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Last Exit Decompositions and Regularity at the Boundary of Transition Probabilities

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Summary. The purpose of this paper is to give a probabilistic approach to studying the regularity at the boundary of the transition probabilities of certain hypoelliptic diffusions with boundary conditions. The main tools are last exit decompositions of Brownian motion, the Malliavin calculus, the theory of excursions, and the calculus of variations on Brownian excursions.

The purpose of this paper is to show how it is possible to use probabilistic methods to prove the regularity at the boundary of the transition densities of certain hypoelliptic diffusions. We essentially use the Malliavin calculus of variations [3, 14, 21–23, 27, 30–32] and the calculus of variations on Brownian excursions which we developed in [6].

Before going into details, we first explain how analysts proceed to study the regularity at the boundary of the solutions of certain partial differential equations with boundary conditions. We here follow Treves [34] – Chap. III. Namely, let P be a second order differential operator in the variables $(z,x)\in R^+ \times R^n$ which can be written as

$$P=\frac{\partial^2}{\partial z^2}+Q \tag{0.1}$$

where Q is an elliptic operator of degree 2, whose partial degree in z is $\leqq 1$.

Modulo regularizing operators [34], P can be factored as

$$P\sim\left(\frac{\partial}{\partial z}+P^-\right)\left(\frac{\partial}{\partial z}+P^+\right). \tag{0.2}$$

Using (0.2), boundary problems on the partial differential operator P in the region $R^+\times R^n$ can be transformed into a problem of the type

$$\begin{aligned}&\left(\frac{\partial}{\partial z}+P^+\right)u=v \qquad (\mathscr{B}u)_{z=0}=g\\ &\left(\frac{\partial}{\partial z}+P^-\right)v=Ru+f\end{aligned} \tag{0.3}$$

where $\mathscr{B}$ is the so-called boundary operator of the problem, and R is a regularizing operator. The analysis of the boundary problem is then easier in the form (0.3).

In the simple case where Q is the Laplacian Δ in the variable $x\in R^n$, (0.2) is the exact

$$\frac{\partial^2}{\partial z^2}+\Delta=\left(\frac{\partial}{\partial z}-\sqrt{-\Delta}\right)\left(\frac{\partial}{\partial z}+\sqrt{-\Delta}\right). \tag{0.4}$$

Now a factorization of the type (0.2) has some obvious formal connections with the Wiener-Hopf factorization of the generators of certain independent increment processes studied by Prabhu [25], Greenwood and Pitman [12], Silverstein [28], and also considered by H. Kaspi [19] for certain Markov processes with discrete state space.

In this paper, we will obtain a factorization of hypoelliptic operators in the form (0.2) which is associated to a certain Williams decomposition of the trajectories of Brownian motion in terms of its local maxima. It is then natural to put this description of Brownian motion at work simultaneously with the Malliavin calculus of variations and the calculus of variations on Brownian excursions, to give a direct proof of the regularity at the boundary of the transition probabilities of certain hypoelliptic diffusions with boundary conditions (this problem is not covered by the technique of (0.2)–(0.3)). The product form of the generator is then a consequence of one Williams decomposition, but is not used as such in the proof of regularity, since we directly work on the paths.

Also note that in [12, 19, 25, 28], techniques of duality on Markov processes are used to establish a factorization of the type (0.2). We will instead use excursion theory on Brownian motion, which lends itself to an easy analytic treatment, without any potential theoretic formalism. Still the reader will notice some obvious connections with results on the time reversal of Brownian motion. In particular the result of Pitman [24], stating that if z is a reflecting Brownian motion and L its standard local time at 0, $z+L$ is a Bes(3) process, plays a key role in our factorization of the form (0.2). Of course, it is known by Ikeda-Watanabe [14], Williams [36] that Pitman's result is strongly connected with the time reversal of Brownian motion.

The paper is divided in three sections. In Sect. 1, we calculate the excursion law of the Brownian motion out of 0, by assuming that a certain time t is chosen at random (using the Lebesgue measure) during the excursion. This leads us to still another description of the excursion law of Brownian motion, besides the one by Lévy-Itô-McKean [15], and the other by Williams [35–36], Rogers [26].

In Sect. 2, we exhibit several explicit factorizations of second order differential operators, using local maxima or last exit decompositions.

If P_0 denotes the law of the Brownian motion z starting at 0, local maxima decompositions and last exit decompositions for $z_{\cdot\wedge t}$ are found under the σ-finite measure $1_{t\geqq 0}\,dt\,dP_0(z)$.

In Sect. 3, these results are applied to studying the regularity at the boundary of transition probabilities.

We use much some now well known results of the Malliavin calculus, the techniques of excursion theory (Williams [35, 36], Ikeda-Watanabe [14]) as well as our results on Brownian excursions [6].

Let us also point out that Dynkin-Vanderbei [11] have studied some of the processes considered here in the elliptic case.

Also observe that Derridj [37] studied the Dirichlet problem for second order differential operators verifying Hörmander's assumptions [13], and this by using a priori estimates. In our paper, we work under conditions which are slightly less general than Hörmander's to study the regularity at the boundary of the transition probabilities with Dirichlet or Neumann boundary conditions.

Finally we have recently received a paper by Ben Arous, Kusuoka and Stroock [38], in which these authors consider the Dirichlet problem and prove the regularity of the Poisson kernel on the boundary under the assumptions of Hörmander [13]. It may well be that the techniques of [38] can be adapted so that our results would also hold under Hörmander's assumptions, although our problems do not fall within the reach of the techniques of [38].

In the whole text $C_b^\infty(R^n)$ (resp. $C_c^\infty(R^n)$) denotes the set of C^∞ functions which are bounded and have bounded differentials (resp. which are C^∞ with compact support).

If X_t is a semi martingale, dX will denote the differential of X in the sense of Stratonovitch, and δX its differential in the sense of Itô.

I. Another Description of Brownian Excursions

At the present stage, there are two descriptions of the Brownian excursion measure n^+ of a reflecting Brownian motion out of 0:

- One by Lévy, Itô-McKean [14, 15] describes the Brownian excursion, conditionally on its length σ, as a Bes(3) bridge, the law of σ being itself known.
- The other by Williams [35, 36], Rogers [26] in which the excursion is described by means of two independent Bes(3) processes, stopped when they first hit the maximum of the excursion, whose law is also known.

In this section, we give a description of the measure $1_{t\leqq\sigma}dt\,dn^+$ by means of two independent Bes(3) processes stopped at their last exit from $a\in R^+$ chosen at random with the law $1_{a\geqq 0}2da$. Of course this gives us a third descriptions of n^+.

As pointed out in the Introduction, our need is to find the structure of a Brownian excursion cut "at random", or more precisely to know the structure of the excursion before a certain time chosen at random.

Of course, the derivation of this third description is obtained by using the results of Williams [35, 36] – Rogers [26].

In a), we first recall without proof a few facts on standard Brownian motion, which are mostly taken from Itô-McKean [15], in the hope that a few explicit calculations will make clear the remainder of the paper.

In b), we give the description of Brownian excursions which we will later need.

a) A Few Facts Concerning Brownian Motion

We first detail a few facts concerning Brownian motion, some of which are proved in Itô-McKean [15] Chap. 1, and others are consequences of last exit decompositions or of excursion theory. Since these facts will receive a proof using excursion theory, we do not give any justification for the moment. However, we feel that what follows will give some intuition for the less intuitive analytic part of the paper.

On $\mathscr{C}(R^+; R)$, whose standard element is z, we consider the filtration $\{F_t\}_{t\geqq 0}$ associated to the σ-fields

$$F_t = \mathscr{B}(z_s | s \leqq t). \tag{1.1}$$

Let P_0 be the Wiener measure on $\mathscr{C}(R^+, R)$, with $P_0(z(0)=0)=1$.

L_t is the local time of z at 0 (i.e. L is twice the standard local time of z at 0). A $\{F_t\}_{t\geqq 0}$-Brownian martingale B_t exists such that

$$|z_t| = L_t + B_t \tag{1.2}$$

and moreover

$$L_t = \sup_{0\leqq s\leqq t} (-B_s). \tag{1.3}$$

M_t is the process

$$M_t = \sup_{0\leqq s\leqq t} z_s. \tag{1.4}$$

If $\bar{z}_t$ is the process

$$\bar{z}_t = M_t - z_t. \tag{1.5}$$

$\bar{z}_t$ is a reflecting Brownian motion (which has the same law as $|z_t|$). Set

$$\begin{aligned} l_t &= \sup\{s<t;\ z_s=0\} \\ m_t &= \sup\{s<t;\ \bar{z}_t=0\}. \end{aligned} \tag{1.6}$$

We also define

$$\begin{aligned} p_t(x) &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \qquad t>0,\ x\in R \\ q_x(t) &= \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \qquad t>0,\ x>0. \end{aligned} \tag{1.7}$$

Clearly,

$$q_x(t) = -\frac{\partial p_t(x)}{\partial x} \qquad t>0,\ x>0. \tag{1.8}$$

Moreover for $t>0$, $p_t(x)\,dx$ is a semi-group of probability laws on R whose Laplace transform is $e^{\frac{t\alpha^2}{2}}$. Similarly for $x>0$, $q_x(t)\,1_{t\geqq 0}\,dt$ is a semi-group of probability laws on R^+, whose Laplace transform is $e^{-x\sqrt{2\alpha}}$.

For $t>0$, the law of (L_t, l_t, z_t) is given by

$$d P_{L_t, l_t, z_t}(L, l, z) = 1_{L\geqq 0,\, 0\leqq l\leqq t}\, q_L(l)\, q_{|z|}(t-l)\, dL\, dl\, dz \tag{1.9}$$

so that by integrating in l

$$d P_{L_t, z_t}(L, z) = 1_{L \geqq 0}\, q_{L+|z|}(t)\, dL\, dz. \tag{1.10}$$

Of course (1.8) is also a consequence of (1.10).

Moreover, conditionally on $(l_t = l)$, $(z_s | s \leqq l)$ and $(z_s | l \leqq s \leqq t)$ are independent. Conditionally on $(l_t = l)$, $(z_s | s \leqq l)$ is a Brownian bridge on the time interval $[0, l]$ (with $z_0 = z_l = 0$). Conditionally on $(l_t = l,\ z_t = z)$, $(|z_{l+s}|\ | 0 \leqq s \leqq t - l)$ is a Bes(3) bridge on the time interval $[0, t - l]$, starting at 0, and ending at $|z|$.

Assume now that t is also a random variable independent of z, whose law is the Lebesgue measure $1_{t \geqq 0}\, dt$ (which is *not* a probability measure).

Set

$$l'_t = t - l_t. \tag{1.11}$$

Inspection of (1.9) shows that

- (L_t, l_t) and (l'_t, z_t) are independent.
- The law of (L_t, z_t) is

$$1_{L \geqq 0}\, dL\, dz.$$

- Conditionally on (L_t, z_t), the law of (l_t, l'_t) is

$$1_{l \geqq 0,\, l' \geqq 0}\, q_{L_t}(l)\, q_{|z_t|}(l')\, dl\, dl'.$$

- Conditionally on (L_t, z_t), $(z_s | s \leqq l_t)$ and $(z_{l_t + s} | 0 \leqq s \leqq l'_t)$ are independent. Their laws will be determined in Theorem 2.17.

Similarly, under P_0, the probability law of $(M_t, m_t, \bar{z}_t)$ is given by

$$d P_{(M_t, m_t, \bar{z}_t)}(M, m, \bar{z}) = 1_{M \geqq 0,\, 0 \leqq m \leqq t,\, \bar{z} \geqq 0}\, 2 q_M(m)\, q_{\bar{z}}(t - m)\, dM\, dm\, d\bar{z} \tag{1.12}$$

and the law of $(M_t, \bar{z}_t)$ is

$$d P_{(M_t, \bar{z}_t)}(M, \bar{z}) = 1_{M \geqq 0,\, \bar{z} \geqq 0}\, 2 q_{M + \bar{z}}(t)\, dM\, d\bar{z}. \tag{1.13}$$

Conditionally on $(m_t = m)$, $(z_s | 0 \leqq s \leqq m)$ and $(\bar{z}_{m+s} | 0 \leqq s \leqq t - m)$ are independent. Conditionally on $(m_t = m, \bar{z}_t = \bar{z})$, $(\bar{z}_{m+s} | 0 \leqq s \leqq t - m)$ is a Bes(3) bridge starting at 0 at time 0, ending at $\bar{z}$ at time $t - m$ (this fact follows from the theory of excursions on z). A time reversal argument shows that conditionally on $(m_t = m, M_t = M)$, $M - z_s$ $(0 \leqq s \leqq m)$ is a Bes(3) bridge starting at M at time 0, ending at 0 at time m.

If t is a random variable independent of z, whose law is $1_{t \geqq 0}\, dt$, if

$$m'_t = t - m_t \tag{1.14}$$

inspection of (1.12) shows that

- (M_t, m_t) and $(m'_t, \bar{z}_t)$ are independent.
- The law of $(M_t, \bar{z}_t)$ is

$$1_{M \geqq 0,\, \bar{z} \geqq 0}\, 2\, dM\, d\bar{z}. \tag{1.15}$$

- Conditionally on $(M_t, \bar{z}_t)$, the law of (m_t, m'_t) is

$$1_{m \geqq 0,\, m' \geqq 0}\, q_{M_t}(m)\, q_{\bar{z}_t}(m')\, dm\, dm'. \tag{1.16}$$

• Conditionally on $(M_t, \bar{z}_t)$, $(z_s|0 \leqq s \leqq m_t)$ and $(M - z_{m_t+s}|0 \leqq s \leqq m'_t)$ are independent. Their laws will be determined in Theorem 2.16.

Of course, since $\bar{z}$ has the same law as $|z.|$, these two series of results are obviously connected. Moreover observe that under the measure $1_{t \geqq 0} dt\, dP_0$, conditioned by $(l+l'=t)$ or by $(m+m'=t)$ (we have dropped the subscripts t for obvious reasons), we go back to the Brownian motion z on the time interval $[0, t]$ considered at the beginning. By making t a random variable, we introduce more flexibility in the description of z.

b) *How to Cut at Random an Excursion Bridge*

Let $\bar{z}$ be a reflecting Brownian motion on $[0, +\infty[$. $\bar{L}_t$ denotes its standard local time at 0. If $\bar{B}_t$ is defined by

$$\bar{z}_t = \bar{L}_t + \bar{B}_t. \tag{1.17}$$

$\bar{B}_t$ is a Brownian martingale.

Set

$$A_s = \inf\{A > 0;\ \bar{L}_A > s\}. \tag{1.18}$$

$\mathscr{W}^+$ is the set of continuous functions $e(s)$ defined on R^+ with values in R^+ such that

- $e(0) = 0$.
- There is $\sigma > 0$ such that if $0 < s < \sigma$, $e(s) > 0$, and if $s \geqq \sigma$, $e(s) = 0$.

$\{G_s\}_{s \geqq 0}$ denotes the filtration in $\mathscr{W}^+$ associated to the σ-fields

$$G_s = \mathscr{B}(e(u)|u \leqq s). \tag{1.19}$$

δ is a cemetery point.

Let e_t be the process valued in $\mathscr{W}^+ \cup \{\delta\}$ defined by

$$\begin{aligned} e_t(s) &= \delta \quad \text{if } A_{t-} = A_t \\ &= \bar{z}_{A_t^- + s} \text{ on } [0, A_t - A_{t^-}]; \quad 0 \text{ for } s > A_t - A_{t^-}, \text{ if } A_t - A_{t^-} \neq 0. \end{aligned}$$

The theory of excursions of Itô (see Itô-McKean [15], Ikeda-Watanabe [14]) shows that e_t is a Poisson point process, whose characteristic measure on $\mathscr{W}^+$ is noted n^+.

Definition 1.1. $\bar{n}^+$ is the σ-finite measure on $R^+ \times \mathscr{W}^+$

$$d\bar{n}^+(t, e) = 1_{0 \leqq t \leqq \sigma(e)}\, dt\, dn^+(e). \tag{1.20}$$

For $a \in R$, P_a denotes the probability law on $\mathscr{C}(R^+; R)$ of the Brownian motion z with $z(0) = a$.

For $r_0 \in R^+$, Q_{r_0} is the probability law on $\mathscr{C}(R^+; R^+)$ of the Bes(3) process $r.$, with $r(0) = r_0$.

We then have the following result.

Theorem 1.2. *On $R^+ \times \mathscr{C}(R^+; R^+) \times \mathscr{C}(R^+; R^+)$, consider the σ-finite measure*

$$dR(a, r, r') = 1_{a \geqq 0}\, 2\, dQ_0(r)\, dQ_0(r')\, da. \tag{1.21}$$

Let A, A' be the random variables

$$\begin{aligned} A &= \sup\{s \in R^+, r_s = a\} \\ A' &= \sup\{s \in R^+, r'_s = a\}. \end{aligned} \tag{1.22}$$

Set

$$\begin{aligned} e(s) &= r_s && \text{if } 0 \leqq s \leqq A \\ & r'_{A+A'-s} && \text{if } A < s \leqq A + A' \\ & 0 && \text{if } s > A + A'. \end{aligned} \tag{1.23}$$

Then the law of (A, e) is exactly $d\bar{n}^+(t, e)$.

Proof. Let f be a C^∞ function on R, whose compact support is included in $]0, +\infty[$. There is $\varepsilon > 0$ such that $f(x) = 0$ if $x < \varepsilon$.

For $t \in R^+$, θ_t is the usual translation operator on $\mathscr{C}(R^+; R)$ which to $z_{\cdot} \in \mathscr{C}(R^+; R)$ associates $(\theta_t z) = z_{\cdot + t} \in \mathscr{C}(R^+; R)$.

$H_s(z)$, $H'_s(z)$ are two processes defined on $R^+ \times \mathscr{C}(R^+; R)$ which are bounded and predictable (with respect to the filtration $\{F_t\}_{t \geqq 0}$). We will now evaluate

$$\begin{aligned} &\int_{R^+ \times \mathscr{W}^+} f(e(t))\, H_t(e)\, H_\infty(\theta_t e)\, d\bar{n}^+(t, e) \\ &\quad = \int_{R^+ \times \mathscr{W}^+} 1_{t \geqq 0} f(e(t))\, H_t(e)\, H_\infty(\theta_t e)\, dt\, dn^+(e). \end{aligned} \tag{1.24}$$

Let $L_t(a)$ be the local time at $a \in R^+$ of $e(.)$ (i.e. $L_t(a)$ is twice the standard local time of $e(.)$ at a).

(1.24) is equal to

$$\begin{aligned} &\int_{\mathscr{W}^+} \Big[\int_{R^+ \times R^+} 1_{t \geqq 0} f(e(t))\, H_t(e)\, H_\infty(\theta_t e)\, dL_t(a)\, da \Big]\, dn^+(e) \\ &\quad = \int_{R^+} da \int_{\mathscr{W}^+} dn^+(e) \left[\int_0^{+\infty} 1_{T_a < +\infty} f(a)\, H_t(e)\, H_\infty(\theta_t e)\, dL_t(a) \right]. \end{aligned} \tag{1.25}$$

Of course in (1.25), T_a is the stopping time

$$T_a = \inf\{s \geqq 0;\ e(s) = a\}. \tag{1.26}$$

By Ikeda-Watanabe [14], under n^+ and conditionally on G_t, the law of $\theta_t e$ is the law of $z_{\cdot \wedge T_0}$ under $P_{e(t)}$. Since the support of $dL_t(a)$ is $\{e(\cdot) = a\}$, (1.25) is equal to

$$\int_{R^+} da \int_{\mathscr{W}^+} dn^+(e) \int_0^{+\infty} 1_{T_a < +\infty} f(a)\, H_t(e)\, E^{P_a}[H_\infty(z_{\cdot \wedge T_0})]\, dL_t(a). \tag{1.27}$$

By a result of Williams, Rogers [26, 35, 36] under n^+, and conditionally on $(T_a < +\infty)$, $e(s)$ $(0 \leqq s \leqq T_a)$ is a Bes(3) process starting at 0 and stopped when it hits a, $e(s)$ $(T_a \leqq s \leqq \sigma)$ is a Brownian motion independent of $e(s)$ $(0 \leqq s \leqq T_a)$

starting at a and stopped when it hits 0. Clearly, if $x \geqq 0$

$$P_x(T_a < T_0) = 1 \quad \text{if } x \geqq a$$
$$\frac{x}{a} \quad \text{if } x < a. \tag{1.28}$$

Set

$$A^a(z) = \sup\{s \geqq 0;\ s < T_0 : z_s = a\}. \tag{1.29}$$

From (1.28) it is not hard to see for P_a, the dual predictable projection of $d(1_{t \geqq A^a(z)})$ is $\frac{dL_{\cdot \wedge T_0}(a)}{2a}$. (1.27) writes

$$2 \int_{R^+} a\,da \int_{\mathscr{W}^+} dn^+(e) 1_{T_a < +\infty} f(a) H_{A^a(e)}(e) E^{P_a}(H_\infty(z_{\cdot \wedge T_0})). \tag{1.30}$$

Now using (1.28), it is easy to see that under P_a, z_s $(0 \leqq s \leqq A^a(z))$ has the same law as r_s $(0 \leqq s \leqq A^a(r))$ under Q_a. It follows that under n^+, conditionally on $(T_a < +\infty)$, the law of $e(s)$ $(0 \leqq s \leqq A^a(e))$ is equal to the law of r_s $(0 \leqq s \leqq A^a(r))$ under Q_0.

Then, classically [36]

$$n^+(T_a < +\infty) = \frac{1}{a}$$

(1.30) is then equal to

$$2 \int_{R^+} da \int f(a) H_{A^a(r)}(r) [E^{P_a} H_\infty(z_{\cdot \wedge T_0})]\, dQ_0(r). \tag{1.31}$$

Finally, a result in [14–36] (which is also a consequence of Pitman [24]) shows that under P_a, the law of $z_{T_0 - s}$ $(0 \leqq s \leqq T_0)$ is the same as the law of r_s $(0 \leqq s \leqq A^a(r))$ under Q_0. The Theorem is proved. □

Corollary. *Under the σ-finite measure*

$$\frac{dR(a, r, r')}{A + A'} \tag{1.32}$$

the law of e is exactly n^+.

Proof. Clearly for one $e \in \mathscr{W}^+$

$$\int_{R^+} d\bar{n}^+(t, e) = \sigma(e)\, dn^+(e). \tag{1.33}$$

The result is now obvious from Theorem 1.2. □

Remark 1. Of course under $dR(a, r, r')$, r_{A-s} $(0 \leqq s \leqq A)$ and $r'_{A'-s}$ $(0 \leqq s \leqq A')$ are two independent Brownian motions starting at a and stopped when they hit 0. This gives another useful description of $d\bar{n}^+(t, e)$.

Moreover conditionally on A, A', r_s $(0 \leqq s \leqq A)$ and r'_s $(0 \leqq s \leqq A')$ are two

Bes(3) bridges (with the obvious end points), as well as r_{A-s} $(0 \leqq s \leqq A)$ and $r'_{A'-s}$ $(0 \leqq s \leqq A')$.

Finally, it is striking that in Williams's description of n^+ [35–36] A, A' are replaced by the corresponding hitting times.

II. Williams Decompositions of Brownian Motion and Factorization of Second Order Differential Operators

In this section we consider a $n+1$ dimensional diffusion (x_t, z_t) where z_t is a Brownian motion with a drift $b(x, z)$. The generator of the process (x_t, z_t) is

$$\frac{1}{2}\frac{\partial^2}{\partial z^2}+b\frac{\partial}{\partial z}+\mathscr{L}'+\frac{\partial}{\partial t} \tag{2.1}$$

where $\mathscr{L}'$ acts only on the variables x.

In this section, we show how to factor *exactly* the operator (2.1) in the form

$$\frac{1}{2}\left(\frac{\partial}{\partial z}+K'\right)\left(\frac{\partial}{\partial z}+K\right). \tag{2.2}$$

It is in fact shown that this factorization of (2.1) is directly related to the decomposition of Williams of the Brownian motion $z._{\wedge t}$ under the σ-finite measure $1_{t \geqq 0}\, dt\, dP_0(z)$ ($dP_0(z)$ is the Brownian measure). K is shown to be the generator of the jump process associated to the local maximums of z (which Dynkin-Vanderbei [11] call a stochastic wave). K' is obtained from K by using the results of Sect. 1 and Pitman's construction [24] of a Bes(3) process by means of a Brownian motion. We also use some results in Jacod [16] on the Girsanov transformation on point processes, and our results [6] on the effect of a Girsanov transformation on the excursion measure of a Brownian motion.

In a), we introduce the main assumptions and notations. In b), a new Markov jump process is introduced, and its generator is explicitly found. This is the stochastic wave of Dynkin-Vanderbei [11]. In c) the stochastic wave is turned upside down, and a new jump process is defined. In d), the factorization (2.2) is derived. In e) we describe some other factorizations of (2.1) which are related to last exits of $z.$, and explain the interrelations of these different factorizations. Finally in f), we give a Williams decomposition and a last exit decomposition of $z._{\wedge t}$ under the σ-finite measure $1_{t \geqq 0}\, dt\, dP_0(z)$. In f), we still use the results of Sect. 1.

Let us point out that we have not tried at all to justify analytically the various decompositions (i.e. for instance to show $K'K$ is well defined), but directly interpret what is *obvious* on the paths of the Brownian motion in terms of operators.

This section is useful to understand the connection between what we will do in Sect. 3 and the work done by analysts (see Treves [34]), but of course, in Sect. 3, all the analysis will be done on paths without using (2.2).

Note that the various operators are given in a form explicit enough so that the proof that (2.2) is not purely formal should raise no difficulty.

Let us again point out that our product form (2.2) is nothing else that an extension of the Wiener-Hopf factorization results of Prabhu [25], Greenwood-Pitman [12], Silverstein [28], Kaspi [19] in the context of multidimensional diffusions.

a) Assumptions and Notations

$X_0(x, z) \dots X_m(x, z)$ denote $m+1$ C^∞ vector fields defined on $R^n \times R$ with values in R^n, whose components belong to $C_b^\infty(R^n \times R)$.

$b(x, z)$ is a C^∞ function defined on $R^n \times R$ with values in R, which belongs to $C_b^\infty(R^n \times R)$.

Ω' is the space $\mathscr{C}(R^+; R^m)$. The standard element of Ω' is $w=(w^1 \dots w^m)$. The filtration $\{F'_t\}_{t \geqq 0}$ is defined on Ω' by

$$F'_t = \mathscr{B}(w_s | s \leqq t).$$

P' denotes the Brownian measure on Ω', with $P'(w_0=0)=0$.

$\bar{\Omega}$ is the probability space $\mathscr{C}(R^+; R) \times \Omega'$ endowed with the filtration $\{\bar{F}_t\}_{t \geqq 0}$ defined by

$$\bar{F}_t = F_t \otimes F'_t.$$

The general element of $\bar{\Omega}$ will be denoted $\bar{\omega}$. Take $(x_0, \bar{z}_0) \in R^n \times R$. On $(\bar{\Omega}, P_0 \otimes P')$, consider the stochastic differential equation

$$\begin{aligned} dx &= X_0(x, \bar{z})\, dt + X_i(x, \bar{z}) . dw^i \\ x(0) &= x_0 \\ d\bar{z} &= dz \\ \bar{z}(0) &= \bar{z}_0 \end{aligned} \tag{2.3}$$

$\left(\text{we omit the summation sign } \sum_{i=1}^{m} \text{ in (2.3)}\right)$.

Let $\varphi.(\bar{\omega}, .)$ be the associated flow of C^∞ diffeomorphisms of $R^n \times R$ onto itself (Bismut [1] – Theorems I.1.2 and I.2.1), so that in (2.1), $P_0 \otimes P'$ a.s.

$$(x_t, \bar{z}_t) = \varphi_t(\bar{\omega}, x_0, z_0)$$

$(x_0, \bar{z}_0)$ are now temporarily fixed, $(x_t, \bar{z}_t)$ are defined by (2.3).

Definition 2.1. On $(\bar{\Omega}, P_0 \otimes P')$, Z_t is the Girsanov martingale

$$Z_t = \exp\left\{\int_0^t b(x_s, \bar{z}_s)\, \delta z - \tfrac{1}{2} \int_0^t b^2(x_s, \bar{z}_s)\, ds\right\}. \tag{2.4}$$

$\bar{P}$ is the probability measure on $\bar{\Omega}$ such that for any $t>0$

$$\left.\frac{d\bar{P}}{d(P_0 \otimes P')}\right|_{\bar{F}_t} = Z_t.$$

Under $\bar{P}$, $(x_t, \bar{z}_t)$ is a Markov process, whose generator $\mathscr{L}$ is given by

$$\mathscr{L}=\frac{1}{2}\frac{\partial^2}{\partial z^2}+b(x, z)\frac{\partial}{\partial z}+X_0(x, z)+\frac{1}{2}\sum_{i=1}^{m} X_i^2(x, z). \tag{2.5}$$

For a discussion of how to reduce locally a second order differential operator to the form (2.5) when $(z=cst)$ is non characteristic, see Remark 2 in Sect. 3.

For $a>\bar{z}_0$, set

$$T_a=\inf\{t>0, \bar{z}_t=a\}.$$

To simplify the argument which follows, we will assume that for any $(x_0, \bar{z}_0)$, $\bar{P}$ a.s., $T_a<+\infty$, or equivalently (see [6]) that

$$E^{P_0\otimes P'} Z_{T_a}=1. \tag{2.6}$$

This is the case if $b\geqq 0$, or if the support of b is compact. The discussion which follows can be extended even if (2.6) is not verified, but the argument is slightly more involved. For a discussion of (2.6) in terms of Föllmer measures, we refer to [6].

b) *The Time-Changed Process*

$$(\bar{z}_0, T, x_0)\in R\times R^+\times R^n \text{ is now fixed.}$$

Definition 2.2. On $(\bar{\Omega}, \bar{P})$, the process Y_a with values in $R\times R^+\times R^n$ is defined by

$$Y_a=(\bar{z}_0+a,\ T_{\bar{z}_0+a}+T,\ x_{T_{\bar{z}_0+a}}).$$

Of course Y_a is right continuous with left hand limits. The strong Markov property of $(\bar{z}, x)$ shows immediately that Y_a is also strong Markov.

Y_a is exactly the stochastic wave process considered by Dynkin and Vanderbei [11].

We will now find its formal generator. Recall that under n^+, conditionally on σ, we know that e is a semi-martingale, since it is a Bes(3) bridge (see Ikeda-Watanabe [14] p. 225). Of course this result also follows from Theorem 1.2 and its corollary.

Definition 2.3. Take $(\bar{z}'_0, x'_0)\in R\times R^n$. On $(\mathscr{W}^+\times\Omega', dn^+(e)\otimes dP'(w))$, we consider the stochastic differential equation

$$\begin{aligned} &dx'=X_0(x', \bar{z}')\,dt+X_i(x', \bar{z}').dw^i; \qquad x'(0)=x'_0\\ &d\bar{z}'=-de; \qquad \bar{z}'(0)=\bar{z}'_0. \end{aligned} \tag{2.7}$$

$Z'_t(e, w)$ is the process

$$Z'_t(e, w)=\exp\left\{-\int_0^t b(x', \bar{z}')\,\delta e-\tfrac{1}{2}\int_0^t b^2(x', \bar{z}')\,ds\right\}. \tag{2.8}$$

Of course $Z'_t(e, w)$ depends on $(\bar{z}'_0, x'_0)$. We do not note this dependence explicitly.

We first have a technical result.

Proposition 2.4. *As* $(\bar{z}'_0, x_0)$ *varies in* $R \times R^n$

$$\int_{\mathscr{W}^+ \times \Omega'} [1_{\sigma \leqq 1} |Z'_\sigma(e, w) - 1|^2 + 1_{\sigma > 1} Z'_\sigma(e, w)] \, dn^+(e) \, dP'(w) \tag{2.9}$$

is uniformly bounded.

Proof. We proceed very much as in the proof of Proposition 3.25 in Bismut [6]. Set

$$f_t = Z'_t - 1. \tag{2.10}$$

Clearly

$$df = -(f+1) b(x', \bar{z}') \, \delta e, \qquad f(0) = 0. \tag{2.11}$$

Let σ_k be the stopping time

$$\sigma_k = \inf\{t \geqq 0; |f_t| \geqq k\} \wedge \sigma.$$

Now by Ikeda-Watanabe [14], the law of σ under n^+ is $\frac{1_{\sigma \geqq 0} \, d\sigma}{\sqrt{2\pi\sigma^3}}$, so that for $t > 0$

$$\int_{\mathscr{W}^+} \sigma \wedge t \, dn^+(e) = \frac{4}{\sqrt{2\pi}} t^{1/2}. \tag{2.12}$$

From Ikeda-Watanabe [14] p. 309, we know that

$$\int_{\mathscr{W}^+ \times \Omega'} |f_{\sigma_k \wedge t}|^2 \, dn^+(e) \, dP'(w) = \int_{\mathscr{W}^+ \times \Omega'} \int_0^{\sigma_k \wedge t} |f+1|^2 b^2(x', \bar{z}') \, ds \, dn^+(e) \, dP'(w). \tag{2.13}$$

Using (2.12), it is clear that the r.h.s. of (2.13) is $< +\infty$. Moreover from (2.13), we see that

$$\int_{\mathscr{W}^+ \times \Omega'} |f_{\sigma_k \wedge t}|^2 \, dn^+(e) \, dP'(w) \leqq C \int_{\mathscr{W}^+ \times \Omega'} \left(\int_0^t |f_{\sigma_k \wedge u}|^2 \, du + \sigma \wedge t \right) dn^+(e) \, dP'(w). \tag{2.14}$$

Using (2.12), (2.14) and Gronwall's lemma, we find that for $k \in N$, $t \leqq 1$,

$$\int_{\mathscr{W}^+ \times \Omega'} |f_{\sigma_k \wedge t}|^2 \, dn^+(e) \, dP'(w)$$

is uniformly bounded. By making $k \to +\infty$, we obtain the boundedness of

$$\int_{\mathscr{W}^+ \times \Omega'} |f_{\sigma \wedge 1}|^2 \, dn^+(e) \, dP'(w). \tag{2.15}$$

Conditionally on $(\sigma>1)$, e_t $(1\leqq t\leqq\sigma)$ is a Brownian motion stopped when it hits 0. Conditionally on $(\sigma\geqq 1)$, $\frac{Z'_t}{Z'_1}$ $(1\leqq t\leqq\sigma)$ is *a* positive martingale, and so

$$E^{n^+\otimes P'}\left[\frac{Z'_\sigma}{Z'_1}\middle| G_1\otimes F'_1;\ \sigma\geqq 1\right]\leqq 1. \tag{2.16}$$

Now

$$\begin{aligned}&\int_{\mathscr{W}^+\times\Omega'} Z'_1\,1_{\sigma\geqq 1}\,dn^+(e)\,dP'(w)\\&\quad=\int_{\mathscr{W}^+\times\Omega'}(Z'_{\sigma\wedge 1}-1)\,1_{\sigma>1}\,dn^+(e)\,dP'(w)+n^+(\sigma>1).\end{aligned} \tag{2.17}$$

Now $n^+(\sigma>1)<+\infty$. Using (2.15) and Cauchy-Schwarz's inequality, we see that

$$\int_{\mathscr{W}^+\times\Omega'}|Z'_{\sigma\wedge 1}-1|\,1_{\sigma>1}\,dn^+(e)\,dP'(w) \tag{2.18}$$

is uniformly bounded. From (2.15)–(2.18), we obtain the Proposition. □

Definition 2.5. If $f\in C_b^\infty(R\times R\times R^n)$, *we define the function* Kf *on* $R\times R\times R^n$ *by*

$$\begin{aligned}Kf(\bar z'_0,T,x'_0)=&\int_{\mathscr{W}^+\times\Omega'}\{1_{\sigma\leqq 1}[f(\bar z'_0,T+\sigma,x'_\sigma)-f(\bar z'_0,T,x'_0)\\&-\int_0^\sigma\langle d_x f(\bar z'_0,T+s,x'_s),X_i(x'_s,\bar z'_s)\rangle\,\delta w^i]\\&+1_{\sigma>1}[f(\bar z'_0,T+\sigma,x'_\sigma)-f(\bar z'_0,T,x'_0)]\}\,dn^+(e)\,dP'(w)\\&+\int_{\mathscr{W}^+\times\Omega'}[f(\bar z'_0,T+\sigma,x'_\sigma)-f(\bar z'_0,T,x'_0)](Z'_\sigma-1)\,dn^+(e)\,dP'(w).\end{aligned} \tag{2.19}$$

Of course

$$\begin{aligned}&f(\bar z'_0,T+\sigma,x'_\sigma)-f(\bar z'_0,T,x'_0)\\&\quad=\int_0^\sigma\left[\frac{\partial f}{\partial T}(\bar z'_0,T+s,x'_s)+X_0(x'_s,\bar z'_s)\,f(\bar z'_0,T+s,x'_s)\right.\\&\quad\left.+\tfrac12 X_i^2(x'_s,\bar z'_s)\,f(\bar z'_0,T+s,x'_s)\right]ds+\int_0^\sigma X_i(x'_s,\bar z'_s)\,f(\bar z'_0,T+s,x'_s)\,\delta w^i\end{aligned} \tag{2.20}$$

(in (2.20) f and its derivatives are calculated at $\bar z'_0$ and X_0, X_i act on the variable x).

From (2.20), we find that

$$\begin{aligned}|Kf(\bar z'_0,T,x'_0)|\leqq c\Bigg[\int_{\mathscr{W}^+\times\Omega'}\Bigg\{&1_{\sigma\leqq 1}\,\sigma+1_{\sigma>1}\\&+1_{\sigma\leqq 1}\left(\left|\int_0^\sigma X_i f\,\delta w^i\right|+\sigma\right)|Z'_\sigma-1|\\&+1_{\sigma>1}\,Z'_\sigma\Bigg\}\,dn^+(e)\,dP'(w).\Bigg].\end{aligned} \tag{2.21}$$

Now

$$\int_{\Omega'} \left| \int_0^\sigma X_i f \delta w^i \right|^2 dP'(w) \leqq D\sigma. \tag{2.22}$$

Using (2.22), Proposition 2.4, and Cauchy-Schwarz's inequality, it is clear that the integrals in the r.h.s. of (2.21) are uniformly bounded, so that Kf is indeed well-defined and uniformly bounded.

Remark 1. It would not be difficult to prove that $Kf \in C_b^\infty(R \times R \times R^n)$, by using the flow properties in (2.7). This is left to the reader. Such a statement is useful if we want to define the product of two operators like K.

We now have:

Theorem 2.6. *Under $\bar{P}$, for any $f \in C_b^\infty(R \times R \times R^n)$,*

$$f(Y_a) - \int_0^a \left(\frac{\partial f}{\partial z} + Kf \right)(Y_c)\, dc \tag{2.23}$$

is a $\{\bar{F}_{T_a}\}_{a \geqq 0}$-martingale.

Proof. To simplify the notations, we will assume that $\bar{z}_0 = 0$. Set

$$\bar{M}_t = \sup_{0 \leqq s \leqq t} \bar{z}_s.$$

First assume that f has compact support and that $b=0$. Under $P_0 \otimes P'$, we know that

$$\begin{aligned} f(\bar{M}_t, T+t, x_t) &- \int_0^t \frac{\partial f}{\partial z}(\bar{M}_s, T+s, x_s)\, d\bar{M}_s - \int_0^t \left(\frac{\partial f}{\partial T} + X_0 f + \tfrac{1}{2} X_i^2 f \right) ds \\ &= f(\bar{z}_0, T, x_0) + \int_0^t X_i f \delta w^i. \end{aligned} \tag{2.24}$$

In the integrals of the r.h.s. of (2.24), f and its differentials are evaluated at $(\bar{M}_s, T+s, x_s)$, while X_0, X_i are evaluated at $(x_s, \bar{z}_s)$. Since f has compact support, all the terms in (2.24) calculated at T_a are integrable, and

$$\int_0^{T_a} (X_i f)\, \delta w^i$$

is a $\{\bar{F}_{T_a}\}_{a \geqq 0}$-martingale.

Now $z'_s = \bar{M}_s - \bar{z}_s$ is a reflecting Brownian motion whose standard local time at 0 is $\bar{M}_s$. $\{\bar{F}_{T_a}\}_{a \geqq 0}$ is exactly the canonical filtration of the corresponding point process (to which w is added as in [6]). It follows that

$$\begin{aligned} & f(\bar{z}_0 + a, T+T_a, x_{T_a}) - \int_0^a \frac{\partial f}{\partial z}(c, T+T_c, x_{T_c})\, dc \\ & - \int_0^a dc \int_{\mathscr{W}^+ \times \Omega'} \left[\int_0^\sigma \left(\frac{\partial f}{\partial T}(c, T+T_c+s, x'_s) + X_0(x', \bar{z}') f(c, T+T_c+s, x'_s) \right.\right. \\ & \left.\left. + \frac{1}{2} X_i^2(x', \bar{z}') f(c, T+T_c+s, x'_s) \right) ds \right] dn^+(e)\, dP'(w) \end{aligned} \tag{2.25}$$

is a $\{\bar{F}_{T_a}\}_{a\geqq 0}$-martingale (in the r.h.s. of (2.25), the integral with respect to $dn^+(e)\,dP'(w)$ is evaluated with $x'_0=x_{T_c}$, $\bar{z}'_0=c$).

Using (2.20) and the fact that on $\sigma\geqq 1$, the stochastic integral $\int_0^\sigma X_i f\,\delta w^i$ gives a 0 contribution in the integral $\int_{\mathscr{W}^+\times\Omega'}[\ldots]\,dn^+(e)\,dP'(w)$, we find that if f has compact support and if $b=0$, (2.23) holds.

By approximating $f\in C_b^\infty(R\times R\times R^n)$ by a sequence $f_k\in C_c^\infty(R\times R\times R^n)$ uniformly on compact sets, (2.23) is seen to hold with $b=0$ for a general $f\in C_b^\infty(R\times R\times R^n)$.

Now from Bismut [6]. Theorem 3.24, we know that the Girsanov transformation on each $\bar{F}_{T_a}$ induces a corresponding Girsanov transformation on the excursion measure, so that the new excursion measure is now $Z_\sigma^{a,x_{T_a}}(e,w)\,dn^+(e)\,dP'(w)$ (we here note explicitly the starting point of the excursion (a,x_{T_a})). Since (2.23) is necessarily the compensated sum of the jumps of $f(Y_a)$ for the measure $P_0\otimes P'$, we find from Jacod [16] and Bismut [6] – Sect. 3 that if K^0 is the operator K calculated with $b=0$, then K is the operator acting on $C_b(R\times R\times R^n)$ defined by

$$(Kf)(\bar{z}'_0,T,x'_0)=(K^0f)(\bar{z}'_0,T,x'_0)+\int_{\mathscr{W}^+\times\Omega'}(f(\bar{z}'_0,T+\sigma,x'_\sigma)-f(\bar{z}'_0,T,x'_0))\cdot(Z'_\sigma-1)\,dn^+(e)\,dP'(w) \tag{2.26}$$

so that under $\bar{P}$

$$f(Y_a)-\int_0^a\left(\frac{\partial f}{\partial z}+Kf\right)(Y_c)\,dc \tag{2.27}$$

is a $\{\bar{F}_{T_a}\}_{a\geqq 0}$-martingale. The proof is finished. □

c) How to Push Down a Stochastic Wave

Let B be a one dimensional Brownian motion with $B_0=0$. P_0 still denotes its probability law of $\mathscr{C}(R^+;R)$ endowed with the corresponding filtration $\{F_t\}_{t\geqq 0}$. Set

$$\begin{aligned} N_t&=\sup_{0\leqq s\leqq t} B_s\\ z'_t&=B_t-2N_t. \end{aligned} \tag{2.28}$$

An essential result of Pitman [24] shows that $-z'_t$ is a Bes(3) process starting at 0, and that moreover

$$N_t=\inf_{s\geqq t}-z'_s. \tag{2.29}$$

On $(\bar{\Omega},dP_0(B)\otimes dP'(w))$ we still put the filtration $\{\bar{F}_t\}_{t\geqq 0}$. We consider Eq. (2.3), in which z is replaced by z', i.e.

$$\begin{aligned} dx&=X_0(x,\bar{z})\,dt+X_i(x,\bar{z}).dw^i; \qquad x(0)=x_0\\ d\bar{z}&=dz'; \qquad \bar{z}(0)=\bar{z}_0. \end{aligned} \tag{2.30}$$

For $c \leqq \bar{z}_0$, we define the random variable

$$U_c = \sup\{s \geqq 0,\ \bar{z}_s = c\}. \tag{2.31}$$

If T'_a is the stopping time

$$T'_a = \inf\{s \geqq 0;\ B_s = a\} \tag{2.32}$$

clearly,

$$U_a = T'_{\bar{z}_0 - a} \tag{2.33}$$

so that U_a is also a $\{F_t\}_{t \geqq 0}$ stopping time.

$(\bar{z}_0, T, \bar{x}_0)$ is now given in $R \times R \times R^n$.

Definition 2.7. On $(\bar{\Omega}, dP_0(B) \otimes dP'(w))$, Y'_a denotes the process with values in $R \times R \times R^n$ given by

$$Y'_a = (\bar{z}_0 - a,\ U_{\bar{z}_0 - a} + T,\ x_{U_{\bar{z}_0 - a}}) \tag{2.34}$$

K^0 denotes the operator K defined in Definition 2.5 calculated with $b=0$. To simplify the exposition, we first state a result corresponding to Theorem 2.6 with $b=0$.

Theorem 2.8. *Under $dP_0(B) \otimes dP'(w)$, Y'_a is a strong Markov process with respect to the filtration $\{\bar{F}_{T'_a}\}_{a \geqq 0}$. For any $f \in C_b^\infty(R \times R \times R^n)$,*

$$f(Y'_a) - \int_0^a \left(-\frac{\partial}{\partial z} f + K^0 f\right)(Y'_c)\, dc \tag{2.35}$$

is a $\{\bar{F}_{T'_a}\}_{a \geqq 0}$-martingale.

Proof. The strong Markov property of the Brownian motion shows that $B_{t+T'_a} - B_{T'_a}$ is a Brownian starting at 0 independent of $\bar{F}_{T'_a}$, so that $-(z'_{t+T'_a} - z'_{T'_a})$ is still a Bes(3) process starting at 0 independent of $\{\bar{F}_{T'_a}\}$. The fact that Y'_a is a strong Markov is now obvious.

First assume that f has compact support. Then

$$\begin{aligned} f(\bar{z}_0 - N_t, T+t, x_t) &- \int_0^t \left(-\frac{\partial f}{\partial z}(\bar{z}_0 - N_s, T+s, x_s)\right) dN_s - \int_0^t \left(\frac{\partial f}{\partial T} + X_0 f + \tfrac{1}{2} X_i^2 f\right) ds \\ &= f(\bar{z}_0, T, x_0) + \int_0^t (X_i f)\, \delta w^i. \end{aligned} \tag{2.36}$$

$z''_s = N_s - B_s$ is a reflecting Brownian motion whose local time at 0 is N_s. $\{\bar{F}_{T'_a}\}_{a \geqq 0}$ is still the canonical filtration of the corresponding point process, and of course the characteristic measure is still n^+. By reasoning as in Theorem 2.6, we easily deduce than

$$f(Y'_a) - \int_0^a \left(-\frac{\partial f}{\partial z} + K^0 f\right)(Y'_c)\, dc \tag{2.37}$$

is a $\{\bar{F}_{T'_a}\}_{a \geqq 0}$-martingale. We then proceed as in Theorem 2.6. □

The introduction of the Girsanov transformation has to be done with some care. To make the probabilistic interpretation easier, we will assume that b is $\geqq 0$, but this will be entirely irrelevant in the sequel.

If $(x, \bar{z})$ are given by (2.30), and if we replace $z_{\cdot}$ by $z'_{\cdot}$ in (2.4) we get

$$\bar{Z}_t = \exp\left\{\int_0^t b(x_s, \bar{z}_s)\,\delta z' - \tfrac{1}{2}\int_0^t b^2(x_s, \bar{z}_s)\,ds\right\} \tag{2.38}$$

or equivalently, if we use (2.28)

$$\bar{Z}_t = \exp\left(-\int_0^t 2b(x_s, \bar{z}_s)\,dN_s\right) \exp\left\{\int_0^t b(x_s, \bar{z}_s)\,\delta B_s - \tfrac{1}{2}\int_0^t b^2(x_s, \bar{z}_s)\,ds\right\} \tag{2.39}$$

so that

$$\bar{Z}_{T'_a} = \exp\left(-\int_0^a 2b(x_{T'_c}, \bar{z}_0 - c)\,dc\right) \exp\left\{\int_0^{T'_a} b(x_s, \bar{z}_s)\,\delta B_s - \tfrac{1}{2}\int_0^{T'_a} b^2(x_s, \bar{z}_s)\,ds\right\}. \tag{2.40}$$

Since b is $\geqq 0$, it is easy to find that

$$\int_\Omega \exp\left[\int_0^{T'_a} b(x_s, \bar{z}_s)\,\delta B_s - \tfrac{1}{2}\int_0^{T'_a} b^2(x_s, \bar{z}_s)\,ds\right] dP_0(B)\,dP'(w) = 1. \tag{2.41}$$

The first term in (2.40) introduces an extrakilling at the ($\geqq 0$) rate $2b(x_{T'_a}, \bar{z}_0 - c)$. We will now define a new measure $\bar{P}'$ by means of its density with respect to $dP_0(B) \otimes dP'(a)$. We assume that the reader is familiar with killings.

Of course after being killed, all the processes (including excursions) go to one cemetery $\{\delta\}$ and remain there (b is $\geqq 0$!). Let ζ be the death time.

Definition 2.9. $\bar{P}'$ is the probability measure on $\bar{\Omega}$ such that for each $t > 0$

$$1_{t<\zeta} \left.\frac{d\bar{P}'}{d(P_0 \otimes P')}\right|_{F_t} = \bar{Z}_t. \tag{2.42}$$

Under $\bar{P}'$ the process Y'_a will be killed at time N_ζ. All functions defined on $R \times R \times R^n$ are given the value 0 on $\{\delta\}$. $b'(\bar{z}, t, x)$ is defined to be equal to $b(x, \bar{z})$.

We now have

Theorem 2.10. *Under $\bar{P}'$, the process Y'_a is a strong Markov process with respect to the filtration $\{\bar{F}_{T'_a}\}_{a \geqq 0}$. For any $f \in C_b^\infty(R \times R \times R^n)$,*

$$f(Y'_a) - \int_0^a \left(-\frac{\partial f}{\partial z} + Kf - 2b'f\right)(Y'_c)\,dc \tag{2.43}$$

is a martingale.

Proof. The term $-2b'f(Y'_c)$ in (2.43) comes from the killing in (2.40). Moreover, under $\bar{P}'$, and before killing, the excursions of the processes considered in Theorem 2.8 are the same as in Theorem 2.6. The theorem follows easily from the proof of Theorem 2.6. □

d) Factorization of a Second Order Differential Operator

The second order differential operator $\mathscr{L}$ has been defined in (2.5). We will now use the results of Sect. 1 and the previous results of this section to put $\frac{\partial}{\partial t}+\mathscr{L}$ in product form.

For simplicity we still assume that b is $\geqq 0$. We have the fundamental result.

Theorem 2.11. *Take $g\in C_c^\infty(R\times R\times R^n)$. For $Y_0=(\bar{z}_0, T, \bar{x}_0)\in R\times R\times R^n$, if $\bar{P}$ is the probability measure defined in Definition 2.1, if $\bar{P}'$ is the measure defined in Definition 2.9, set*

$$\begin{aligned} u(Y_0)&=E^{\bar{P}}\int_0^{+\infty} g(\bar{z}_t, T+t, x_t)\,dt\\ v(Y_0)&=2E^{\bar{P}'}\int_0^{+\infty} g(Y_c')\,dc. \end{aligned} \tag{2.44}$$

Then

$$u(Y_0)=E^{\bar{P}}\int_0^{+\infty} v(Y_c)\,dc. \tag{2.45}$$

Proof. Clearly

$$\int_0^{+\infty} g(\bar{z}_t, T+t, x_t)\,dt=\sum_{T_{a+\bar{z}_0}\neq T_{\bar{a}+\bar{z}_0}}\ \int_{T_{\bar{a}+\bar{z}_0}}^{T_{a+z_0}} g(\bar{z}_t, T+t, \bar{x}_t)\,dt.$$

From Theorem 2.6 and its proof, we find that under $\bar{P}$, the excursion measure is $Z_\sigma'^{a,x}(e,w)\,dn^+(e)\,dP(w)$.

Since g has compact support, under the assumptions of Definition 2.3

$$\begin{aligned} &\int_{\mathscr{W}^+\times\Omega'}\left[\int_0^\sigma |g(\bar{z}_u', T+u, x_u')|\,du\right] Z_\sigma'\,dn^+(e)\,dP'(w)\\ &\quad\leqq c\int_{\mathscr{W}^+\times\Omega'}[\sigma\wedge k]\,Z_\sigma'\,dn^+(e)\,dP'(w). \end{aligned} \tag{2.46}$$

By Proposition 2.4, the r.h.s. of (2.46) is uniformly bounded so that using the definition of compensators, the martingale property of Z_u' for $u>0$, which follows from (2.11), and (2.6), we get

$$\begin{aligned} u(Y_0)=E^{\bar{P}}\Bigg\{&\int_0^{+\infty} da\int_{\mathscr{W}^+\times\Omega'}\left[\int_0^\sigma g(\bar{z}_u', T+T_{a+\bar{z}_0}+u, x_u')\,du\right]\\ &\cdot Z_u'^{a,x_{T_{a+\bar{z}_0}}}(e,w)\,dn^+(e)\,dP'(w)\Bigg\}. \end{aligned} \tag{2.47}$$

In (2.47), the integral $\int_{\mathscr{W}^+\times\Omega'}[\ldots]\,Z_u'^{a,x_{T_{a+\bar{z}_0}}}\,dn^+(e)\,dP'(w)$ is calculated as in Definition 2.3 with $x_0'=x_{T_{a+\bar{z}_0}}$, $\bar{z}_0'=a$.

From Theorem 1.2, it is clear that if $Y_a=(a+\bar{z}_0, T+T_{a+\bar{z}_0}, x_{T_{a+\bar{z}_0}})$ then

$$\int_{\mathscr{W}^+ \times \Omega'} \left[\int_0^\sigma g(\bar{z}'_u, T+T_{a+\bar{z}_0}+u, x'_u)\,du\right] Z_u'^{a, x_{T_{a+\bar{z}_0}}}(e, w)$$
$$\cdot\, dn^+(e)\,dP'(w) = v(Y_a). \tag{2.48}$$

The Theorem follows. □

Corollary. *Take* $g \in C_c^\infty(R \times R \times R^n)$. *For* $Y_0 = (\bar{z}_0, T, \bar{x}_0) \in R \times R \times R^n$, *for any* $c \geqq \bar{z}_0$, *if*

$$\begin{aligned} u'(Y_0) &= E^{\bar{P}} \int_0^{T_c} g(\bar{z}_t, T+t, x_t)\,dt \\ v'(Y_0) &= 2E^{\bar{P}'} \int_0^{+\infty} g(Y'_a)\,da \end{aligned} \tag{2.49}$$

then

$$u'(Y_0) = E^{\bar{P}} \int_0^c v'(Y_a)\,da. \tag{2.50}$$

Proof. The proof is identical to the proof of Theorem 2.11. □

At least formally, we may write using Theorems 2.6 and 2.10

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathscr{L}\right) u &= -g \\ \left(\frac{\partial}{\partial z} + K\right) u &= -v \\ \left(-\frac{\partial}{\partial z} + K - 2b'\right) v &= -2g. \end{aligned} \tag{2.51}$$

We then find that at least formally

$$\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial z^2} + b\frac{\partial}{\partial z} + X_0 + \frac{1}{2}X_i^2 = \frac{1}{2}\left[\frac{\partial}{\partial z} - K + 2b'\right]\left[\frac{\partial}{\partial z} + K\right]. \tag{2.52}$$

As pointed out in Remark 1, since K acts on $C_b^\infty(R \times R \times R^n)$, the r.h.s. of (2.52) is indeed well-defined so that (2.52) can be given rigorously justified.

Remark 2. If instead of being $\geqq 0$, b is assumed to have compact support, the previous results are still true. Indeed, for any $a \geqq 0$

$$E^{P_0 \otimes P'} Z_{T_{\bar{z}_0 + a}} = 1.$$

In the definition of v the only difficulty could come from the term

$$\exp - \int_0^a 2b(x_{T'_c}, \bar{z}_0 - c)\,dc. \tag{2.53}$$

However for c large enough, $b(x_{T'_c}, \bar{z}_0 - c) = 0$ so that (2.53) is still integrable for any probability measure. Of course if b is not $\geqq 0$, there is no longer any killing in the definition of $\bar{P}'$, but Theorem 2.11, its corollary and (2.52) extend in the obvious way.

Remark 3. Assume that b has compact support. In Definitions 2.3 and 2.5, replace e by $-e$, and let $\bar{K}$ be the associated operator. Let $\bar{Y}, \bar{Y}'$ be the processes corresponding to Y, Y'.

Instead of using the local maximums of the Brownian motion $\bar{z}$, we now use its local minimums.

(2.52) is still formally true for $\bar{K}$, so that at least formally

$$\left(\frac{\partial}{\partial z}+\bar{K}+2b'\right)\left(\frac{\partial}{\partial z}-\bar{K}\right)=\left(\frac{\partial}{\partial z}-K+2b'\right)\left(\frac{\partial}{\partial z}+K\right)=2\left[\frac{\partial}{\partial t}+\mathscr{L}\right]. \tag{2.54}$$

e) Operator Calculus and Excursion Theory

To simplify the discussion of what follows, we still assume that b has compact support.

We will now derive several relations between K and $\bar{K}$ and give their probabilistic interpretations.

Definition 2.12. $L_t(b)$ denotes the local time at $b\in R$ of $\bar{z}_t$. $A_t(b)$ is defined by

$$A_t(b)=\inf\{A\geqq 0;\ L_A(b)>t\}. \tag{2.55}$$

Besides (2.45) and the similar expression calculated with $\bar{K}$ instead of K, we will still give two other expressions of u, which has been defined in Theorem 2.11.

First of all, we have, using the notations of Theorem 2.11

$$\int_0^{+\infty} g(\bar{z}_t, T+t, x_t)\,dt=\int_{-\infty}^{+\infty} da \int_0^{+\infty} g(a, T+t, x_t)\,dL_t(a) \tag{2.56}$$

or equivalently

$$\int_0^{+\infty} g(\bar{z}_t, T+t, x_t)\,dt=\int_{\bar{z}_0}^{+\infty} da \int_0^{+\infty} g(a, T+A_t(a), x_{A_t(a)})\,dt + \int_{-\infty}^{\bar{z}_0} da \int_0^{+\infty} g(a, T+A_t(a), x_{A_t(a)})\,dt. \tag{2.57}$$

Of course in (2.57), we have used the fact that since b has compact support, for any $a\in R$, $L_\infty(a)=+\infty$ $\bar{P}$ a.s.

Definition 2.13. For $(\bar{z}_0, T, x_0)\in R\times R\times R^n$, set

$$h(\bar{z}_0, T, x_0)=E^{\bar{P}}\int_0^{+\infty} g(\bar{z}_0, T+A_t(\bar{z}_0), x_{A_t(\bar{z}_0)})\,dt. \tag{2.58}$$

We then have the easy

Theorem 2.14. *The following equality holds*

$$u(Y_0)=E^{\bar{P}}\int_{\bar{z}_0}^{+\infty} h(Y_a)\,da+E^{\bar{P}}\int_{-\infty}^{\bar{z}_0} h(\bar{Y}_a)\,da. \tag{2.59}$$

Proof. Integrating (2.57) with respect to $\bar{P}$, the first term in the r.h.s. of (2.57) produces the first term in the r.h.s. of (2.59). $\bar{Y}$ (which is constructed by means of the local minimums of $\bar{z}$) appear similarly by integrating the second term in the r.h.s. of (2.57). □

Remark 4. Under $\bar{P}$, excursion theory shows that if $f\in C_b^\infty(R\times R\times R^n)$

$$f(\bar{z}_0,\,T+A_t(\bar{z}_0),\,x_{A_t}(\bar{z}_0))-\int_0^t\left(\frac{K+\bar{K}}{2}f\right)(\bar{z}_0,\,T+A_s(\bar{z}_0),\,x_{A_s}(\bar{z}_0))\,ds$$

is a $\{\bar{F}_{A_t(\bar{z}_0)}\}_{t\geqq 0}$ local martingale.

At least formally, we can then write that

$$\left(\frac{K+\bar{K}}{2}\right)h=-g \tag{2.60}$$

or equivalently

$$h=2(K+\bar{K})^{-1}(-g). \tag{2.61}$$

Now in (2.59), u is given by

$$u=\left(\frac{\partial}{\partial z}+K\right)^{-1}(-h)+\left(-\frac{\partial}{\partial z}+\bar{K}\right)^{-1}(-h). \tag{2.62}$$

Equivalently

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathscr{L}\right)^{-1}(-g)&=2\left[\left(\frac{\partial}{\partial z}+K\right)^{-1}+\left(-\frac{\partial}{\partial z}+\bar{K}\right)^{-1}\right](K+\bar{K})^{-1}g\\&=2\left(\frac{\partial}{\partial z}+K\right)^{-1}(K+\bar{K})\left(-\frac{\partial}{\partial z}+\bar{K}\right)^{-1}(K+\bar{K})^{-1}g.\end{aligned} \tag{2.63}$$

From (2.63), at least formally, we get

$$\left(\frac{\partial}{\partial t}+\mathscr{L}\right)=\tfrac{1}{2}(K+\bar{K})\left(\frac{\partial}{\partial z}-\bar{K}\right)(K+\bar{K})^{-1}\left(\frac{\partial}{\partial z}+K\right). \tag{2.64}$$

Comparing with (2.54), we see that formally

$$\frac{\partial}{\partial z}-K+2b'=(K+\bar{K})\left(\frac{\partial}{\partial z}-\bar{K}\right)(K+\bar{K})^{-1} \tag{2.65}$$

or equivalently that

$$\left(\frac{\partial}{\partial z}-K+2b'\right)(K+\bar{K})=(K+\bar{K})\left(\frac{\partial}{\partial z}-\bar{K}\right). \tag{2.66}$$

Still (2.66) has a nice probabilistic interpretation. Namely we may write

$$\int_0^{+\infty} 1_{\bar{z}_t \leqq \bar{z}_0} g(\bar{z}_t, T+t, x_t)\, dt = \sum_{A_s^-(\bar{z}_0) \neq A_s(\bar{z}_0)} \int_{A_s^-(\bar{z}_0)}^{A_s(\bar{z}_0)} 1_{\bar{z}_t \leqq \bar{z}_0} g(\bar{z}_t, T+t, x_t)\, dt. \quad (2.67)$$

As in the proof of Theorem 2.11, but working instead with the Poisson point process of the excursions of $\bar{z}$ out of $\bar{z}_0$, we obtain

$$E^{\bar{P}} \int_0^{+\infty} 1_{\bar{z}_t \leqq \bar{z}_0} g(\bar{z}_t, T+t, x_t)\, dt = E^{\bar{P}} \int_0^{+\infty} \frac{v}{2}(\bar{z}_0, T+A_s(\bar{z}_0), x_{A_s(\bar{z}_0)})\, ds. \quad (2.68)$$

The factor 1/2 on the r.h.s. of (2.53) comes form the fact that excursions below $\bar{z}_0$ are chosen with weight 1/2.

Now using (2.59), the 1.h.s. of (2.68) is exactly

$$2\left(-\frac{\partial}{\partial z}+\bar{K}\right)^{-1}(K+\bar{K})^{-1} g. \quad (2.69)$$

The r.h.s. of (2.68) is

$$2(K+\bar{K})^{-1}\left(-\frac{\partial}{\partial z}+K-2b'\right)^{-1} g. \quad (2.70)$$

From (2.69), (2.70), (2.66) follows.

f) Williams Decomposition and Last Exit Decomposition of the Brownian Motion for a σ-finite Measure

As a by product of Theorem 1.2 and of what has been previously done, we will obtain a useful description of $z.$.

Definition 2.15. On $R^+ \times \mathscr{C}(R^+; R)$, $d\bar{P}_0(t, \bar{z})$ is the σ-finite measure

$$d\bar{P}_0(t, z) = 1_{t \geqq 0}\, dt\, dP_0(z). \quad (2.71)$$

We then have a result which is closely related to Theorem 1.2 and which gives a Williams decomposition [31, 32] of $z.$ under $d\bar{P}_0$. This answers a question left open in Sect. 1a).

Theorem 2.16. *On $R^+ \times R^+ \times \Omega \times \Omega$, consider the σ-finite measure*

$$dS(M, a, \zeta, r) = 1_{u \geqq 0, a \geqq 0}\, 2\, dM\, da\, dP_0(\zeta)\, dQ_0(r). \quad (2.72)$$

Let $m, m', \bar{t}$ be the random variables

$$\begin{aligned} m &= \inf\{s \geqq 0, \zeta_s = M\} \\ m' &= \sup\{s \geqq 0; r_s = a\} \\ \bar{t} &= m + m'. \end{aligned} \quad (2.73)$$

Let $\bar{z}.$ be the process defined by

$$\begin{aligned} \bar{z}_s &= \zeta_s & 0 \leqq s \leqq m \\ M - r_{s-m} & & m < s \leqq \bar{t}. \end{aligned} \tag{2.74}$$

Then the law of $(\bar{t}, \bar{z}.)$ is identical to the law of $(t, z._{\wedge t})$ under $\bar{P}_0$.

Proof. Let $H_s(z)$ be a bounded $\{F_s\}_{s \geqq 0}$ predictable process. Assume that for $s \geqq T$, $H_s(z) = 0$.

The process $M.$ has been defined in (1.4) and $m.$ has been defined in (1.6).

For $z \in \mathscr{C}(R^+; R)$, we may write as in Ikeda-Watanabe [14], Bismut [6] -3

$$z = (z \,|s|\, \bar{\theta}_s z) \tag{2.75}$$

where

$$(\bar{\theta}_s z)_u = z_{s+u} - z_s \tag{2.76}$$

i.e. the trajectory of z is decomposed into the part before s and the part after s. If for $M \geqq 0$

$$T_M = \inf\{s \geqq 0;\ z_s > M\} \tag{2.77}$$

we have

$$\int_0^{+\infty} H_s(z)\, ds = \sum_{T_{\bar{M}} < T_M} \int_{T_{\bar{M}}}^{T_M} (H_s(z \,|T_{\bar{M}}|\, \bar{\theta}_{T_{\bar{M}}} z)\, ds \tag{2.78}$$

so that by using excursion theory

$$E^{P_0} \int_0^{+\infty} H_s(z)\, ds = E^{P_0} \int_0^{+\infty} dM \int_{\mathscr{W}^+} \left[\int_0^{\sigma} H_{T_M+u}(z \,|T_{\bar{M}}|- e)\, du\right] dn^+(e). \tag{2.79}$$

From Theorem 1.2, we know that

$$\begin{aligned} &\int_{\mathscr{W}^+} \left[\int_0^{\sigma} H_{T_M+u}(z \,|T_M|- e)\, du\right] dn^+(e) \\ &\quad = 2 \int_{\mathscr{W}^+} dQ_0(r) \left[\int_0^{+\infty} da\, H_{T_M+A(a)}(z \,|T_M|- r)\right] \end{aligned} \tag{2.80}$$

where

$$A(a) = \sup\{s \in R^+;\ r_s = a\}. \tag{2.81}$$

From (2.79)–(2.81), we find

$$\begin{aligned} E^{P_0} \int_0^{+\infty} H_s(z)\, ds = &\int_{R^+ \times R^+ \times \Omega \times \mathscr{W}^+} 1_{M \geqq 0,\, a \geqq 0}\, 2\, dM\, da\, dP_0(z)\, dQ_0(r) \\ &\cdot H_{T_M(z)+A(a)}(z \,|T_M|- r). \end{aligned} \tag{2.82}$$

The Theorem follows. □

Corollary. *For $c > 0$, set*

$$T_c = \inf\{s \geqq 0;\ z_s = c\}. \tag{2.83}$$

Then under the measure $1_{M \leqq c} dS$, *the law of* $(\bar{t}, \bar{z}.)$ *is identical to the law of* $(t, z._{\wedge t})$ *under* $1_{t \leqq T_c} d\bar{P}_0$.

Proof. Clearly, using the notations of Theorem 2.16

$$(t \leqq T_c) = (M \leqq c). \tag{2.84}$$

The corollary follows. □

Remark 5. From a result which is equivalent to Pitman [29] (see Ikeda-Watanabe [14]), we know that $r_{m'-s}$ $(0 \leqq s \leqq m')$ is a Brownian motion stopped when it hits a. The description of $(t, z._{\wedge t})$ given in Theorem 2.16 is then in a sense time reversible.

Recall that conditionally on $(m+m'=t, \bar{z}_{m+m'}=z)$, $\bar{z}_s$ $(0 \leqq s \leqq t)$ is a Brownian bridge.

We now give a last exit decomposition of z under $d\bar{P}_0(t, z)$ which answers a second question raised in Sect. 1a).

Theorem 2.17. *On* $R^+ \times R^+ \times \Omega \times \Omega \times \{-1, +1\}$, *consider the* σ*-finite measure*

$$dS'(L, a, \zeta, r, \varepsilon) = 1_{L \geqq 0, a \geqq 0}\, dL\, da\, dP_0(\zeta)\, dQ_0(r)(\delta_1 + \delta_{-1})(\varepsilon). \tag{2.85}$$

Let λ_s *be the local time at* 0 *of* ζ. *Set*

$$\begin{aligned} l &= \inf\{s \geqq 0;\ \lambda_s = L\} \\ l' &= \sup\{s \geqq 0;\ r_s = a\} \\ \bar{t} &= l + l'. \end{aligned} \tag{2.86}$$

Let $\bar{z}$ *be the process defined by*

$$\begin{aligned} \bar{z}_s &= \zeta \qquad 0 \leqq s \leqq l \\ &= \varepsilon r_{s-l} \qquad l < s \leqq \bar{t}. \end{aligned} \tag{2.87}$$

Then the law of $(\bar{t}, \bar{z}.)$ *is identical to the law of* $(t, z._{\wedge t})$ *under* $\bar{P}_0$.

Proof. The proof still uses Theorem 1.2 and excursion theory as in Theorem 2.16. It is left to the reader. □

Remark 6. Results very much like Theorem 2.16 and 2.17 still hold if certain boundary conditions (Itô-McKean [15]) are imposed on z, like reflection or killing on $(z=-1)$.

Also observe that if m is a C^∞ function and if g is $a>0$ function such that

$$\frac{g''}{2} - mg = 0 \tag{2.88}$$

then

$$\frac{\partial^2}{\partial z^2} - 2m = \left(\frac{\partial}{\partial z} + \frac{g'}{g}\right)\left(\frac{\partial}{\partial z} - \frac{g'}{g}\right). \tag{2.89}$$

(2.89) is a decomposition of the operator in the l.h.s. of (2.89) very much like (2.52). Of course the probabilistic interpretation is basically the same.

III. Regularity at the Boundary of Transition Densities

In this section, we use the decompositions of the Brownian motion which have been found in Sect. 2 to study the regularity at the boundary of the transition probabilities of certain hypoelliptic diffusion with boundary conditions.

In a), we consider the case of diffusions which live in the half space $(z<1)$ and are killed at $(z=1)$. In b) we consider diffusions with elastic reflection on $(z=0)$.

The method consists in first using the transversal coordinate z as a new time, use the Malliavin calculus on certain components of the new process, and the calculus of variations on Brownian excursions (which we have developed in [6] – Sect. 4) on the other so as to control the transition densities at the boundary. In the manner of Stroock [30], we show how these results, which have been proved on a half space, can be localized so as to apply to general boundaries. We assume that the boundary is non characteristic for the considered diffusion, and also that a slightly less general assumption than Hörmander's [13] is verified.

Although the method is entirely probabilistic, it has close connections with what the analysts are accustomed to do for boundary problems in the elliptic case [34], that is to change the problem into the analysis of certain pseudo-differential operators acting on the boundary.

As pointed out in the Introduction, Derridj [37] studied the Dirichlet problem for hypoelliptic operators verifying Hörmander's assumptions. In [37]. Derridj uses purely analytic techniques.

a) Killing on a Boundary

The notations are the same as in Sect. 2a).

To simplify, we assume that $\bar{z}_0=0$. T_1 denotes the stopping time

$$T_1=\inf\{t\geqq 0;\ \bar{z}_t=1\}. \tag{3.1}$$

We consider the following assumption

H1: At $(x,z)\in R^n\times R$, the vector space spanned by $X_1(x,z),\ldots,X_m(x,z)$ and the Lie brackets at (x,z) of $\left(X_0,X_1,\ldots,X_m,\frac{\partial}{\partial z}\right)$ where one of the vector fields $(X_1,\ldots,X_m)$ appears at least once is equal to R^n.

H1 is slightly more restrictive than [3] and [21]. It implies that

$$\frac{\partial}{\partial t}+\mathscr{L}$$

verifies Hörmander's assumptions, but $\frac{\partial}{\partial z}$ plays a special role, like in Bismut [6] (for the precise relationship of H1 with Hörmander's assumptions, see Remark 2). $C^\infty(R^n\times]-\infty,1])$ is the set of C^∞ functions $f(x,z)$ on $R^n\times]-\infty,1]$ such that f and all its derivatives extend continuously to $(z=1)$.

Theorem 3.1. *If* H1 *is verified on* $R^n \times]-\infty, +\infty[$, *for* $t>0$, *the law of* $(x_t, \bar{z}_t)$ *under the measure* $1_{t<T_1} d\bar{P}$ *is given by* $1_{z \leqq 1} p_t(x,z)\,dx\,dz$, *where* $p_t(x,z) \in C^\infty(R^n \times]-\infty, 1])$ *and is* 0 *on* $(z=1)$.

Proof. Using the localization procedure of Stroock [30] it is easy to see that $p_t(x,z)$ exists and is C^∞ on $]0,+\infty[\times R^n \times]-\infty, 1[$. We now concentrate on the behavior of $p_t(.,.)$ at the boundary $(z=1)$.

Take $f \in C_c^\infty(R \times R^n \times R)$ whose support is included in $]0,+\infty[\times R^n \times [0, \infty[$.

Clearly

$$E^{\bar{P}} \int_0^{T_1} f(t, x_t, \bar{z}_t)\,dt = E^{P_0 \otimes P'} \int 1_{t<T_1} Z_t f(t, x_t, \bar{z}_t)\,dt. \tag{3.2}$$

We will now use Theorem 2.16, its corollary and the notations therein. (3.2) is equal to

$$2\int \Big[\int_{\substack{0 \leqq M \leqq 1 \\ a \geqq 0}} Z_{\bar{t}} f(\bar{t}, x_{\bar{t}}, M-a)\,dM\,da \Big]\,dP_0(\zeta)\,dQ_0(r)\,dP'(w). \tag{3.3}$$

Using the variables $(c,a)=(M-a,a)$ instead of (M,a), (3.3) is equal to

$$2\int \Big[\int_{\substack{0 \leqq a+c \leqq 1 \\ 0 \leqq a}} Z_{\bar{t}} f(\bar{t}, x_{\bar{t}}, c)\,dc\,da \Big]\,dP_0(\zeta)\,dQ_0(r)\,dP'(w). \tag{3.4}$$

Of course we can assume that in (3.4), a Brownian motion B (independent of (ζ, w)) has been given such that if

$$N_s = \sup_{v \leqq s} B_v \tag{3.5}$$

then

$$r_s = 2N_s - B_s \tag{3.6}$$

$$m' = \inf\{s \geqq 0; B_s = a\}.$$

Instead of (3.4), we will write

$$2\int \Big[\int_{\substack{0 \leqq a+c \leqq 1 \\ 0 \leqq a}} Z_{\bar{t}} f(\bar{t}, x_{\bar{t}}, c)\,dc\,da \Big]\,dP_0(\zeta)\,dP_0(B)\,dP'(w). \tag{3.7}$$

Let H be the measure

$$dH(\zeta, B, w) = 2\,dP_0(\zeta)\,dP_0(B)\,dP'(w).$$

Take $\varepsilon>0$, and $g \in C_c^\infty(]0,+\infty[\times R^n)$ whose support is included in a compact subset K of $[\varepsilon, +\infty[\times R^n$.

For one given c such that $0 \leqq c < 1$, we consider the integral

$$\int \Big[\int_{0 \leqq a \leqq 1-c} Z_{\bar{t}}\, g(\bar{t}, x_{\bar{t}})\,da \Big]\,dH. \tag{3.8}$$

We claim that for any multi-index $m=(m^1 \dots m^l)$, any $k \in N$, then

$$\left|\int\left[\int_{0\leqq a\leqq 1-c} Z_{\bar t}\frac{\partial^{|m|}g}{\partial x^m}(\bar t, x_{\bar t})\,da\right]dH\right|\leqq C_K^m \sup|g(t,x)| \tag{3.9}$$

$$\left|\int\left[\int_{0\leqq a\leqq 1-c} Z_{\bar t}\frac{\partial^{k}g}{\partial t^k}(\bar t, x_{\bar t})\,da\right]dH\right|\leqq C_K'^k \sup|g(t,x)|.$$

We first obtain the first line of (3.9). Note that

$$\begin{aligned} Z_{\bar t}=&\exp\left\{\int_0^m b(x_s,\zeta_s)\,\delta\zeta_s-\tfrac{1}{2}\int_0^m b^2(x_s,\zeta_s)\,ds\right\}\\ &\cdot\exp\left\{-2\int_0^{m'} b(x_{s+m}, c+a+B-2N)\,dN\right\}\\ &\cdot\exp\left\{\int_0^{m'} b(x_{s+m}, c+a+B-2N)\,\delta B-\tfrac{1}{2}\int_0^{m'} b^2(x_{s+m}, c+a+B-2N)\,ds\right\}. \end{aligned} \tag{3.10}$$

Now if $s\leqq m'$, $N_s\leqq a\leqq 1-c\leqq 1$. Moreover since g has compact support if $\bar t\geqq T$, $g(\bar t, x_{\bar t})=0$.

Since the first and last exponentials in (3.10) are standard Girsanov martingales, we see that $1_{\bar t\leqq T}Z_{\bar t}$ is in all the $L_p(H)\,(1\leqq p<+\infty)$ with bounds on the L_p-norms only depending on T.

Moreover observe that $g(\bar t, x_{\bar t})$ is $\neq 0$ only if $\bar t\geqq\varepsilon$. As in Bismut-Michel [9], Bismut [6], we can then use the partial Malliavin calculus on w (which leaves ζ, B unchanged) so that the first line in (3.9) is easily obtained.

We now come to the second line of (3.9). Set

$$M_s=\sup_{v\leqq s}\zeta_v.$$

Then if η_s, η'_s, are defined by

$$\eta_s=M_s-\zeta_s$$

$$\eta'_s=N_s-B_s$$

η_s, η'_s are two independent reflecting Brownian motions on $[0,+\infty[$, and η'' defined by

$$\eta''_s=\begin{cases}\eta_s & 0\leqq s\leqq m\\ \eta'_{s-m} & m\leqq s\leqq m'\end{cases}$$

is a reflecting Brownian motion, whose standard local time at 0 is M_s for $0\leqq s\leqq m$, $c+a+N_{s-m}$ for $m\leqq s\leqq m'$.

On η'', we can then apply the calculus of variations of Bismut [6] Section 4, which produces separate variations of η, η' and so respects the description of x on the time intervals $[0,m]$ and $[m,m']$.

For one given $a>0$, we find that

$$\int Z_{\bar t}\frac{\partial^k g}{\partial t^k}(\bar t, x_{\bar t})\,dH=\int Z_{\bar t}\,g(\bar t, x_{\bar t})\,R^k\,dH \tag{3.11}$$

(both sides of (3.11) depend on a).

Moreover from [6] – Sect. 4, we know that the L_p norm of $1_{t \leqq T} R^k$ is controlled by the L_p norm of $\frac{1}{\int_0^{m+m'} \eta'' ds}$.

Now if $g(m+m', x_{\bar{t}}) \neq 0$, $m+m' \geqq \varepsilon$. Since by Malliavin [23], Ikeda-Watanabe [14], we know that

$$\left[\int_0^\varepsilon \eta'' ds\right]^{-1}$$

is in all the $L_p(H)(1 \leqq p < +\infty)$, we find that the L_p norm of $1_{\varepsilon \leqq \bar{t} \leqq T} R^k$ is bounded uniformly. The second line in (3.9) immediately obtains by integrating in $a \in [0, 1-c]$.

Using the properties of Fourier transform as in Malliavin [22], Bismut [6]. Section 4, we find from (3.9) that for any c such that $0 \leqq c \leqq 1$, there is a function $q(.,.,c)$ on $]0, +\infty[\times R^n$ which has the following properties:

- It is C^∞ on $]0, +\infty[\times R^n$.
- For any ε, T such that $0 \leqq \varepsilon < T < +\infty$, the k^{th} derivatives of $q(t, x, c)$ in (t, x) are bounded on $[\varepsilon, T] \times R^n$ independently of $c \in [0, 1]$.
- For any $g \in C_c^\infty(R \times R^n)$

$$\int \Big[\int_{0 \leqq a \leqq 1-c} Z_{\bar{t}}\, g(\bar{t}, x_{\bar{t}})\, da \Big]\, dH = \int g(t, x)\, q(t, x, c)\, dx\, dt. \tag{3.12}$$

Moreover using the properties of the l.h.s. of (3.12), it is trivial to prove that $c \in [0, 1] \to q(t, x, c)\, dx\, dt$ (considered as a σ-finite measure on $R^+ \times R^n$) is continuous (the set of σ-finite measures on $R^+ \times R^n$, considered as the dual of the set of continuous functions on $R^+ \times R^n$ with compact support, is endowed with the corresponding weak topology).

Inspection of (3.3), (3.7), (3.12) shows that

$$p_t(x, c) = q(t, x, c) \qquad \text{a.e. on } R^+ \times R^n \times [0, 1]. \tag{3.13}$$

Since for a given $c \in [0, 1]$, $p_t(x, c)$ and $q(t, x, c)$ are C^∞ on $R^+ \times R^n$, we deduce from (3.13) that for a.e. $c \in]0, 1[$

$$p_t(x, c) = q(t, x, c) \qquad \text{on }]0, +\infty[\times R^n. \tag{3.14}$$

Since $c \to q(t, x, c)\, dx\, dt$ is continuous from $]0, 1[$ in the set of σ-finite measures, and since $p_t(x, c)$ is C^∞ on $]0, +\infty[\times R^n \times]0, 1[$, (3.14) holds for every $c \in]0, 1[$.

Let $\mathscr{L}^*$ be the formal adjoint of $\mathscr{L}$ with respect to the Lebesgue measure. The function $p_t(x, c)$ which is smooth on $]0, +\infty[\times R^n \times]0, 1[$ verifies the Fokker-Planck equation

$$\frac{\partial p_t}{\partial t}(x, c) = \mathscr{L}^* p_t(x, c)$$

which also writes

$$\frac{\partial p_t}{\partial t}(x,c)=\frac{1}{2}\frac{\partial^2}{\partial c^2}p_t(x,c)-b(x,c)\frac{\partial p_t}{\partial c}(x,c)+M_x p_t(x,c) \tag{3.15}$$

where M_x is a differential operator which only acts on the variable x.

Since as $c\in]0,1[$, $q(t,x,c)$ has bounded derivatives in (t,x) for $\varepsilon\leqq t\leqq T$, we find from (3.15) that

$$\frac{1}{2}\frac{\partial^2 p_t}{\partial c^2}(x,c)-b(x,c)\frac{\partial p_t}{\partial c}(x,c)=d(t,x,c) \tag{3.16}$$

where for $\varepsilon\leqq t\leqq T$, $d(t,x,c)$ is smooth in (t,x) with uniformly bounded derivatives.

Now by taking initial conditions in (3.16) on $(c=0)$, (3.16) can be explicitly integrated. We find that for $c<1$

$$\frac{\partial p_t}{\partial c}(x,c)=\exp\left\{\int_0^c 2b(x,a)\,da\right\}\left[\frac{\partial p_t}{\partial c}(x,0)+\int_0^c \exp\left\{-\int_0^c 2b(x,a)\,da\right\}2d(t,x,h)\,dh\right] \tag{3.17}$$

(3.17) shows that $\frac{\partial p_t}{\partial c}(x,c)$ extends continuously up to $c=1$ as well as all its derivatives in $(t,x)\in[\varepsilon,T]\times R^n$. Of course the same result holds for $p_t(x,c)$.

Because of what has been proved before, $d(t,x,c)$ extends continuously up to $c=1$. (3.16) show that the same result holds for $\frac{\partial^2 p_t}{\partial c^2}(x,c)$.

By differentiating (3.16) as many times as needed, and proceeding by induction, we see that $p_t(x,c)$ extends to a C^∞ function on $]0,+\infty[\times R^n\times]-\infty,1]$. (3.12) shows that as $c\uparrow\uparrow 1$, $q(t,x,c)\,dt\,dx$ converges weakly to 0. $p_t(x,1)$ is then necessarily 0. □

Remark 1. In this special case, it would have been possible to control $\frac{\partial p_t}{\partial c}(x,c)$ by using (3.8) without using the explicit form of the Fokker-Planck equation.

Namely by writing $l=l_M$, $l'=l'_a$, recall that when $b=0$, l_M has the same law as $M^2 l_1$, and l'_a has the same law as $a^2 l_1$. We could then have renormalized the time scale of the equation giving x, so that direct differentiation of (3.8) in c would have been possible.

However recall that l_1 is *not* integrable, and l_1 would appear in differentiating in c. In this case the support property of g would do the necessary cutoff. This is not the case in the theorem which follows. This is why we have preferred to directly use the Fokker-Planck equation.

b) The Case of a Reflecting Process

We now consider the case of a reflecting process as in Ikeda-Watanabe [14], Bismut [6].

$X_0, X_1, \ldots, X_m, b$ are taken as before.

$D(x)$ is a C^∞ vector field defined in R^n which is bounded with bounded differentials.

z is now a reflecting Brownian motion on $[0, +\infty[$ starting at $z_0 \in R^+$. $P^r_{z_0}$ denotes its law on $\mathscr{C}(R^+; R)$. L is the standard local time of z. at 0. The Brownian martingale B is defined by

$$z_t = L_t + B_t.$$

On $(\bar{\Omega}, P^r_{z_0} \otimes P')$, we consider the stochastic differential equation

$$dx = X_0(x,z)\,dt + X_i(x,z)\cdot dw^i + D(x)\,dL \tag{3.18}$$

$$x(0) = x_0.$$

Set

$$Z_t = \exp\left\{\int_0^t b(x,z)\,\delta B - \tfrac{1}{2}\int_0^t b^2(x,z)\,ds\right\}.$$

A new probability measure $\bar{P}$ is defined on $\bar{\Omega}$ by

$$\left.\frac{d\bar{P}}{d(P^r_{z_0} \otimes P')}\right|_{F_t} = Z_t.$$

We now have

Theorem 3.2. *If Assumption $H1$ is verified on $R^n \times R$, for any $t > 0$, the law of (x_t, z_t) under $\bar{P}$ is given by $1_{z>0}\, p_t(x,z)\,dx\,dz$, where $p_t(x,z) \in C^\infty(R^n \times [0, +\infty[)$.*

Proof. The same argument as in Theorem 3.1 shows that $p_t(x,z)$ is C^∞ on $]0, +\infty[\times R^n \times]0, +\infty[$. We will concentrate on the behavior of $p_t(x,z)$ at $(z=0)$. Moreover we will assume that $z_0 = 0$. The case where $z_0 > 0$ can be easily dealt with, by the same technique.

Take $f \in C_c^\infty(R \times R^n \times R)$. Obviously

$$E^{\bar{P}} \int_0^{+\infty} f(t, x_t, z_t)\,dt = E^{P^r_0 \otimes P'} \int_0^{+\infty} Z_t f(t, x_t, z_t)\,dt. \tag{3.19}$$

We now will use Theorem 2.17 (which trivially applies to the reflecting Brownian motion z).

Let ζ_t be a reflecting Brownian motion on $[0, +\infty[$, λ_t its standard local time at 0. Let B_t be the Brownian motion

$$B_t = \zeta_t - \lambda_t. \tag{3.20}$$

Let B'_t be a Brownian motion independent of ζ. Set

$$N'_s = \sup_{v \leqq s} B'_v \tag{3.21}$$

$$r_s = 2N'_s - B'_s$$

$$\zeta'_s = N'_s - B'_s.$$

For $L \geqq 0$, $a \geqq 0$, we define

$$\begin{aligned} l &= \inf\{s \geqq 0; \lambda_s = L\} \\ l' &= \inf\{s \geqq 0; B'_s = a\}. \end{aligned} \tag{3.22}$$

We finally define

$$\begin{aligned} Z'_s &= \exp\left\{\int_0^s b(x_u, \zeta_u)\,\delta B_u - \tfrac{1}{2}\int_0^s b^2(x_u, \zeta_u)\,du\right\} \quad \text{if } s \leqq l \\ &= Z'_l \exp\left\{2\int_0^{s-l} b(x_{l+u}, r_u)\,dN'\right\} \\ &\quad \cdot \exp\left\{-\int_0^{s-l} b(x_{l+u}, r_u)\,\delta B' - \tfrac{1}{2}\int_0^{s-l} b^2(x_{l+u}, r_u)\,du\right\} \quad \text{if } l \leqq s \leqq l + l'. \end{aligned} \tag{3.23}$$

Theorem 2.17 shows that (3.19) is equal to

$$2\int\Big[\int_{\substack{0 \leqq L < +\infty \\ 0 \leqq a < +\infty}} Z'_{l+l'} f(l+l', x_{l+l'}, a)\,dL\,da\Big]\,dP_0^r(\zeta)\,dP_0(B)\,dP'(w). \tag{3.24}$$

The basic difference with (3.7) is that L is integrated on the unbounded $[0, +\infty[$.

Let dH' be the measure

$$dH'(\zeta, B', w) = dP_0^r(\zeta)\,dP_0(B')\,dP'(w). \tag{3.25}$$

Take ε, T such that $0 < \varepsilon < T < +\infty$. Take $g \in C_c(R \times R^n)$ whose support is included in a compact subset K of $[\varepsilon, T] \times R^n$. For a such that $0 < a \leqq 1$, consider the integral

$$\int\Big[\int_{0 \leqq L} Z'_{l+l'}\,g(l+l', x_{l+l'})\,dL\Big]\,dH'. \tag{3.26}$$

Now $g(l+l', x_{l+l'})$ is 0 if $l + l' > T$, and so $g(l+l', x_{l+l'})$ is 0 if $L > \lambda_T$. (3.26) is then equal to

$$\int\Big[\int_{0 \leqq L \leqq \lambda_T} Z'_{l+l'}\,g(l+l', x_{l+l'})\,dL\Big]\,dH'. \tag{3.27}$$

Now λ_T is in all the $L_p(H')$ $(1 \leqq p < +\infty)$. This fact permits us to proceed exactly as in the proof of Theorem 3.1 and obtain the required result. □

Remark 2. The techniques of the proof of Theorem 3.2 can be made to work on a manifold.

Namely assume that $\bar{\mathscr{L}}$ is a second order differential operator on a manifold M, written in Hörmander form

$$\bar{\mathscr{L}}=\bar{X}_0+\tfrac{1}{2}\sum_1^r \bar{X}_i^2 \tag{3.28}$$

that ∂D is a smooth submanifold of M of codimension 1, and that $y\in\partial D$ is such that ∂D is non characteristic at y for $\bar{\mathscr{L}}$, i.e. that at least one of the vector fields $X_1(y)\dots X_m(y)$ is not tangent to ∂D.

Consider the Hamilton-Jacobi equation on a function z

$$\sum_1^r(\bar{X}_i z)^2=1; \qquad z=0 \quad \text{on } \partial D. \tag{3.29}$$

Since D is non characteristic for $\bar{\mathscr{L}}$, (3.45) has a single solution on a neighborhood of y. The fibration $(z=cst)$ is then intrinsically defined on a neighborhood of y. By proceeding as in Ikeda-Watanabe [14], Bismut [6]. Section 1, we can rotate the vector fields $\bar{X}_i$. New vector fields $\bar{X}'_1\dots\bar{X}'_r$ are then obtained locally, such that $\bar{X}'_1\dots\bar{X}'_{i-1}$ are tangent to the fibration $(z=cst)$, and $\bar{X}'_r$ is such that $[\bar{X}'_r z]^2=1$. We may then express $\bar{X}_0$ in the form

$$\bar{X}_0=X_0+b\bar{X}'_r \tag{3.30}$$

where X_0 is tangent for the fibration $(z=cst)$. By setting $\bar{X}'_r=\frac{\partial}{\partial z}$, $\bar{\mathscr{L}}$ writes

$$\bar{\mathscr{L}}=X_0+b\frac{\partial}{\partial z}+\frac{\partial^2}{\partial z^2}+\frac{1}{2}\sum_1^{r-1}\bar{X}_i'^2. \tag{3.31}$$

It is then a trivial matter to check that $H1$ expressed on $\bar{\mathscr{L}}$ in the form (3.31) is indeed intrinsically defined (the key fact is that the fibration $(z=cst)$ is intrinsic). Note that if the first order part of $\bar{\mathscr{L}}$ is not necessary to fullfill Hörmander's theorem, i.e. if $\bar{X}_1\dots\bar{X}_r$ and their Lie brackets at $y\in\partial D$ span R^n, if ∂D is non characteristic for $\bar{\mathscr{L}}$ at y, then $H1$ is verified at y.

Assume that ∂D is the boundary of an open domain D, that $H1$ is verified on a neighborhood of $y\in\partial D$, and that $y_0\in D$.

Let y_t be the Markov diffusion whose generator in D is $\bar{\mathscr{L}}$, and which either is killed on ∂D, or reflects on D. If y_t starts at y_0, for any $t>0$, the law of y_t is smooth in $D\cup\partial D$ in the two cases which we have considered.

To see this, we may use the technique of localization of Stroock [30], which basically amounts to proving uniform estimates in the two situations considered in Theorems 3.1 and 3.2. Instead of assuming that $\bar{t}$ is $\geqq\varepsilon>0$, we will assume that the starting point and the final point are far enough. The estimates are not very different from those which we have given.

In the case of reflection on the boundary ∂D, we can also assume that $y_0\in\partial D$, with $y\neq y_0$.

Using the methods of Kusuoka-Stroock [21] and adequate estimates on the transition probabilities, we could also prove regularity results at the boundary of the solution of Dirichlet or Neumann problems for the operator $\bar{\mathscr{L}}$. Of course we would have to work under assumption $H1$, which is stronger than Hörmander's [13]. At least in the case of Dirichlet problems, the results of Derridj [37] (who works under Hörmander's assumptions and also assumes that the boundary is non characteristic) are stronger than ours.

Remark 3. We could also assume that y diffuses on the boundary as in Ikeda-Watanabe [14] as long as reflection is elastic. In the case of inelastic reflection, there are some technical difficulties which we do not know for the moment how to solve.

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