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Note on Continuous Additive Functionals of the 1-Dimensional Brownian Path

By

HIROSHI TANAKA

1. Introduction

A functional $\mathfrak{a}(t) = \mathfrak{a}(t, w)$ of $t \geqq 0$ and a Brownian path $w: t \to x(t)$ is called *additive* if

1.1a $\qquad$ $\mathfrak{a}(t, w)$ *depends on* t *and* $x(s): 0 \leqq s \leqq t$ *only,*

1.1b $\qquad$ $\mathfrak{a}(t, w) = \mathfrak{a}(s, w) + \mathfrak{a}(t - s, w_s^+), \quad 0 \leqq s \leqq t,$

where w_s^+ is the *shifted* path $w_s^+: t \to x(t + s)$.

Additive functionals play an important rôle in transforming a Markov process, such as the substitution of time by the inverse functional of a non-negative additive functional (K. ITÔ and H. P. McKEAN [*7*], V. A. VOLKONSKII [*10*]) and the transformation of a Brownian motion to a diffusion with a drift (E. B. DYNKIN [*2*], M. MOTOO [*8*]). As for the structure of additive functionals of the several dimensional Brownian motion, A. D. VENTSEL [*9*] proved that such a functional $\mathfrak{a}$ can be written in the form:

1.2
$$\mathfrak{a}(t, w) = f(x(t)) - f(x(0)) + \int_0^t g(x(s)) \cdot dx(s),$$

under the assumption that $\mathfrak{a}(t, w)$ has a finite expectation for each $t < \infty$. Here, f and g are Borel functions on the state space, g satisfies $\int_0^t g(x(s))^2\, ds < \infty$ $(t < \infty)$ with probability one and the integral in 1.2 is a stochastic integral of K. ITÔ [*5*].

In this note, it will be proved that, for *every continuous* additive functional $\mathfrak{a}$ of the 1-dimensional Brownian path, $\mathfrak{a}(\tau, w): \tau = \min(t, \mathfrak{m})$ has finite moments of all orders $\geqq 0$, where $\mathfrak{m}$ is the first time $\mathrm{x}(\cdot)$ hits the complement of a bounded interval. This result combined with a slight modification of A. D. VENTSEL's proof implies that *any continuous additive functional of the* 1*-dimensional Brownian path has the representation* 1.2 *with continuous* f.

I wish to thank Professor H. P. McKEAN for helpful suggestions.

2. Existence of moments

Given the space W of continuous sample paths $w: t \in [0, \infty) \to w_t \in R^1$, write $w_t = x(t, w)$ (or $= x(t) = x_t$ for short), and introducing the corresponding coordinate fields $\boldsymbol{B}_t = \boldsymbol{B}[x(s): s \leqq t]$ and $\boldsymbol{B} = \boldsymbol{B}[x(s): s < \infty]$, let $P_a(\cdot)$ be the *Wiener measure* on $\boldsymbol{B}$ with initial position $a \in R^1$. $[x, \boldsymbol{B}, P.]$ is the so-called (standard) 1-dimensional *Brownian motion.* We use the notation $E_a(f, B)$ for the integral over $B \in \boldsymbol{B}$ of a $\boldsymbol{B}$-measurable function f with respect to P_a and write $E_a(f, B) = E_a(f)$ when $B = W$. The *first passage time* $\inf\{t : x(t) = r\}$ through r and the *first exit time* $\inf\{t : x(t) \notin I\}$ for an interval I are denoted respectively by $\mathfrak{m}_r$ and $\mathfrak{m}_I$. The following *strong Markov property** will be used often: if $\mathfrak{m}$ is a *Markov time***, then for $a \in R^1$ and $B \in \boldsymbol{B}$

$$2.1 \qquad P_a[(P_a(w_{\mathfrak{m}}^+ \in B \,|\, \boldsymbol{B}_{\mathfrak{m}+}) = P_{x(\mathfrak{m})}\,(B), \mathfrak{m} < \infty] = P_a(\mathfrak{m} < \infty),$$

where $w_{\mathfrak{m}}^+$ is the *shifted* path $w_{\mathfrak{m}}^+ : t \to x(t + \mathfrak{m})$ and $\boldsymbol{B}_{\mathfrak{m}+}$ is the field generated by the events $B \in \boldsymbol{B}$ such that $B \cap (w : \mathfrak{m} < t) \in \boldsymbol{B}_t$, $t \geqq 0$. By definition, a *continuous additive functional* is a functional $\mathfrak{a}(t, w)$ of t and w which satisfies the following conditions 2.2a—2.2c:

2.2a $\qquad \mathfrak{a}(t, w)$ *is* $\boldsymbol{B}_t$*-measurable for each* $t \geqq 0$,

2.2b $\qquad |\mathfrak{a}(t, w)| < \infty$ and $\mathfrak{a}(t, w)$ *is continuous in* t (a.e.),

2.2c $\qquad \mathfrak{a}(t, w) = \mathfrak{a}(s, w) + \mathfrak{a}(t - s, w_s^+), \quad 0 \leqq s \leqq t$ (a.e.),

where (a.e.) means (for every w outside a certain set which has P_a-measure zero for all $a \in R^1$).

Now, consider a continuous additive functional $\mathfrak{a}$ of the 1-dimensional Brownian path, and put

$$\bar{\mathfrak{a}}(t, w) = \max_{0 \leqq s \leqq t} |\mathfrak{a}(s, w)|.$$

Then, we have the following

Theorem 1. *For any bounded interval* $I = [r_1, r_2]$ *we have*

$$2.3 \qquad P_a[\bar{\mathfrak{a}}(\mathfrak{m}) > t] < c_1 e^{-c_2 t}, \quad a \in I,$$

where $\mathfrak{m} = \mathfrak{m}_I$, *and* c_1, c_2 *are some positive constants depending on* r_1 *and* r_2 *but not on* $a \in I$. *In particular,* $\mathfrak{a}(t \wedge \mathfrak{m})$*** *has finite moments of all orders* $\geqq 0$.

Proof. First we note that

$$2.4 \qquad -\bar{\mathfrak{a}}(t, w) + \bar{\mathfrak{a}}(s, w_t^+) \leqq \bar{\mathfrak{a}}(t + s, w) \leqq \bar{\mathfrak{a}}(t, w) + \bar{\mathfrak{a}}(s, w_t^+), \quad s, t \geqq 0 \text{ (a.e.)},$$

which follows from the additivity 2.2c of $\mathfrak{a}$. Let $\dot{I}$ be a bounded open interval containing I, and, for each $a \in \dot{I}$, choose a constant $t_a < \infty$ such that $P_a[\bar{\mathfrak{a}}(\dot{\mathfrak{m}}) > t_a] < 1/2$, where $\dot{\mathfrak{m}} = \mathfrak{m}_{\dot{I}}$. Then, using 2.4 and the strong Markov property 2.1, we have

$$\begin{aligned} 1/2 &\geqq P_a[\bar{\mathfrak{a}}(\dot{\mathfrak{m}}) > t_a] \\ &\geqq P_a[\bar{\mathfrak{a}}(\dot{\mathfrak{m}}) > t_a, \mathfrak{m}_b < \dot{\mathfrak{m}}], \qquad b \in \dot{I} \\ &\geqq P_a[\bar{\mathfrak{a}}(\dot{\mathfrak{m}}(w_{\mathfrak{m}_b}^+), w_{\mathfrak{m}_b}^+) - \bar{\mathfrak{a}}(\mathfrak{m}_b, w) > t_a, \mathfrak{m}_b < \dot{\mathfrak{m}}] \\ &\geqq P_a[\bar{\mathfrak{a}}(\dot{\mathfrak{m}}(w_{\mathfrak{m}_b}^+), w_{\mathfrak{m}_b}^+) > 2t_a, \mathfrak{m}_b < \dot{\mathfrak{m}}, \bar{\mathfrak{a}}(\mathfrak{m}_b) < t_a] \\ &= P_a[\mathfrak{m}_b < \dot{\mathfrak{m}}, \bar{\mathfrak{a}}(\mathfrak{m}_b) < t_a]\, P_b[\bar{\mathfrak{a}}(\dot{\mathfrak{m}}) > 2t_a]. \end{aligned}$$

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* See G. A. HUNT [4] or R. BLUMENTHAL [1].

** $\mathfrak{m}$ is called a Markov time if $\mathfrak{m} \geqq 0$ and $(w : \mathfrak{m} < t) \in \boldsymbol{B}_t$ for each $t \geqq 0$.

*** $a \wedge b\,(a \vee b)$ is the smaller (larger) of a and b.

Now, choosing a neighborhood U_a of a such that for all $b \in U_a$

$$P_a[\mathfrak{m}_b < \dot{\mathfrak{m}}, \bar{\mathfrak{a}}(\mathfrak{m}_b) < t_a] > 3/4,$$

it follows that

2.5 $$P_b[\bar{\mathfrak{a}}(\dot{\mathfrak{m}}) > 2t_a] < 2/3, \quad b \in U_a.$$

Because $\bar{\mathfrak{a}}(\mathfrak{m}) \leqq \bar{\mathfrak{a}}(\dot{\mathfrak{m}})$, 2.5 is true for $a \in I$ and $b \in U_a \cap I$ when $\dot{\mathfrak{m}}$ is replaced by $\mathfrak{m}$. Therefore, using Heine-Borel's covering theorem, it is clear that

2.6 $$P_a[\bar{\mathfrak{a}}(\mathfrak{m}) > T] < 2/3, \quad a \in I,$$

for some constant $T < \infty$ independent of a.

Next, put $\sigma_n = \inf\{t : \bar{\mathfrak{a}}(t) \geqq nT\}$ if $\bar{\mathfrak{a}}(t) \geqq nT$ for some $t < \infty$, and $= \infty$ if there is no such t ($n \geqq 1$). Because σ_n ($n \geqq 1$) is a Markov time and

$$\sigma_n \geqq \sigma_{n-1} + \sigma_1(w^+{\sigma_{n-1}}), \text{ (a.e.)}, \ n \geqq 2,$$

by 2.4, it follows from 2.6 and by induction that

$$\begin{aligned} P_a[\bar{\mathfrak{a}}(\mathfrak{m}) > nT] &= P_a[\sigma_n < \mathfrak{m}] \\ &\leqq P_a[\sigma_{n-1} + \sigma_1(w^+{\sigma_{n-1}}) < \mathfrak{m}] \\ &= E_a[P_{x(\sigma_{n-1})}(\sigma_1 < \mathfrak{m}), \ \sigma_{n-1} < \mathfrak{m}] \\ &\leqq (2/3)\, P_a(\sigma_{n-1} < \mathfrak{m}) \leqq (2/3)^n, \end{aligned}$$

and this implies 2.3.

3. Representation

Let g be a Borel function on R^1 such that

3.1 $$P_a[\int_0^t g^2(x_s)\,ds < \infty, \ 0 \leqq t < \infty] = 1, \qquad a \in R^1.$$

Then, the stochastic integral $\int_0^t g(x_s)\,dx_s$ is defined (K. Itô [5]).

We first remark that a version of this stochastic integral can be chosen so that it gives a continuous additive functional. When g is bounded, such a version exists, as was discussed by E. B. Dynkin [3]. When g is unbounded, we use the following simple lemma, which corresponds to lemma 1 in [3] and can be proved similarly.

Lemma. *Suppose the stochastic integral $\int_0^t g_n(x_s)\,dx_s$ has a version of continuous additive functional ($n = 1, 2, \ldots$). Under the notation $e(a, h) = E_a[\int_0^{\mathfrak{m}_I} h^2(x_s)\,ds]$, suppose, for each compact interval I, that $e(a, g)$ and $e(a, g_n)$ are finite for $a \in I$ and that $e(a, g - g_n)$ converges to zero uniformly in $a \in I$ as $n \to \infty$. Then, the stochastic integral $\int_0^t g(x_s)\,dx_s$ has a version of continuous additive functional.*

Given g satisfying 3.1, putting $g_n(b) = g(b)$ for $|g(b)| < n$ and $= 0$ for $|g(b)| \geqq n$, we show that the assumptions of the lemma are satisfied. Theorem 1 applied to the additive functional $\mathfrak{f}(t) = \int_0^t g^2(x_s)\,ds$ implies that $e(a, g)$ is finite, and using the additivity of $\mathfrak{f} \geqq 0$ and the strong Markov property, we see that e is concave and hence continuous on I. By the same reason $e(a, g_n)$ is continuous on I, and since $e(a, g_n)$ increases to $e(a, g)$ as $n \uparrow \infty$ on I, $e(a, g - g_n) = e(a, g) - e(a, g_n)$ tends to zero uniformly on I as $n \to \infty$ Thus the lemma is applicable.

From now on, we take a version of continuous additive functional for a stochastic integral.

Theorem 2. *A continuous additive functional* $\mathfrak{a}$ *of the* 1*-dimensional Brownian path can be written in the following form:*

3.2 $$\mathfrak{a}(t, w) = f(x_t) - f(x_0) + \int_0^t g(x_s)\, dx_s, \quad 0 \leqq t < \infty \qquad \text{(a.e.)},$$

where f is a continuous function on R^1 *and g satisfies* 3.1.

Because of theorem 1, the method of A. D. VENTSEL is applicable, but here we will give a proof which is different from A. D. VENTSEL's except as regards 3.12. Our method of obtaining the function g in 3.2 seems to be simpler.

Proof. Take $I = [r_1, r_2]$ as before, let $\mathfrak{m} = \mathfrak{m}_I$, and put

3.3a $$\mathfrak{s}(t, w) = \mathfrak{a}(t, w) - [f(x_t) - f(x_0)], \quad t \leqq \mathfrak{m},$$

3.3b $$f = f_I = -E_{\cdot}[\mathfrak{a}(\mathfrak{m})].$$

Denote by $G(a, b)$ the Green function:

$$G(a, b) = 2(a \wedge b - r_1)(r_2 - a \vee b)/(r_2 - r_1),$$

and by $L_2(a)$ $(r_1 < a < r_2)$ the space of those functions on I which are square integrable with respect to the measure $G(a, b)db$. $L_2(a)$ is a Hilbert space with inner product $(\Phi_1, \Phi_2)_a = \int_{r_1}^{r_2} G(a, b)\Phi_1(b)\, \Phi_2(b)\, db$ for each a; it is independent of $a \in (r_1, r_2)$ as a set. For each $\Phi \in L_2(a)$ let $\mathfrak{s}_\Phi(t \wedge \mathfrak{m})$ be the stochastic integral $\int_0^{t \wedge \mathfrak{m}} \Phi(x_s)\, dx_s$ and note that (see K. ITÔ [5])

3.4a $$E_a[\mathfrak{s}(t \wedge \mathfrak{m})] = E_a[\mathfrak{s}_\Phi(t \wedge \mathfrak{m})] = 0$$

3.4b $$E_a[\mathfrak{s}_{\Phi_1}(\mathfrak{m})\, \mathfrak{s}_{\Phi_2}(\mathfrak{m})] = E_a\Big[\int_0^{\mathfrak{m}} \Phi_1(x_s)\, \Phi_2(x_s)\, ds\Big] = (\Phi_1, \Phi_2)_a.$$

Now, consider the functions $p_\pm(a, \Phi) = E_a[|\mathfrak{s}(\mathfrak{m}) \pm \mathfrak{s}_\Phi(\mathfrak{m})|^2]$ and $p(a, \Phi) = E_a[\mathfrak{s}(\mathfrak{m})\, \mathfrak{s}_\Phi(\mathfrak{m})] = (1/4)(p_+ - p_-)$. Because $\mathfrak{s}$ and $\mathfrak{s}_\Phi$ satisfy 2.2 for $t \leqq \mathfrak{m}$, it follows, using 3.4a and the strong Markov property, that for any compact subinterval $J = [\varrho_1, \varrho_2]$ in (r_1, r_2)

3.5 $$\begin{aligned} p_\pm(a, \Phi) &= E_a[|\mathfrak{s}(\mathfrak{n}) \pm \mathfrak{s}_\Phi(\mathfrak{n})|^2] + E_a[p_+(x(\mathfrak{n}), \Phi)] \\ &\geqq \frac{\varrho_2 - a}{\varrho_2 - \varrho_1} p_\pm(\varrho_1, \Phi) + \frac{a - \varrho_1}{\varrho_2 - \varrho_1} p_\pm(\varrho_2, \Phi), \quad a \in (\varrho_1, \varrho_2), \quad \mathfrak{n} = \mathfrak{m}_J, \end{aligned}$$

and hence $p_\pm$ is concave in (r_1, r_2) and $p_\pm(a, \Phi) \downarrow 0$ as $a \downarrow r_1$ or $a \uparrow r_2$. Therefore, there exists a non-negative measure $\mu_\Phi^\pm$ finite on compact subsets in (r_1, r_2) such that

3.6 $$p_\pm(a, \Phi) = \int_{r_1}^{r_2} G(a, b)\, \mu_\Phi^\pm(db), \quad a \in (r_1, r_2);$$

this $\mu_\Phi^\pm$ is uniquely determined by $p_\pm$; in fact $d\mu_\Phi^\pm = d(-D^+ p_\pm)^\star$. Hence we have**

3.7
$$p(a, \Phi) = \int_{r_1}^{r_2} G(a, b)\, \mu_\Phi(db), \quad a \in (r_1, r_2)$$
$$d\mu_\Phi = (1/4)\,(d\mu_\Phi^+ - d\mu_\Phi^-) = d(-D^+ p);$$

μ_Φ is the unique signed measure finite on compact subsets in (r_1, r_2) and satisfying 3.7.

Next, we prove that there exists a unique function $g(b)$ independent of a and belonging to $L_2(a)$ such that

3.8
$$p(a, \Phi) = (\Phi, g)_a, \quad \Phi \in L_2(a), \quad a \in (r_1, r_2).$$

This results from the following three steps ((1)—(3)).

(1) If Φ vanishes identically in $J = (\varrho_1, \varrho_2) \subset I$, then μ_Φ has no mass inside J. In fact, in this case $\mathfrak{s}_\Phi = 0$ up to the exist time $\mathfrak{m}_J$ and hence from 3.5

$$p(a, \Phi) = \frac{\varrho_2 - a}{\varrho_2 - \varrho_1} p(\varrho_1, \Phi) + \frac{a - \varrho_1}{\varrho_2 - \varrho_1} p(\varrho_2, \Phi),$$

i.e., p is linear inside J, proving $d\mu_\Phi = d(-D^+ p) = 0$ there.

(2) If χ_E is the indicator function of a Borel set $E \subset I$, then

3.9
$$p(a, \chi_E) = \int_E G(a, b)\, \mu_{\mathbf{1}}(db),$$

where $\mathbf{1}$ is the function identically equal to one on I. First, from 3.4b we note that

$$p_+(a, \Phi) + p_-(a, \Phi) = 2 p_+(a, \mathbf{0}) + 2 \int G(a, b)\, \Phi^2(b)\, db$$

and hence from 3.6 that

3.10
$$d\mu_\Phi^+ + d\mu_\Phi^- = 2\, d\mu_{\mathbf{0}}^+ + 2\, \Phi^2\, db.$$

Now, if E is an open interval with $\mu_{\mathbf{0}}^+(\partial E) = 0$ and if $\Phi = \chi_E$, then from (1) and 3.10 $d\mu_\Phi = 0$ on $I - E$ and $d\mu_{\mathbf{1}-\Phi} = 0$ on $\bar{E}$ and hence $d\mu_\Phi = \Phi\, d\mu_{\mathbf{1}}$ follows from the identity $d\mu_\Phi + d\mu_{\mathbf{1}-\Phi} = d\mu_{\mathbf{1}}$. Thus we have 3.9 for such an E. But, because $|p(a, \chi_E)| \leqq \sqrt{E_a[\mathfrak{s}(\mathfrak{m})^2]} \cdot \|\chi_E\|_a$*** by Schwarz's inequality and 3.4b, $p(a, \chi_E)$ is a signed measure in E. Hence 3.9 must hold for any Borel set E in I because it holds for open intervals E with $\mu_0^+(\partial E) = 0$ and these intervals generate all Borel sets in I.

(3) From (2) we have

3.11
$$p(a, \Phi) = \int G(a, b)\, \Phi(b)\, \mu_{\mathbf{1}}(db), \quad a \in (r_1, r_2),$$

if Φ is a linear combination of finite numbers of indicator functions. On the other hand, for each a, $p(a, \Phi)$ is a linear functional on $L_2(a)$ because of the bound $|p(a, \Phi)| \leqq \sqrt{E_a[\mathfrak{s}(\mathfrak{m})^2]} \cdot \|\Phi\|_a$, and hence by Riesz's theorem there is a unique

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* D^+ means the right derivative, and $d(-D^+ p_\pm)$ is the measure induced by the function $-D^+ p_\pm$ of bounded variation.

** I owe this reasoning to H. P. McKean.

*** $\|\Phi\|_a = \sqrt{(\Phi, \Phi)_a}$.

$g_a \in L_2(a)$ such that $p(a, \Phi) = (\Phi, g_a)_a$ for $\Phi \in L_2(a)$. Comparing this with 3.11, it follows that $d\mu_1 = g_a db$ and that g_a must be independent of a, which we denote by g. Thus we have 3.8 with a unique $g \in L_2(a)$.

Next, we sketch the proof that $\mathfrak{s}(t) = \mathfrak{s}_g(t)$ for $t \leqq \mathfrak{m}$ (a.e.), following A. D. VENTSEL [9]. Putting $\bar{\mathfrak{s}} = \mathfrak{s} - \mathfrak{s}_g$, it is sufficient to show that

$$E_a[F_1(x(t_1 \wedge \mathfrak{m})) \cdots F_n(x(t_n \wedge \mathfrak{m}))\bar{\mathfrak{s}}(t \wedge \mathfrak{m})] = 0,$$
$$0 \leqq t_1 < \cdots < t_n < \infty, \quad 0 \leqq t < \infty,$$

for any bounded Borel functions $F_1, \ldots, F_n$, and by the additive property of $\bar{\mathfrak{s}}$ and the Markovian property of Brownian motion, it is also enough to prove that

3.12 $$E_a[F(x(t \wedge \mathfrak{m}))\bar{\mathfrak{s}}(t \wedge \mathfrak{m})] = 0$$

for any continuous function F on I. From 3.4b and 3.8 we note that $E_a[\mathfrak{s}_\Phi(\mathfrak{m})\bar{\mathfrak{s}}(\mathfrak{m})] = 0$ and hence by the additive property and 3.4a that

3.13 $$E_a[\mathfrak{s}_\Phi(t \wedge \mathfrak{m})\bar{\mathfrak{s}}(t \wedge \mathfrak{m})] = 0, \quad \Phi \in L_2(a).$$

Now, if F is continuous on I, writing $F = F_0 + h$ where $F_0(r_1) = F_0(r_2) = 0$ and h is a linear function, we have $E_a[h(x(t \wedge \mathfrak{m}))\bar{\mathfrak{s}}(t \wedge \mathfrak{m})] = 0$ by 3.13 and 3.4a. On the other hand, if q is an eigenfunction for the problem $(1/2)q'' = \lambda q$ with $q(r_1) = q(r_2) = 0$, then applying the formula of stochastic integrals (K. ITÔ [6]) to $q(x_t)$ and using 3.13, we have

$$E_a[q(x(t \wedge \mathfrak{m}))\bar{\mathfrak{s}}(t \wedge \mathfrak{m})] = \tfrac{1}{2}\int_0^t E_a[q''(x(\mathfrak{s} \wedge \mathfrak{m}))\bar{\mathfrak{s}}(\mathfrak{s} \wedge \mathfrak{m})]\,ds$$
$$= \lambda \int_0^t E_a[q(x(s \wedge \mathfrak{m}))\bar{\mathfrak{s}}(\mathfrak{s} \wedge \mathfrak{m})]\,ds,$$

and hence $E_a[q(x(t \wedge \mathfrak{m}))\bar{\mathfrak{s}}(t \wedge \mathfrak{m})] = 0$. Now, $E_a[F_0(x(t \wedge \mathfrak{m}))\bar{\mathfrak{s}}(t \wedge \mathfrak{m})] = 0$ follows by approximating F_0 uniformly by a linear combination of q's. Thus 3.12 holds.

Finally, to obtain 3.2, take an increasing sequence of bounded intervals $I_n = [r_{n1}, r_{n2}]$, $n \geqq 1$, with union R^1. We have obtained already

3.14a $$\mathfrak{a}(t, w) = f_n(x_t) - f_n(x_0) + \int_0^t g_n(x_s)\,dx_s, \quad t \leqq \mathfrak{n}_n \qquad \text{(a.e.)},$$

3.14b $$f_n = -E_\cdot[\mathfrak{a}(\mathfrak{n}_n)], \quad \mathfrak{n}_n = \mathfrak{m}_{I_n}.$$

Put $l_1(a) = 0$ and for $n \geqq 2$

$$l_n(a) = \sum_{k=1}^{n-1} (r_{k2} - r_{k1})^{-1}[(r_{k2} - a) f_{k+1}(r_{k1}) + (a - r_{k1}) f_{k+1}(r_{k2})].$$

Then, because $l_n(x_t) - l_n(x_0)$ can be written as the stochastic integral $\int_0^t l'(x_s)\,dx_s$, 3.14a remains valid when f_n and g_n are replaced respectively by $\bar{f}_n \equiv f_n - l_n$ and $\bar{g}_n \equiv g_n + l_n'$. On the other hand, from 3.14b, it follows that $\bar{f}_n = \bar{f}_{n+1}$ inside I_n and hence $\bar{g}_n = \bar{g}_{n+1}$ inside I_n. Thus, defining $f = \bar{f}_n$ and $g = \bar{g}_n$ inside I_n, we obtain 3.2. 3.1 is clear from the construction of g. Because $\mathfrak{a}$ and the stochastic integral term are continuous in t (a.e.), f must be continuous on R^1. Thus the theorem is completely proved.

Note added in proof. As for the multidimensional case, the author found, in the course of proof reading, A. B. SKOROHOD's paper (Teor. Verojatn. Primen **6**, 430–439 (1961)) and A. D. VENTSEL's paper (Doklady Akad. Nauk SSSR n. Ser. **142**, 1223–1226 (1962)). These papers treats the same representation as 3.2 for continuous additive functionals, and especially the latter paper treats the most general continuous addive functionals of a Brownian motion, but the proof given here is different from theirs. The multidimensional version of Theorem 1 was obtained also by H. P. MCKEAN (private communication) where I (in the theorem) must be replaced by a suitable *fine* open set.

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Mathematicae Institute
Kyusyu University
Fukuoka, Japan

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