Z. Wahrscheinlichkeitstheorie verw. Gebiete
64, 1–5 (1983)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Some Exchangeable Probabilities are Singular With Respect to All Presentable Probabilities

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Introduction

As has been reported in Dubins and Freedman (1979), there are exchangeable probabilities which are not presentable, that is, there are some which are not mixtures of power probabilities. The exchangeable probabilities constructed in Dubins and Freedman (1979) are, in fact, singular with respect to all presentable probabilities. This will become evident once a simple condition given below, meaningful for members of a certain class of exchangeable processes, is shown to be necessary and sufficient for one of its members to be singular with respect to all presentable processes.

(In order for an exchangeable probability, P, not to be a mixture of power probabilities, it is necessary that P be defined for events or random entities that are not determined by any finite number of coordinates. This necessity is formally demonstrated in Dubins (1981); a weaker version of this necessity is in Hewitt and Savage (1955), and it is implicit in de Finetti's earlier work on exchangeability.)

§ 1. Statement of the Condition for Singularity with Respect to Presentable Probabilities

Let S be a subset of the closed unit interval I endowed with the sigma field $\mathscr{S}$ of its Borel subsets, and let $\mathscr{P}(S)$ be the countably additive probabilities on $(S, \mathscr{S})$. Let $\mathscr{S}^{(2)}$ be the smallest sigma-field of subsets of $\mathscr{P}(S)$ such that, for each $A \in \mathscr{S}$, the map: $\phi \to \phi A$ of $\mathscr{P}(\mathscr{S}) \to \mathbb{R}$ is measurable and, using a notational device introduced by de Finetti, let $\mathscr{P}^2(S)$ be the set of all countably additive probabilities μ on $(\mathscr{P}(\mathscr{S}), \mathscr{S}^{(2)})$. Each $\mu \in \mathscr{P}^2(\mathscr{L})$ will, in this note, be called a *prior*, or an *S-prior* when greater precision is needed.

As usual, S^∞ denotes the cartesian-product of a denumerable number of copies of S, and $\mathscr{S}^\infty$ is the corresponding product of copies of $\mathscr{S}$.

* Research supported by National Science Foundation grant no. MCS-80-02535

For f a bounded, measurable, $\mathbb{R}$-valued function defined on S^∞ and $\phi \in \mathscr{P}(S)$, let $\hat{f}\phi$ be the expectation of f under the power probability ϕ^∞.

As writers subsequent to de Finetti use the term, for a probability P on $(S^\infty, \mathscr{S}^\infty)$ to be *exchangeable*, or *S-exchangeable*, it must not only be invariant under the usual permutation group, but it must also be countably additive and defined for all elements of $\mathscr{S}^\infty$.

For $\mu \in \mathscr{P}^2(S)$, let $\beta\mu \in \mathscr{P}(S^\infty)$ be defined, thus.

$$(\beta\,\mu)(f) = \mu \hat{f}, \tag{1.1}$$

for all bounded Borel, $\mathbb{R}$-valued functions, f, defined on S^∞. In (1.1), the convention, also introduced by de Finetti, of using the same symbol for a probability measure as for the expectation it determines is, of course, being used. In more familiar, but less compact notation,

$$(\beta\,\mu) f = \int \phi^\infty f\, \mu(d\phi) \tag{1.1*}$$

or

$$\beta\,\mu = \int \phi^\infty\, \mu(d\phi).$$

As accords with the definition in Hewitt and Savage (1955), an S-exchangeable, P, is *S-presentable* if

$$P = \beta\,\mu \quad \text{for some } \mu \in \mathscr{P}^2(S). \tag{1.2}$$

For $\mu \in \mathscr{P}^2(I)$ and $\mathbb{B}$ ranging over the Borel subsets of I^∞, the formula

$$P(\mathbb{B} \cap S^\infty) = (\beta\,\mu)(\mathbb{B}) \tag{1.3}$$

defines a countably additive probability, P, on $(S^\infty, \mathscr{S}^\infty)$ if, and only if, $(\beta\mu)(\mathbb{B}) = 1$ whenever $\mathbb{B} \supset S^\infty$. When this holds, $\beta\,\mu$ is said to have a *trace* on S^∞, and P is its trace.

(1.4) **Theorem.** *Let $\mu \in \mathscr{P}^2(I)$, $S \subset I$, such that $\beta\,\mu$ has a trace on S^∞, say P. Then, for P to be singular with respect to every S-presentable probability, it is necessary and sufficient that the set, Φ, of $\phi \in \mathscr{P}(I)$ under which S has outer probability one, be a μ-null set.*

(The meaning intended by various terms used in the statement of Theorem (1.4) is probably clear, but to dissipate certain possible ambiguities, they are defined thus: Countably additive P and Q are *mutually singular* if, for some event f, $P(f) = Q(1-f) = 1$. For $\phi \in \mathscr{P}(I)$ and $S \subset I$, S has *outer probability* one under ϕ if each Borel subset of I which includes S has ϕ-probability one. Finally, a set, N, is a null set under μ if N must receive probability zero under every nonnegative finitely additive extension of μ to N.)

§ 2. Preliminaries to the Proof of Theorem (1.4)

For $\theta \in \mathscr{P}(S)$, let $l\theta \in \mathscr{P}(I)$ be defined for Borel subsets B of I by:

$$(l\,\theta)(B) = \theta(B \cap S). \tag{2.1}$$

Verify that ϕ is in the range of l if, and only if, the outer ϕ-probability of S, namely $\phi^* S$, is 1; that is, if and only if $\phi \in \Phi$.

Similarly, for each $P \in \mathscr{P}(S^\infty)$, let lP be defined for Borel subsets $\mathbb{B}$ of I^∞ by:

$$(lP)(\mathbb{B}) = P(\mathbb{B} \cap S^\infty). \tag{2.2}$$

Finally, for $\gamma \in \mathscr{P}^2(S)$, let $l\gamma \in \mathscr{P}^2(I)$ be defined as the γ-distribution of l, Plainly, $\mu' \in \mathscr{P}^2(I)$ is $l\gamma$ for some $\gamma \in \mathscr{P}^2(S)$ if, and only if, for every Borel subset Π of $\mathscr{P}(I)$ which includes Φ, $\mu'(\Pi) = 1$.

With the conventions now in force, the l operation commutes with the operation of taking cartesian products and with the barycentric operator β, that is,

$$l(\theta^\infty) = (l\,\theta)^\infty \qquad (\forall \theta \in \mathscr{P}(S)), \tag{2.3}$$

and

$$(l\,\beta)\,\gamma = (\beta\,l)\,\gamma \qquad (\forall \gamma \in \mathscr{P}^2(S)). \tag{2.4}$$

The last equality can be visualized in terms of the commutative diagram:

$$\begin{array}{ccc} \mathscr{P}^2(S) & \xrightarrow{\;l\;} & \mathscr{P}^2(I) \\ \beta \big\downarrow & & \big\downarrow \beta \\ \mathscr{P}(S) & \xrightarrow[\;l\;]{} & \mathscr{P}(I) \end{array}$$

(2.5) **Lemma.** *For μ and μ' elements of $\mathscr{P}^2(I)$, these two assertions are equivalent:*

(a) *μ and μ' are mutually singular;*

(b) *$\beta\,\mu$ and $\beta\,\mu'$ are mutually singular.*

Proof. Suppose (b) holds. Then, for some Borel subset $\mathbb{B}$ of I^∞, $(\beta\,\mu)(\mathbb{B}) = 0$ and $(\beta\,\mu')(\mathbb{B}) = 1$. That is,

$$\int \phi^\infty(\mathbb{B})\,\mu(d\phi) = 0 = 1 - \int \phi^\infty(\mathbb{B})\,\mu'(d\phi). \tag{2.6}$$

Since μ and μ' are countably additive probability measures, the set of ϕ such that $\phi^\infty(\mathbb{B}) = 1$ has μ-measure zero but μ'-measure 1. Therefore, (a) holds.

Now suppose that (a) holds. For some Borel subset, Π, of $\mathscr{P}(I)$, $\mu(\Pi) = 0$ and $\mu'(\Pi) = 1$. Let $\mathbb{B}$ be that subset of I^∞ consisting of all $(x_1, x_2, \ldots)$ whose empirical distribution converges to a $\phi \in \Pi$, that is, such that, for all $f \in C(I)$, $\frac{1}{n} \sum f(x_i)$ $(1 \leqq i \leqq n)$ converges to $\int_0^1 f(t)\,\phi(dt)$. Apply the strong law of large numbers to each f in a countable dense subset of $C(I)$ to conclude that $\mathbb{B}$ has probability zero under $\beta\,\mu$, but probability one under $\beta\,\mu'$. Hence (b) obtains. □

For future reference, record here this easily verified fact.

(2.7) **Lemma.** *For P and Q elements of $\mathscr{P}(S)$, these two assertions are equivalent:*

(a) *P and Q are mutually singular;*
(b) *lP and lQ are mutually singular.*

(2.8) **Lemma.** *Let μ be a probability, Φ a set, and suppose that μ is singular with respect to every probability that assigns Φ outer probability one. Then Φ is a null set for μ.*

Proof. Suppose Φ is not null for μ, let $\varepsilon>0$ be the infimum of the μ-probability of all measurable sets which include Φ, and let A be a measurable set, $A\supset\Phi$ for which $\mu(A)=\varepsilon$. Let μ' be a probability measure which vanishes off A and is proportional to μ on A. Then Φ has outer-probability one under μ', but μ and μ' are not mutually singular, for their infimum is simply μ on A and zero off A. □

§ 3. Proof of Theorem (1.4)

Sufficiency. Suppose the condition holds. So, for some Borel subset Π of $\mathscr{P}(I)$, Π includes Φ and $\mu(\Pi)=0$. Let Q be S-presentable or, equivalently, let Q be $\beta\gamma$ for some S-prior γ, and let μ' be the I-prior $l\gamma$. As noted early in § 2, $\mu'(\Pi)=1$. So μ and μ' are mutually singular. As Lemma (2.5) now implies, $\beta\mu$ and $\beta\mu'$ are mutually singular. Next, note that

$$\beta\mu'=\beta l\gamma=l\beta\gamma=lQ, \tag{3.1}$$

where the second equality holds in view of the general commutativity relation (2.4). Since $\beta\mu$ is, of course, lP, lP and lQ are mutually singular. Now Lemma (2.7) applies.

Necessity. Suppose $\mu\in\mathscr{P}^2(I)$, $P\in\mathscr{P}(S^\infty)$, $lP=\beta\mu$, and P is singular with respect to every S-presentable Q, or equivalently, suppose P is singular with respect to $\beta\gamma$ for all $\gamma\in\mathscr{P}^2(S)$. By Lemma (2.7), lP is singular with respect to $l\beta\gamma$ for all $\gamma\in\mathscr{P}^2(S)$. By the commutativity relation (2.4), lP is singular with respect to $\beta l\gamma$ for all γ or, equivalently, lP is singular with respect to $\beta\mu'$ for every $\mu'\in\mathscr{P}^2(I)$ which assigns outer-probability one to Φ. Since $lP=\beta\mu$, $\beta\mu$ is singular with respect to all such $\beta\mu'$. As is implied by Lemma (2.5), μ is singular with respect to all such μ'. Hence by Lemma (2.8), Φ is a null set for μ. □

§ 4. Existence of (μ, S) Which Satisfy the Condition of Theorem (1.4)

Call $\mu\in\mathscr{P}^2(I)$ *doubly nonatomic* if μ is nonatomic and the set of *nonatomic* $\phi\in\mathscr{P}(I)$ has probability 1 under μ. Call a Borel subset, Σ, of $\mathscr{P}(I)$ *distal* if there is a Borel map ψ of I onto Σ such that, for each $\phi\in\Sigma$, the set, B_ϕ, of all $x\in I$ such that $\psi(x)=\phi$ has ϕ-probability one; and call $\mu\in\mathscr{P}^2(\mathrm{I})$ *distal* if, for some distal subset Σ of $\mathscr{P}(I)$, $\mu(\Sigma)=1$.

(4.1) **Theorem.** *Suppose $\mu\in\mathscr{P}^2(I)$ is distal and doubly nonatomic. Then there is a subset, S, of I such that the trace of $\int\phi^\infty\,\mu(d\phi)$ on S^∞ is an exchangeable process which is singular with respect to each countably additive S-presentable process.*

Proof of Theorem (4.1). A transfinite argument given in Dubins and Freedman (1979) delivers an S with these two properties: (a) S^∞ has outer probability one under $\beta\mu$; (b) for μ-almost all ϕ, the intersection of S with B_ϕ is a countable subset of I; hence, for μ-almost all ϕ, S is a ϕ-null set. By (a), the trace of $\beta\mu$ on S^∞ is an S-exchangeable probability P. By (b), the condition of Theorem (1.4) obtains. □

In view of Theorem (4.1), to see that some exchangeable processes are singular with respect to all presentable ones, it is only necessary to see that there is a $\mu\in\mathscr{P}^2(I)$ which is distal and doubly nonatomic. That there are such becomes evident by considering any nonatomic μ which is distributed over the set of biased-coin distributions, as in Dubins and Freedman (1979).

For a conclusion related to Theorem (1.4), see Freedman (1980).

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Received November 11, 1981