completes the induction.

Theorem (4.3). *Suppose a, b, and M satisfy the conditions of Eq. (4.1). Then the martingale problem for $\mathcal{L}_t$ is well-posed. Moreover, if $P_{s,x}$ denotes the unique solution starting from (s, x), then the family $\{P_{s,x}: (s,x)\in[0,\infty)\times R^d\}$ is strong Markov.*

Proof. The existence assertion is contained in Theorem (2.2). To prove uniqueness, first assume that the conditions in Eq. (4.2) are met. Then uniqueness is an immediate consequence of Corollary (4.2.1) and Corollary (3.3.1). As a consequence of uniqueness and weak compactness, it follows that in this situation $P_{s_n,x_n}\to P_{s,x}$ weakly as $(s_n,x_n)\to(s,x)$, and therefore $\{P_{s,x}: (s,x)\in[0,\infty)\times R^d\}$ is measurable. Note that, by Corollary (1.3.4), uniqueness of solutions and measurability of $\{P_{s,x}: (s,x)\in[0,\infty)\times R^d\}$ continues to hold even when $b\neq 0$. We can now drop the assumptions of uniformity on a and M. In fact, given (s,x), choose a_N and M_N for $N\geqq 1$ so that $a_N=a$ and $M_N=M$ on $[s,s+N]\times B(x,N)$ and a_N and M_N satisfy the conditions of (4.2). Then for each N, the martingale problem associated with a_N, b, and M_N is well-posed. Moreover, if $P_{s,x}$ and $P^{(N)}_{s,x}$ are solutions associated with a, b, and M and a_N, b, and M_N, respectively, starting at (s,x), then $P_{s,x}=P^{(N)}_{s,x}$ until $\zeta^{(N)}$, the first exit time from $[s,s+N]\times B(x,N)$ (cf. Corollary (3.1.1)). Since $\zeta^{(N)}\to\infty$ as $N\to\infty$, this show that $P_{s,x}$ is uniquely determined and is measurable with respect to (s,x).

Finally, to show that the strong Markov property holds, one need only take the r.c.p.d. of $P_{s,x}|\mathcal{M}^s_\tau$ and use uniqueness to see that for almost all ω it must equal $P_{\tau(\omega),x(\tau(\omega)),\omega)}$ on $\mathcal{M}^{\tau(\omega)}$.

Remark. Proceeding as in [7], one can improve Theorem (4.2) and show that the process is actually strongly Feller continuous. Moreover, L^p-estimates on $E^{P_{s,x}}\left[\int_s^T f(t,x(t))\,dt\right]$ can be obtained for all $\frac{d+2}{2}<p<\infty$ and coefficients satisfying (4.1). The details of these arguments are very similar to those in [7].

Remark. Using uniqueness and the final assertion in Theorem (2.1), one sees that if $\tau=\inf\{t\geqq s: |x(t)-x(t-)|\geqq\varepsilon\}$, then

$$P_{s,x}(x(\tau)-x(\tau-)\in\Gamma \mid \tau \text{ and } x(t) \text{ for } t<\tau)=\frac{M(\tau,x(\tau-);\Gamma)}{M(\tau,x(\tau-);R^d\setminus B(0,\varepsilon))}$$

on the set $\{\tau<\infty\}$ for $\Gamma\in\mathcal{B}_{R^d\setminus B(0,\varepsilon)}$; and

$$P_{s,x}(\tau>T\,|\,x(t) \text{ for } t<\tau)=e^{-\int_s^T M(u,x_\varepsilon(u\wedge\tau);R^d\setminus B(0,\varepsilon))\,du}$$

where $x_\varepsilon(u)=x(u)-\int_{|y|\geqq\varepsilon} y\,\eta(t,dy)$.

Remark. With a little more work, one can show that the existence and uniqueness of solutions remains true under the assumption that $\int_\Gamma \frac{|y|^2}{1+|y|^2}M(t,x;dy)$

is bounded and measurable for all $\Gamma \in \mathscr{B}_{R^d \setminus \{0\}}$ and $\int_{|y|<\delta} \frac{|y|^2}{1+|y|^2} M(t, x; dy) \to 0$ uniformly on compacts as $\delta \downarrow 0$. In order to do this one has to use L^p-estimates of the sort obtained in Theorem (4.1) above.

5. Uniqueness, the Parabolic Case

In this section, $\Omega = D([0,\infty), R^{d+1})$. Given $x \in R^{d+1}$, we will write $x = (x_0, \bar{x}) \in R \times R^d$. The Lévy operators $\mathscr{L} = L + K$ with which we will be dealing are of the form:

$$\begin{aligned} Lf(x) &= \frac{\partial}{\partial x_0} + \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(x) \frac{\partial}{\partial x_i} \\ Kf(x) &= \int \left(f(x+y) - f(x) - \frac{\langle \bar{y}, \nabla_{\bar{x}} f(x) \rangle}{1+|y|^2} \right) M(x, dy) \end{aligned} \tag{5.1}$$

where

(i) $a: R^{d+1} \to S_d$ is bounded, continuous, and $a(x)$ is positive definite for each x,

(ii) $\bar{b}: R^{d+1} \to R^d$ is bounded and measurable,

(iii) $\int_\Gamma \frac{|y_0| + |\bar{y}|^2}{1+|y_0|+|\bar{y}|^2} M(x, dy)$ is bounded and continuous for all $\Gamma \in \mathscr{B}_{R^d \setminus \{0\}}$. (5.2)

Since $\mathscr{L}$ is independent of time, the martingale problem is that of finding for each $x \in R^{d+1}$ a P on $\langle \Omega, \mathscr{M}^0 \rangle$ such that

$$P(x(0) = x) = 1$$

and

$$f(x(t)) - \int_0^t \mathscr{L} f(x(s))\, ds$$

is a P-martingale for $f \in C_0^\infty(R^{d+1})$.

Obviously, all the results of Sections (1), (2) and (3) apply to this situation. In fact, the present set-up reduces to the one considered up until now by taking:

$$\tilde{a}(x) = \begin{pmatrix} 0 & 0 \\ 0 & a(x) \end{pmatrix},$$

$$\tilde{b}(x) = \left(1 + \int \frac{y_0}{1+|y|^2} M(x, dy), \bar{b}(x) - \int \bar{y} \frac{y_0^2}{(1+|y|^2)(1+|\bar{y}|^2)} M(x, dy) \right),$$

$$\tilde{L} f(x) = \frac{1}{2} \sum_{i,j=0}^{d} \tilde{a}^{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=0}^{d} \tilde{b}^i(x) \frac{\partial f}{\partial x_i},$$

$$\tilde{K} f(x) = \int \left(f(x+y) - f(x) - \frac{\langle y, \nabla_x f(x) \rangle}{1+|y|^2} \right) M(x, dy),$$

and noting that $\mathscr{L} = \tilde{L} + \tilde{K}$. In particular, existence of solutions has been established. Also, the localization procedure worked out in Section (3) can be used here. Hence, our method of proving uniqueness will parallel very closely the one developed in the preceding section. For that reason, the proofs in this section will not be as detailed of those given there.

Lemma (5.1). *Let* $a\colon [0,\infty)\times\Omega\to S_d$ *and* $M\colon [0,\infty)\times\Omega\times\mathscr{B}_{R^d\setminus\{0\}}\to[0,\infty)$ *be non-anticipating functions. Assume that*

(i) $\dfrac{|\bar\theta|^2}{\mu}\leqq\langle\bar\theta, a(t)\bar\theta\rangle\leqq\mu|\bar\theta|^2$, $\bar\theta\in R^d$, for some $\mu>0$,

(ii) $a(t)=a\left(\dfrac{[Nt]}{N}\right)$, $t\geqq 0$, *for some* $N\geqq 1$,

(iii) $M(t, R^{d+1}\setminus\{0\})\leqq B$ *for some* $B<\infty$.

Let $\alpha\colon [0,\infty)+\Omega\to R^{d+1}$ *be a non-anticipating, right-continuous function having left limits and suppose P is a probability measure on* $\langle\Omega,\mathscr{M}^0\rangle$ *such that:*

$$\exp\left[i\langle\theta,\alpha(t)-\alpha(0)-t\hat e_0\rangle+\frac{1}{2}\int_0^t\langle\bar\theta, a(s)\bar\theta\rangle\,ds-\int_0^t ds\int(e^{i\langle\theta,y\rangle}-1)M(s,dy)\right]$$

is a P-martingale for all $\theta\in R^{d+1}$ ($\hat e_0$ *is the unit vector* $(1,0)\in R\times R^d$). *Then for* $\lambda>0$ *and* $p>\dfrac{d+2}{2}$,

$$\left|E^P\left[\int_0^\infty e^{-\lambda t}f(\alpha(t))\,dt\right]\right|\leqq C_p\|f\|_{L^p(R^{d+1})},\qquad f\in C_0(R^{d+1}),$$

for some $C_p<\infty$.

Proof. As in the proof of Lemma (4.1), we see that

$$\alpha(t)-\alpha(0)=t\hat e_0+\int_0^t a^{\frac{1}{2}}(s)\,d\bar\beta(s)+\int y\,\eta(t,dy),$$

where $\bar\beta(\cdot)$ is a d-dimensional P-Brownian motion and $\eta(t,\cdot)$ is defined as in Section (1). Now set $\tau_0\equiv 0$ and define

$$\tau_{n+1}=\left(\inf\{t\geqq\tau_n\colon \eta(\tau,R^{d+1})>\eta(\tau_n,R^{d+1})\}\right)\wedge\left(\frac{[N\tau_n]+1}{N}\right).$$

Given a non-negative $f\in C_0(R^{d+1})$, we have:

$$\begin{aligned}E^P\left[\int_0^\infty e^{-\lambda t}f(\alpha(t))\,dt\right]&=\sum_0^\infty E^P\left[\int_{\tau_n}^{\tau_{n+1}}e^{-\lambda t}f(\alpha(t))\,dt\right]\\&=\sum_0^\infty E^P\Big[e^{-\lambda\tau_n}\int_{\tau_n}^\infty dt\,e^{-\lambda(t-\tau_n)}\\&\qquad\cdot\int g_{a(\tau_n)}(t-\tau_n, y-\bar\alpha_n(\tau_n))\,f(\alpha_0(\tau_0)+t-\tau_n,\bar y)\,dy\Big]\\&\leqq A\|f\|_{L^p(R^{d+1})}\sum_0^\infty E^P[e^{-\lambda\tau_n}]\end{aligned}$$

for $p>\dfrac{d+2}{2}$. Finally,

$$\begin{aligned}\sum_0^\infty E^P[e^{-\lambda\tau_n}]&=1+\lambda\int_0^\infty e^{-\lambda\tau}E^P\left[\sum_1^\infty\mathscr{X}_{[0,t)}(\tau_n)\right]dt\\&\leqq 1+\lambda NB\int_0^\infty e^{-\lambda t}t^2\,dt.\end{aligned}$$

Lemma (5.2). *Let* $a\colon [0,\infty)\times\Omega\to S_d$, $\bar{b}\colon [0,\infty)\times\Omega\to R^d$ *and* $M\colon [s,\infty)\times\Omega\times\mathscr{B}_{R^{d+1}\setminus\{0\}}\to[0,\infty)$ *be non-anticipating functions. Assume that* a, b, *and* $\int \frac{|y_0|+|y|^2}{1+|y_0|+|y|^2} M(t,dy)$ *are bounded. Let* $\alpha\colon [0,\infty)\times\Omega\to R^{d+1}$ *be a non-anticipating, right continuous function having left limits. Suppose that there exist* $T>0$, $\lambda>0$, $1<p_1<p_2<\infty$, *and* $C_p<\infty$, $p_1<p<p_2$, *such that*

$$\left|E^Q\left[\int_0^T e^{-\lambda t} f(\alpha(t))\,dt\right]\right| \leqq C_p \|f\|_{L^p(R^{d+1})}, \quad f\in C_0(R^{d+1}),$$

whenever Q *is a probability measure on* $\langle\Omega,\mathscr{M}^0\rangle$ *for which*

$$\exp\left[i\langle\theta,\alpha(t)-\alpha(0)-\int_0^t(\hat{e}_0+\bar{b}(u))\,du\rangle+\frac{1}{2}\int_0^t\langle\theta,a(u)\theta\rangle\,du - \int_0^t du\int\left(e^{i\langle\theta,y\rangle}-1-\frac{i\langle\bar{\theta},\bar{y}\rangle}{1+|\bar{y}|^2}\right)M(u,dy)\right]$$

is a Q*-martingale for all* $\theta\in R^{d+1}$. *Then for* $p_1<p<p_2$:

$$\left|E^P\left[\int_0^T e^{-\lambda t} f(\alpha(t))\,dt\right]\right| \leqq C_{\frac{p-p_1}{2}} \frac{e^{(B_p-\lambda)T}-1}{B_p-\lambda}\|f\|_{L^p(R^{d+1})},$$

whenever P *makes*

$$\exp\left[i\left\langle\theta,\alpha(t)-\alpha(0)-\int_0^t(\hat{e}_0+\bar{b}'(u))\,du\right\rangle+\frac{1}{2}\int_0^t\langle\theta,a(u)\theta\rangle\,du - \int_0^t du\int\left(e^{i\langle\theta,y\rangle}-1-\frac{i\langle\bar{\theta},\bar{y}\rangle}{1+|y|^2}\right)M(u,dy)\right]$$

into a martingale for all $\theta\in R^{d+1}$, *where* $\bar{b}'=\bar{b}+a\bar{c}$ *and* $\bar{c}\colon [0,\infty)\times\Omega\to R^d$ *is a bounded non-anticipating function. The constant* B_p *depends on the interval* (p_1,p_2) *and the bounds on* a *and* $\bar{c}$.

Proof. The proof is essentially the same as that of Lemma (4.2).

Theorem (5.1). *Let* $a\colon [0,\infty)\times\Omega\to S_d$ *and* $M\colon [0,\infty)\times\Omega\times\mathscr{B}_{R^{d+1}\setminus\{0\}}\to[0,\infty)$ *be non-anticipating functions, and let* $\alpha\colon [0,\infty)\times\Omega\to R^{d+1}$ *be a non-anticipating, right continuous function having left limits. Given a positive definite* $A\in S_d$ *and* $\frac{d+2}{2}<p_1<p_2<\infty$, *there exists an* $\varepsilon>0$ *such that*

(i) $\sup_s \|a(s)-A\|<\varepsilon$,

(ii) $\sup_s \int \frac{|y_0|+|\bar{y}|^2}{1+|y_0|+|\bar{y}|^2} M(s,dy)<\varepsilon$

imply for $p_1<p<p_2$ *and* $\lambda>0$:

$$\left|E^P\left[\int_0^\infty e^{-\lambda t} f(x(t))\,dt\right]\right| \leqq C_{p,\lambda}\|f\|_{L^p(R^{d+1})}, \quad f\in C_0(R^{d+1}),$$

whenever P is a probability measure on $\langle \Omega, \mathscr{M}^0\rangle$ *for which*

$$\exp\Big[i\langle\theta, \alpha(t)-\alpha(0)-t\hat{e}_0\rangle+\frac{1}{2}\int_0^t\langle\theta, a(u)\theta\rangle du - \int_0^t du\int\big(e^{i\langle\theta,y\rangle}-1-i\langle\bar{\theta},\bar{y}\rangle\mathscr{X}_{B(0,1)}(\bar{y})\big)M(u,dy)\Big]$$

is a P-martingale for all $\theta\in R^{d+1}$. *The* ε *depends only on the interval* (p_1, p_2) *and the largest* μ *for which* $\mu|\bar{\theta}|^2\leqq\langle\bar{\theta}, A\bar{\theta}\rangle\leqq\mu|\bar{\theta}|^2$, $\bar{\theta}\in R^d$. *The* $C_{p,\lambda}$ *depends on* ε, (p_1, p_2) *and* μ.

Proof. The outline of the proof is the same as that of Theorem (4.1). The only place that it differs at all is in the proof that the estimate holds when a and M satisfy the conditions of Lemma (5.1) as well as those of the theorem. To carry out this step, define

$$v_{\lambda,T}(\psi)=E^P\Big[\int_0^T e^{-\lambda t}\psi(\alpha(t))\,dt\Big],\qquad \psi\in C_0(R^{d+1}).$$

Then, by Lemma (5.1) and (5.2), $v_{\lambda,T}\in L^q(R^{d+1})$ for all $1<q<\frac{d+2}{2}$. We have to show that for small enough ε, the norm of $v_{\lambda,T}$ is independent of the quantities N and B in Lemma (5.1). To do this, define

$$G_\lambda\phi(x)=\int_{x_0}^\infty e^{-\lambda(y_0-x_0)}\int g_A(y_0-x_0, \bar{y}-\bar{x})\phi(y)\,dy.$$

For $\phi\in C_0^\infty(R^{d+1})$, if $f=G_\lambda\phi$, then $f\in C_b^\infty(R^{d+1})$ and

$$\Big(\lambda-\frac{\partial}{\partial x_0}-\frac{1}{2}\Delta_A\Big)f=\phi.$$

Hence

$$\begin{aligned}e^{-\lambda T}E^P[f(x(T))]-E^P[f(x(0))]&=-E^P\Big[\int_0^T e^{-\lambda t}\phi(x(t))\,dt\Big]\\&+\frac{1}{2}E^P\Big[\int_0^T e^{-\lambda t}\Big(\sum_{i,j=1}^d(a^{ij}(t)-A^{ij})\Big)\frac{\partial^2 f}{\partial x_i\partial x_j}(\alpha(t))\,dt\Big]\\&+E^P\Big[\int_0^T e^{-\lambda t}dt\int\big(f(\alpha(t)+y)-f(\alpha(t)-\langle\bar{y},\nabla_{\bar{x}}f(\alpha(t))\rangle\mathscr{X}_{B(0,1)}(\bar{y})\big)M(t,dy)\Big].\end{aligned}$$

If $\phi\geqq 0$, then $f\geqq 0$, and so (cf. the appendix for the L^p-estimates):

$$\begin{aligned}E^P\Big[\int_0^T e^{-\lambda t}\phi(x(t))\,dt\Big]&\leqq\|f\|_{L^\infty}+\frac{\varepsilon}{2}E^P\Big[\int_0^T e^{-\lambda t}\Big(\sum_1^d\Big(\frac{\partial^2 f}{\partial x_i\partial x_j}\Big)^2(\alpha(t))\Big)^{\frac{1}{2}}dt\Big]\\&+2\varepsilon E^P\Big[\int_0^T e^{-\lambda t}\sup_{|y|\leqq 1}\frac{|f(\alpha(t)+y)-f(\alpha(t))-\langle\bar{y},\nabla_{\bar{x}}f(\alpha(t))\rangle|}{|y_0|+|\bar{y}|^2}\,dt\Big]\\&+\frac{2\varepsilon}{\lambda}\|f\|_{L^\infty}\\&\leqq\Big(1+\frac{2\varepsilon}{\lambda}\Big)\frac{A_p}{\lambda^{\frac{d}{2p}+\frac{1}{q}}}\|\phi\|_{L^p}+\varepsilon\Big(\frac{A_p'}{2}+2A_p''\Big)\|\phi\|_{L^p}\|v_{\lambda,T}\|_{L^q}.\end{aligned}$$

Hence if

$$\varepsilon\left(\frac{A'_p}{2}+2A''_p\right)\leqq\frac{1}{2}$$

for $p_1<p<p_2$, then

$$\|v_{\lambda,T}\|_{L^q}\leqq 2\left(1+\frac{2\varepsilon}{\lambda}\right)\frac{A_p}{\lambda^{\frac{d}{2p}+\frac{1}{q}}}$$

for $q_2<q<q_1$. Since this estimate is independent of T, we conclude that

$$E^P\left[\int_0^\infty e^{-\lambda t}\phi(t)\,dt\right]\Bigg|\leqq 2\left(1+\frac{2\varepsilon}{\lambda}\right)\frac{A_p}{\lambda^{\frac{d}{2p}+\frac{1}{q}}}.$$

The rest of the argument is exactly like that given in the proof of Theorem (4.1).

Theorem (5.2). *Let $\mathscr{L}=L+K$, where L and K are given as in Eq.* (5.1) *and the coefficients satisfy the conditions in Eq.* (5.2). *Then the martingale problem for $\mathscr{L}$ is well-posed. Moreover, if P_x is the solution starting at x, then $\{P_x: x\in R^{d+1}\}$ is a strong Markov, temporally homogeneous family.*

Proof. Starting from Theorem (5.1), the proof of this theorem can be accomplished by exactly the same reasoning as we used to pass from Theorem (4.1) to Theorem (4.3). The only difference is that G_λ plays the role here that G^T did there.

Remark. With a little more work one can show that $\{P_x: x\in R^{d+1}\}$ is Feller continuous, even when $\bar{b}(x)$ is just bounded and measurable. However, it will no longer be true, in general, that the family is strongly Feller continuous. The same equations hold for the conditional distributions of the first jump place and time as we gave in the second remark at the end of Section (4).

Appendix

Compactness of Measures

Let $\Omega=D([0,\infty),R^d)$ be the endowed with the Skorohod metric. Then Ω becomes a complete, separable metric space, and so the Prokhorov theory applies. That is, a family $\mathscr{P}$ of probability measures on Ω is relatively weakly compact if and only if it is tight. That is for $\varepsilon>0$ there is a $K\subset\subset\Omega$ ("$\subset\subset$" is used to denote compact subsets) such that $\sup_{P\in\mathscr{P}} P(K)\geqq 1-\varepsilon$. Our purpose here is to state a sufficient condition for thightness in terms of the Lévy parameters.

Theorem (A.1). *Let I be an index set and for each $\alpha\in I$ let a_α: $[0,\infty)\times\Omega\to S_d$, b_α: $[0,\infty)\to R^d$, and M_α: $[0,\infty)\times\Omega\times\mathscr{B}_{R^d\setminus\{0\}}\to[0,\infty)$ be non-anticipating functions. Assume that:*

$$\sup_{\alpha\in I}\sup_{s\geqq 0}\|a_\alpha(s)\|\leqq A<\infty,$$

$$\sup_{\alpha\in I}\sup_{s\geqq 0}|b_\alpha(s)|\leqq A<\infty,$$

$$\sup_{\alpha\in I}\sup_{s\geqq 0}\int\frac{|y|^2}{1+|y|^2}M(s,dy)\leqq A.$$

For each $\alpha\in I$, let P_α be a probability measure on $\langle\Omega,\mathscr{M}^0\rangle$ such that

$$\exp\left[i\left\langle\theta, x(t)-x(0)-\int_0^t b_\alpha(s)\,ds\right\rangle+\tfrac{1}{2}\int_0^t\langle\theta,a_\alpha(s)\,\theta\rangle\,ds\right.$$
$$\left.-\int_0^t ds\int\left(e^{i\langle\theta,y\rangle}-1-\frac{i\langle\theta,y\rangle}{1+|y|^2}\right)M_\alpha(s,dy)\right]$$

is a P_α martingale for all θ, and assume that

$$\lim_{R\to\infty} \sup_{\alpha\in I} P_\alpha(|x(0)|\geqq R)=0.$$

Then $\{P_\alpha: \alpha\in I\}$ is relatively weakly compact.

Proof. The idea of the proof is this. Let $\tau_0\equiv 0$ and

$$\tau_{n+1}=\inf\{t\geqq\tau_n: |x(t)-x(\tau_n)|\geqq\varepsilon\}.$$

Define

$$\delta_T^\varepsilon=\inf\{\tau_n-\tau_{n-1}: \tau_{n-1}\leqq T\}.$$

We must show that

$$\lim_{\delta\downarrow 0} \sup_{\alpha\in I} P_\alpha(\delta_T^\varepsilon\leqq\delta)=0 \tag{A.1}$$

for all $T>0$ and $\varepsilon>0$. The critical step is the following observation. Let $\phi\in C_b^\infty(R^d)$ be chosen so that $0\leqq\phi\leqq 1$, $\phi(0)=0$, and $\phi\equiv 1$ off $B(0,\varepsilon)$. Given $n\geqq 0$, let $P_{\alpha,\omega}^{(n)}$ be the r.c.p.d. of $P_\alpha|\mathscr{M}_{\tau_n}^0$. Then (a.s., P_α)

$$\phi\big(x(t\wedge\tau_{n+1})-x(\tau_n(\omega))\big)-\int_{\tau_n(\omega)}^{t\wedge\tau_{n+1}} \mathscr{L}_{\alpha,u}\,\phi\big(x(u)-x(\tau_n(\omega))\big)\,du$$

is a $P_{\alpha,\omega}^{(n)}$-martingale. Thus:

$$\begin{aligned}
P_{\alpha,\omega}^{(n)}\big(\tau_{n+1}-\tau_n(\omega)\leqq\delta\big)&\leqq P_{\alpha,\omega}^{(n)}\big(|x(\tau_{n+1}\wedge(\tau_n(\omega)+\delta))-x(\tau_n(\omega))|\geqq\varepsilon\big)\\
&\leqq E^{P_{\alpha,\omega}^{(n)}}\big[\phi\big(x(\tau_{n+1}\wedge(\tau_n(\omega)+\delta))-x(\tau_n(\omega))\big)\big]\\
&\leqq E^{P_{\alpha,\omega}^{(n)}}\left[\int_{\tau_m(\omega)}^{\tau_n(\omega)+\delta} |\mathscr{L}_u\,\phi(x(u))|\,du\right]\leqq C\delta.
\end{aligned}$$

The constant C here depends only on the bound A and the C^2-norm of ϕ.

From this estimate, it is reasonably easy to get (A.1).

L^p-Estimates

The estimates which we want to derive are:

$$|G^T\phi(s,x)|\leqq A_p(T-s)^{\frac{1}{q}-\frac{d}{2p}}\,\|\phi\|_{L^p([s,T)\times R^d)}, \qquad p>\frac{d+2}{2}, \tag{A.2}$$

$$\left\|\left(\sum_1^d\left(\frac{\partial G^T\phi}{\partial x_i\,\partial x_j}\right)^2\right)^{\frac{1}{2}}\right\|_{L^p([0,T)\times R^d)}\leqq A_p'\,\|\phi\|_{L^p([0,T)\times R^d)}, \qquad 1<p<\infty, \tag{A.3}$$

$$\left\|\sup_{|y|>0}\frac{G^T\phi(t,x+y)-G^T\phi(t,x)-\langle y,\nabla_x G^T\phi(x)\rangle}{|y|^2}\right\|_{L^p([0,T]\times R^d)}$$

$$\leqq A_p''\,\|\phi\|_{L^p([0,T]\times R^d)}, \qquad \frac{d+2}{2}<p<\infty, \tag{A.4}$$

and

$$|G_\lambda \phi(x)| \leqq A_p \lambda^{\frac{1}{q}-\frac{d}{2p}} \|\phi\|_{L^p(R^{d+1})}, \qquad p > \frac{d+2}{2}, \tag{A.5}$$

$$\left\| \left(\sum_1^d \left(\frac{\partial G_\lambda \phi(x)}{\partial x_i \partial x_j} \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p(R^{d+1})} \leqq A'_p \|\phi\|_{L^p(R^{d+1})}, \qquad 1 < p < \infty, \tag{A.6}$$

$$\left\| \sup_{y \in R^{d+1}} \frac{G_\lambda \phi(x+y) - G_\lambda \phi(x) - \langle \bar{y}, \nabla_{\bar{x}} G_\lambda \phi(x) \rangle}{|y_0| + |\bar{y}|^2} \right\|_{L^p(R^{d+1})} \leqq A''_p \|\phi\|_{L^p(R^{d+1})}, \qquad \frac{d+2}{2} < p < \infty. \tag{A.7}$$

Here

$$G^T \phi(s, x) = \int_s^T dt \int_{R^d} g_A(t-s, y-x) \phi(y) \, dy,$$

$$G_\lambda \phi(x) = \int_{x_0}^\infty e^{-\lambda(y_0 - x_0)} dy_0 \int_{R^d} g_A(y_0 - x_0, \bar{y} - \bar{x}) \phi(y) \, d\bar{y},$$

$$g_A(t, x) = \frac{\mathscr{X}_{[0,\infty)}(t)}{(2\pi t)^{d/2} (\det A)^{\frac{1}{2}}} e^{-\langle x, A^{-1} x \rangle / 2t},$$

and $A \in S_d$ is positive definite. The constants A_p, A'_p and A''_p depend only on the greatest and least eigen values of A. (This can be seen by making the obvious change of coordinates.) Thus we will always take A to be the identity matrix and will drop the subscript on g_A.

The inequalities (A.2) and (A.5) are easy consequences of Hölder's inequality and don't warrant further comment. Inequality (A.4) follows from (A.7); and (A.3) and (A.6) are really the same thing. Thus we will devote our attention to (A.6) and (A.7). Actually, (A.6) is a special case of a singular integral result first derived by B. Frank Jones [3]. In order to discuss his and related results, we use the function $\rho(x)$ on R^{d+1} defined by

$$\rho(x) = \left(\frac{|\bar{x}|^2 + (|\bar{x}|^4 + 4x_0^2)^{\frac{1}{2}}}{2} \right)^{\frac{1}{2}}.$$

This function was introduced by Fabes and Rivière [2] in their work on singular integrals with mixed homogeneity. Its importance is that $\rho(x)$ is the number ρ satisfying:

$$\frac{x_0^2}{\rho^4} + \frac{|\bar{x}|^2}{\rho^2} = 1,$$

and so if f is a function of R^{d+1} with parabolic homogeneity of order α (i.e. $f(\lambda^2 x_0, \lambda \bar{x}) = \lambda^\alpha f(x_0, x)$, $\lambda > 0$), then

$$f(x) = (\rho(x))^\alpha f\left(\frac{x_0}{\rho^2(x)}, \frac{x}{\rho(x)} \right).$$

That is, $f(x)$ is the product of $\rho^\alpha(x)$ and a function on the d-sphere. For instance, any of the functions:

$$k_0(x)=\frac{\partial}{\partial x_0} g(x_0,\bar{x}),$$

$$k_{ij}(x)=\frac{\partial^2}{\partial x_i\,\partial x_j} g(x_0,\bar{x}), \quad i,j\geqq 1,$$

can be written in the form

$$k_0(x)=\frac{\Gamma_0(x)}{(\rho(x))^{d+2}},$$

$$k_{ij}(x)=\frac{\Gamma_{ij}(x)}{(\rho(x))^{d+2}}$$

where Γ_0 and Γ_{ij} have parabolic homogeneity of degree 0. It is easy to see that Γ_0 and Γ_{ij} are smooth away from the origin, and so for all $x\in R^{d+1}$:

$$\int\limits_{\rho(y)\geqq 2\rho(x)} |k_0(x-y)-k_0(-y)|\,dy\leqq B,$$

$$\int\limits_{\rho(y)\geqq 2\rho(x)} |k_{ij}(x-y)-k_{ij}(-y)|\,dy\leqq B.$$

Moreover, we can compute $\hat{k}_0$ and $\hat{k}_{ij}$ explicitly:

$$\hat{k}_0(\xi)=\frac{i\xi_0}{(i\xi_0+|\xi|^2)},$$

$$\hat{k}_{ij}(\xi)=\frac{-\xi_i\,\xi_j}{(i\xi_0+|\xi|^2)}.$$

In particular, these are bounded. Hence, by Theorem (1) in Fabes and Riviére [2] (cf. Stroock [9] for a more probabilistic proof), we have:

$$\begin{aligned} &\|k_0*\phi\|_{L^p(R^{d+1})}\leqq C_p\,\|\phi\|_{L^p(R^{d+1})}, \quad 1<p<\infty,\\ &\|k_{ij}*\phi\|_{L^p(R^{d+1})}\leqq C_p\,\|\phi\|_{L^p(R^{d+1})}, \quad 1<p<\infty, \end{aligned} \tag{A.8}$$

where "$*$" stands for convolution (in the sense of principle value evaluation). The inequality (A.6) follows from this once one observes that

$$\frac{\partial^2 G_\lambda\phi}{\partial x_i\,\partial x_j}=k_{ij}*(\phi-\lambda G_\lambda\phi) \quad\text{and}\quad \|\lambda G_\lambda\phi\|_p\leqq\|\phi\|_p, \quad 1\leqq p\leqq\infty.$$

In order to obtain (A.7), we need a refinement of inequality (A.8) due to Riviére [6]. For $\varepsilon>0$, define

$$k_0^{(\varepsilon)}(x)=\mathscr{X}_{[\varepsilon,\infty)}(\rho(x))\,k_0(x),$$

$$k_{ij}^{(\varepsilon)}(x)=\mathscr{X}_{[\varepsilon,\infty)}(\rho(x))\,k_{ij}(x).$$

The theorem of Riviére is that

$$\|\sup_{\varepsilon>0}|k_0^{(\varepsilon)}*\phi|\,\|_{L^p}\leqq C_p\|\phi\|_{L^p}, \quad 1<p<\infty,$$

$$\|\sup_{\varepsilon>0}|k_{ij}^{(\varepsilon)}*\phi|\,\|_{L^p}\leqq C_p\|\phi\|_{L^p}, \quad 1<p<\infty.$$

The proof of (A.9), as well as of (A.7), turns on the following variation on the Hardy-Littlewood inequality.

Lemma (A.1) (*cf. Rivière* [6]). *For* $\delta>0$, *let* $B_\rho(\delta)=\{x\in R^{d+1}: \rho(x)<\delta\}$. *Given* $f\in L^1(R^{d+1})\to L^1(R^{d+1})$

$$M_\rho f(x)=\sup_{\delta>0}\frac{1}{|B_\rho|}\int_{B_\rho}|f(x+y)|\,dy,$$

where $|S|$ *is used to denote the Lebesgue measure of the set S. Then*

$$|\{x: M_\rho f(x)\geqq\lambda\}|\leqq\frac{2^{d+1}}{\lambda}\|f\|_{L(R^{d+1})},\qquad \lambda>0.$$

Because the proof of this special case is considerably easier than the general case treated by Rivière, we will include here the derivation of (A.9) from this lemma.

Proof of (A.9). We will carry out the proof for k_0. The first thing we need is to show that:

$$\Big|\int_{\rho(y)\geqq 2\varepsilon}\big(k_0(y-\xi)-k_0(y)\big)f(y+x)\,dy\Big|\leqq CM_\rho f(x) \tag{A.10}$$

for $\rho(\xi)\leqq\varepsilon$. To do this, we note that when $\rho(\xi)\leqq\varepsilon$ and $\rho(y)\geqq 2\varepsilon$:

$$|k_0(y-\xi)-k_0(y)|\leqq C_1\left(\frac{\varepsilon^2}{(\rho(y))^{d+4}}+\frac{\varepsilon}{(\rho(y))^{d+3}}\right).$$

Changing coordinates to $y_0=\rho^2\omega_0$, $y_1=\rho\,\omega_1,\dots,y_2=\rho\,\omega_d$, $\rho>0$ and $\omega\in\sum_{d+1}$, we see that $dy=\rho^{d+1}J(\omega)\,d\rho\,d\omega$, where $J(\omega)\in C^\infty(\sum_{d+1})$. Hence

$$\Big|\int_{\rho(y)\geqq 2\varepsilon}\big(k_0(y-\xi)-k_0(y)\big)f(y+x)\,dy\Big|$$

$$\leqq C_2\left(\varepsilon^2\int_{2\varepsilon}^{\infty}\sigma^{-3}\frac{1}{\sigma^{d+1}}\frac{d}{d\sigma}\Big(\int_{B_\rho}|f(x+y)|\,dy\Big)d\sigma+\varepsilon\int_{2\varepsilon}^{\infty}\sigma^{-2}\frac{1}{\sigma^{d+1}}\frac{d}{d\sigma}\Big(\int_{B_\rho(\sigma)}|f(x+y)|\,d\sigma\Big)d\sigma\right),$$

and integrating by parts we obtain (A.10).

We now write

$$I_0\equiv\int_{\rho(y)\geqq 2\varepsilon}k_0(y)f(y+x)\,dy=-\int_{\rho(y)\geqq 2\varepsilon}\big(k_0(y-\xi)-k_0(y)\big)f(y+x)\,dy$$
$$-\int k_0(y)f(y+\xi+x)\,dy+\int_{\rho(y)\leqq 2\varepsilon}k_0(y)f(y+\xi+x)\,dy=I_1(\xi)+I_2(\xi)+I_3(\xi).$$

Integrate both side as a function of ξ over $B_\rho(\varepsilon)$ and dividing by $|B_\rho(\varepsilon)|$, we get:

$$|I_0|\leqq\frac{1}{|B_\rho(\varepsilon)|}\int_{B_\rho(\varepsilon)}|I_1(\xi)|\,d\xi+\frac{1}{|B_\rho(\varepsilon)|}\int_{B_\rho(\varepsilon)}|I_2(\xi)|\,d\xi+\frac{1}{|B_\rho(\varepsilon)|}\int_{B_\rho(\varepsilon)}|I_3(\xi)|\,d\xi.$$

By (A.10),

$$\frac{1}{|B_\rho(\varepsilon)|}\int_{B_\rho(\varepsilon)}|I_1(\xi)|\,d\xi\leqq CM_\rho f(x)\leqq C\big(M_\rho|f|^p(x)\big)^{1/p}.$$

Also

$$\frac{1}{|B_\rho(\varepsilon)|}\int_{B_\rho(\varepsilon)}|I_2(\xi)|\,d\xi \leqq \left(\frac{1}{|B_\rho(\varepsilon)|}\int_{B_\rho(\varepsilon)}|k_0 * f|^p(\xi+x)\,d\xi\right)^{1/p}$$
$$\leqq (M_\rho |k_0 * f|^p(x))^{1/p}$$

and

$$\frac{1}{|B_\rho(\varepsilon)|}\int_{B_\rho(\varepsilon)}|I_3(\xi)|\,d\xi \leqq \left(\frac{1}{|B_\rho(\varepsilon)|}\int_{B_\rho(\varepsilon)}|k_0 *(\mathscr{X}_{x+B_\rho(2\varepsilon)} f)|^p(\xi+x)\,d\xi\right)^{1/p}$$
$$\leqq \left(\frac{1}{|B_\rho(\varepsilon)|}\int_{R^d}|k_0 *(\mathscr{X}_{x+B_\rho(2\varepsilon)} f)(\xi+x)|^p\right)^{1/p}$$
$$\leqq C_p\left(\frac{1}{|B_\rho(\varepsilon)|}\|\mathscr{X}_{x+B_\rho(2\varepsilon)} f\|_{L^p}^p\right)^{1/p} \leqq C_p 2^{\frac{d+2}{p}}(M_\rho |f|^p(x))^{1/p}.$$

Hence

$$\sup_{\varepsilon>0}|k_0^{(\varepsilon)} * f|(x) \leqq C_p'(M_\rho |f|^p(x))^{1/p}, \qquad 1<p<\infty.$$

But, by Lemma (A.1),

$$|\{x\colon (M_\rho |f|^p(x))^{1/p} \geqq \lambda\}| \leqq \frac{2^{d+1}}{\lambda^p}\|f\|_{L^p}^p,$$

and so, by the Marcinkiewiez interpolation theorem, (A.9) follows.

Theorem (A.2). *Let* $\phi\in C_0^\infty(R^{d+1})$ *and* $f(x)=g*\phi(x)$. *Then for* $\frac{d+2}{2}<p<\infty$,

$$\left\|\sup_{y\in R^{d+1}}\left|\frac{f(x+y)-f(x)-\langle\bar{y}, \nabla_{\bar{x}} f(x)\rangle}{(\rho(y))^2}\right|\right\|_{L^p(R^{d+1})} \leqq C_p\|\phi\|_{L^p(R^{d+1})}.$$

Proof. Let $y=(\rho^2\gamma_0, p\bar{\gamma})$, $\gamma=(\gamma_0, \bar{\gamma})\in\sum_{d+1}$.

$$\frac{f(x+y)-f(x)-\langle\bar{y}, \nabla_{\bar{x}} f(x)\rangle}{(\rho(y))^2}$$
$$=\frac{1}{\rho^2}\int_{\rho(\xi)\leqq 2\rho} g(\xi_0-\rho^2\gamma_0, \bar{\xi}-\rho^{\bar{\gamma}})\,\phi(\xi+g)\,d\xi$$
$$-\frac{1}{\rho^2}\int_{\rho(\xi)\leqq 2\rho} g(\xi)\,\phi(\xi+x)\,d\xi$$
$$-\frac{1}{\rho}\int_{\rho(\xi)\leqq 2\rho}\langle\bar{\gamma}, \nabla_{\bar{x}} g(\xi)\rangle\,\phi(\xi+x)\,d\xi$$
$$+\frac{1}{\rho^2}\int_{\rho(\xi)\geqq 2\rho}\left(g(\xi_0-\rho^2\gamma_0, \bar{\xi}-\rho\bar{\gamma})-g(\xi)-\rho\langle\bar{\gamma}, \nabla_{\bar{x}} g(\xi)\rangle\right)\phi(\xi+x)\,d\xi$$
$$=I_1(x)+I_2(x)+I_3(x)+I_4(x).$$

For $\frac{d+2}{2} < p < \infty$:

$$|I_1(x)| \leqq \frac{1}{\rho^2} \Big(\int_{\rho(\xi) \leqq 2\rho} |g(\xi_0 - \rho^2 \gamma_0, \bar{\xi} - \rho^{\bar{\gamma}})|^q \, d\xi \Big)^{1/q} \Big(\int_{\rho(\xi) \leqq 2\rho} |\phi(\xi + x)|^p \, d\xi \Big)^{1/p}$$

$$\leqq \frac{B_p}{\rho^2} \left(\int_0^{3\rho} \sigma^{d(1-q)+1} \, d\sigma \right)^{1/q} \Big(\int_{\rho(\xi) \leqq 2\rho} |\phi(\xi + x)|^p \Big)^{1/p}$$

$$\leqq \frac{B'_p}{\rho^2} \rho^{\frac{2}{q} - \frac{d}{p}} \Big(\int_{\rho(\xi) \leqq 2\rho} |\phi(\xi + x)|^p \Big)^{1/p} \leqq B''_p M_\rho |\phi|^p(x).$$

Similarly,

$$|I_2(x)| \leqq B''_p M_\rho |\phi|^p(x).$$

Also,

$$|I_3(x)| \leqq \frac{1}{\rho} \int_{\rho(\xi) \leqq 2\rho} |\nabla_{\bar{x}} g(\xi)| \, |\phi(\xi + x)| \, d\xi \leqq \frac{C}{\rho} \int_0^{2\rho} \frac{1}{\sigma^{d+1}} \Big(\int_{B_\rho(\sigma)} |\phi(x + \xi)| \, d\xi \Big) \, d\sigma$$

$$\leqq C' M_\rho \phi(x) \leqq C' (M_\rho |\phi|^p)^{1/p}.$$

Finally, by Taylor's Theorem:

$$\big(g(\xi_0 - \rho^2 \gamma_0, \xi - \rho\bar{\gamma}) - g(\xi) - \rho \langle \bar{\gamma}, \nabla_{\bar{x}} g(\xi) \rangle\big)$$

$$= \rho^2 \gamma_0 k_0(\xi) + \frac{\rho^2}{2} \sum_1^d k_{ij}(\xi) + \rho^4 \Delta_1(\xi) + \rho^3 \Delta_2(\xi),$$

where $|\Delta_1(\xi)| \leqq \frac{D_1}{(\rho(\xi))^{d+4}}$ and $|\Delta_2(\xi)| \leqq \frac{D_2}{(\rho(\xi))^{d+3}}$. Hence,

$$|I_4(x)| \leqq \Big| \int_{\rho(\xi) \geqq 2\rho} k_0(\xi) \, \phi(\xi + x) \, d\xi \Big| + \tfrac{1}{2} \sum_1^d \Big| \int_{\rho(\xi) \geqq 2\rho} \Delta_2(\xi) \, \phi(\xi + x) \, d\xi \Big|$$

$$+ \rho^2 \Big| \int_{\rho(\xi) \geqq 2\rho} \Delta_1(\xi) \, \phi(\xi + x) \, d\xi \Big| + \rho \Big| \int_{\rho(\xi) \geqq 2\rho} \Delta_2(\xi) \, \phi(\xi + x) \, d\xi \Big|.$$

The supremum over ρ of the first two terms was estimated in Theorem (A.2). The last two terms are handled in the same way as we handled $I_1(x)$. The rest of the proof is completed in the same way as we completed the proof of Theorem (A.2).

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(Received July 5, 1974)