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The Exit Measure of a Supermartingale

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Let $X=(X_t)_{t\geqq 0}$ be a nonnegative right continuous supermartingale relative to an increasing family $(\mathscr{F}_t)_{t\geqq 0}$ of σ-fields on a probability space $(\Omega, \mathscr{F}, P)$. It is well known that X has a *boundary function* $X_\infty := \lim_{t\uparrow\infty} X_t$, and if X is a uniformly integrable martingale then the boundary function actually determines X in the sense that X_t is the conditional expectation of X_∞ with respect to $\mathscr{F}_t$. In general however the boundary function may provide very little information on X. In this paper we give a characterization of X by its *terminal behaviour* which applies to the general case. Here the terminal behaviour is specified by a certain measure, the *exit measure* of X. It reduces to the measure $X_\infty\, dP$ whenever X is a uniformly integrable martingale.

As an illustration consider classical potential theory on the unit disc where the supermartingale X arises by observing a superharmonic function $u>0$ along Brownian motion paths. Any such u is characterized by a finite measure on the closed unit disc (Poisson-Riesz representation), which may be viewed as the terminal distribution of a certain Markov process associated to u, the so called *u-path process* introduced by Doob in [3]. Analytically, this *exit measure* is just the Choquet measure which represents u as a mixture of extremal rays in the cone of superharmonic functions >0. This example will serve as a guideline for our discussion.

In Section 1 we construct a probability measure P^X on the σ-field $\mathscr{P}$ of *previsible sets* in $\bar{\Omega} := \Omega \times [0, \infty]$ such that

$$P^X[A\times(t,\infty]] = \frac{1}{E[X_0]} E[X_t; A] \qquad (A\in\mathscr{F}_t,\ t\geqq 0).$$

The second coordinate of $\bar{\Omega}$ serves as a *lifetime*. The measure P^X generalizes the notion of a u-path process; a more explicit discussion of this can be found in [8]. The construction is based on the Ito-Watanabe factorization of X into a local martingale and a decreasing process. This factorization allows to define P^X consistently on an increasing sequence of σ-fields in P, similarly to the construction of α-subprocesses of a Hunt process in [5]. In order to extend P^X to $\mathscr{P}$ we introduce the following regularity assumption on the underlying σ-fields: $(\mathscr{F}_t)$ is the right continuous modification of a *standard system* $(\mathscr{F}_t^0)$. The notion of a standard system is essentially due to Parthasarathy in [9]. It means that (i) each σ-field $\mathscr{F}_t^0$ is standard Borel and that (ii) decreasing sequences of atoms have a non void intersection (cf. Appendix). Condition (ii) could be dropped if we were ready to replace $\bar{\Omega}$ by an inverse limit space. Path spaces of type $D(0,\infty)$ satisfy (i) but not (ii). However, and this remark is due to Meyer, if we allow for 'explosion in finite time' (cf. Appendix) then we obtain indeed a standard system, and this example is basic in the study of stochastic processes. A construction of the measure P^X which is directly adapted to this case can be found in [8]. Let us note that for a potential of class (D) the measure has been obtained by Doleans in [2] with a different method.

In Section 2 we show how the behaviour of X is reflected in the behaviour of the lifetime under P^X: X is a potential if and only if the lifetime is almost surely finite, and the more 'previsible' the lifetime the less 'regular' the potential. In Section 3 we decompose the measure P^X into a 'regular' part and a series of 'jump' parts. Section 4 reduces ratios Y/X of supermartingales on Ω to supermartingales on $\bar{\Omega}$ with respect to P^X.

Section 5 may be viewed as a contribution to Doob's program [4] of writing probabilistic potential theory purely in terms of supermartingales. We consider the cone S of supermartingales $X \neq 0$ which are adapted to a fixed (not necessarily increasing) family of σ-fields $\mathscr{E}_t \subseteq \mathscr{F}_t$. The system $(\mathscr{E}_t)$ is supposed to be *invariant under conditioning* (cf. (5.1)). Any $X \in S$ is then determined, up to a constant factor, by the restriction μ^X of the measure P^X to a certain *exit field* $\bar{\mathscr{E}} \subseteq \mathscr{P}$; if X is a martingale then we may identify the exit field with the tail field $\mathscr{E}_\infty := \bigcap_t \bigvee_{u>t} \mathscr{E}_u$. The *exit measure* μ^X is $0-1$ if and only if X lies on an extremal ray in S, and it is essentially the Choquet measure of X whenever the latter exists i.e. whenever X can be represented as a mixture of extremal rays in S.

The relation to potential theory is this. Consider a stochastic process (ξ_t) with nice paths on a state space E, let $\mathscr{G}_t$ be the σ-field generated by ξ_t (the *present at time t*) and define $\mathscr{F}_t^0 := \bigvee_{s \leqq t} \mathscr{G}_s$. If we now take $\mathscr{E}_t = \mathscr{G}_t$ then the invariance under conditioning (5.1) is just the Markov property, and the above results reduce to well known facts in the boundary theory of Markov processes. But there are quite different applications e.g. in Statistical Mechanics, and this will be discussed elsewhere.

Notation and vocabulary are essentially the same as in [6]. '$p(P)$' means that the property p holds for P-almost all sample points. Let us recall the definition of a *local martingale* $(Y_t)_{t \geqq 0}$: there is a sequence of stopping times T_n increasing almost surely to ∞ such that the stopped process $(Y_{T_n \wedge t})_{t \geqq 0}$ is a uniformly integrable martingale for any $n \geqq 1$.

1. Construction of the Measure P^X

Let $(\mathscr{F}_t^0)_{t \geqq 0}$ be a standard system (cf. Appendix) on a probability space $(\Omega, \mathscr{F}, P)$ with $\bigvee_{t \geqq 0} \mathscr{F}_t^0 = \mathscr{F}$ and define $(\mathscr{F}_t)_{t \geqq 0}$ as its right continuous modification: $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s^0$.

Suppose that $X = (X_t^0)_{t \geqq 0}$ is a nonnegative supermartingale adapted to $(\mathscr{F}_t^0)_{t \geqq 0}$. This means that X_t^0 is $\mathscr{F}_t^0$-measurable, nonnegative and integrable, and that we have the supermartingale inequality

$$X_s^0 \geqq E[X_t^0 \mid \mathscr{F}_s^0] \ (P)$$

whenever $s \leqq t$. We assume that $E[X_t^0]$ is right continuous in t.

(1.1) **Lemma.** *There is a right continuous supermartingale* $(X_t)_{t\geqq 0}$ *adapted to* $(\mathscr{F}_t)_{t\geqq 0}$ *which is a version of* X *in the sense that*

$$X_t^0 = E[X_t|\mathscr{F}_t^0]\ (P). \tag{1.2}$$

Proof. Define $H_{t,a,b}$ as the set of all $\omega\in\Omega$ such that the function $X_r^0(\omega)$ $(0\leqq r\leqq t$, r rational) is either unbounded or performs infinitely many upcrossings over the interval $[a,b]$ (cf. [6] VI 3, but note that we do not assume completeness of the σ-fields). Let $X_t(\omega):=\lim_{r\downarrow t} X_r^0(\omega)$ (r rational) for $\omega\notin H_t$ and $=0$ for $\omega\in H_t$ where

$$H_t = \bigcap_{s>t} \bigcup_{\substack{a<b\\ a,b \text{ rational}}} H_{s,a,b}\in\mathscr{F}_t.$$

The process $(X_t)_{t\geqq 0}$ has right continuous paths in $[0,\infty)$, limits from the left at least before hitting 0, and is adapted to $(\mathscr{F}_t)_{t\geqq 0}$. (1.2) and the supermartingale inequality follow now as in [6] VI T4.

The measure P^X will be defined on the product space $\bar{\Omega}:=\Omega\times[0,\infty]$. Let us introduce the product fields $\bar{\mathscr{F}}_t^0:=\mathscr{F}_t^0\times\mathscr{R}_t$ and $\bar{\mathscr{F}}:=\bigvee_{t\geqq 0}\bar{\mathscr{F}}_t^0$ where $\mathscr{R}_t$ denotes the σ-field on $[0,\infty]$ generated by the intervals $[0,s]$ with $s\leqq t$, and let $(\bar{\mathscr{F}}_t)_{t\geqq 0}$ be the right continuous modification of $(\bar{\mathscr{F}}_t^0)_{t\geqq 0}$. Suppose that T is a *stopping time* on Ω, and by that we mean a $(\mathscr{F}_t)$-stopping time if not specified otherwise, i.e. $\{T\leqq t\}\in\mathscr{F}_t$ $(t\geqq 0)$. Then we treat it simultaneously as a $(\bar{\mathscr{F}}_t)$-stopping time on $\bar{\Omega}$ with $T(\omega,t):=T(\omega)$. In particular we define a σ-field

$$\bar{\mathscr{F}}_T:=\{\bar{A}\in\bar{\mathscr{F}}|\bar{A}\cap\{T\leqq t\}\in\bar{\mathscr{F}}_t\ (t\geqq 0)\}$$

on $\bar{\Omega}$ in the same way as the usual σ-field $\mathscr{F}_T$ on Ω (obtained by omitting the bars). Let us write $A\times(S,T]$ instead of $\{(\omega,t)|\omega\in A,\ S(\omega)<t\leqq T(\omega)\}$ etc. Furthermore let us call $\zeta(\bar{\omega}):=t$ the *lifetime* of $\bar{\omega}=(\omega,t)\in\bar{\Omega}$.

(1.3) **Lemma.** *If T is a stopping time and $\bar{A}\in\bar{\mathscr{F}}_T$ then there is a set $A\in\mathscr{F}_T$ such that*

$$\bar{A}\cap\{\zeta>T\}=A\times(T,\infty]. \tag{1.4}$$

Proof. If T is constant then (1.4) is easy to check. For a general T define A as the union of the sets $A_r\in\mathscr{F}_r$ for rational r such that $\bar{A}\cap\{T\leqq r\}\cap\{\zeta>r\}=A_r\times(r,\infty]$ and use the fact that $A_r=A_s\cap\{T\leqq r\}$ for $r<s$.

Following Chung and Doob we introduce for any stopping time T the σ-field $\mathscr{F}_{T-}$ on Ω generated by $\mathscr{F}_0^0$ and the sets $A\cap\{T>t\}$ $(t>0,\ A\in\mathscr{F}_t)$, and in the same manner we define $\bar{\mathscr{F}}_{T-}$ on $\bar{\Omega}$. In this way we obtain in particular the σ-field $\bar{\mathscr{F}}_{\zeta-}$ which coincides with the class $\mathscr{P}$ of *previsible sets* in [7] 203. It is easy to see that $\mathscr{P}$ is generated as well by the sets $A_0\times\{0\}$ and $A_t\times(t,\infty]$ with $A_t\in\mathscr{F}_t^0$ $(t\geqq 0)$.

(1.5) **Theorem.** *If $E[X_0]\neq 0$ then there is exactly one probability measure P^X on $\mathscr{P}$ such that*

$$P^X[A\times(t,\infty]]=\frac{1}{E[X_0]}E[X_t;A] \quad (A\in\mathscr{F}_t^0,\ t>0). \tag{1.6}$$

Remark. Note that in (1.6) we could replace X_t by X_t^0 due to (1.2). Thus P^X depends only on X, not on our special version $(X_t)_{t\geqq 0}$.

Proof. We are going to combine the Ito-Watanabe factorization of X with an extension theorem for standard systems. Let us observe first that the additive decomposition of X (Doob-Meyer) can be obtained in the form $X_t = L_t - I_t$ where (L_t) and (I_t) are right continuous and adapted to (F_t), (L_t) is a local martingale and (I_t) is a natural increasing process (cf. [6] T29 where we use (1.1) to choose a suitable version of the arising martingale, and [5] Lemma 2). By right continuity

$$S_n := n \wedge \inf\left\{t>0 \,|\, X_t \notin \left[\frac{1}{n}, n\right] \text{ or } I_t > n\right\}$$

is a $(\mathscr{F}_t)$-stopping time. If we now follow the construction in [5] where we define $D_t = 0$ for $t \geqq S_\infty := \sup S_n$ then we obtain the Ito-Watanabe factorization of X in the form $X_t = M_t D_t$ on $\{S_\infty > t\}$, where (M_t) and (D_t) are both right continuous and adapted to $(\mathscr{F}_t)$, (D_t) is decreasing with $D_0 = 1$ and $(M_{S_n \wedge t})_{t \geqq 0}$ is a uniformly integrable martingale for any $n \geqq 1$. Moreover we have $X_t = M_t D_t\ (P)$ for any $t \geqq 0$. We can now define the measure P^X on $\bar{\mathscr{F}}_{S_n}$ by

$$P^X[\bar{A}] := \frac{1}{E[X_0]} E[M_{S_n}(\omega)\, \alpha(\omega, \bar{A}_\omega)] \tag{1.7}$$

where $\alpha(\omega, dt)$ is the random measure on $[0, \infty]$ defined by

$$\alpha(\omega, (s, t]) := D_s(\omega) - D_t(\omega) \quad \text{and} \quad \bar{A}_\omega := \{t \,|\, (\omega, t) \in \bar{A}\}$$

is the ω-section of $\bar{A} \in \bar{\mathscr{F}}_{S_n}$. Since $(M_{S_n})_{n \geqq 1}$ is a martingale due to the stopping theorem, (1.7) defines P^X consistently on the sequence $(\bar{\mathscr{F}}_{S_n})$ and in particular on the sequence $(\bar{\mathscr{F}}_{S_n -})$. But $(\bar{\mathscr{F}}_{S_n -})$ is a standard system by (6.1), and the extension theorem (6.2) guarantees that P^X is actually well defined on $\bigvee_{n \geqq 1} \bar{\mathscr{F}}_{S_n -}$. It is now easy to extend P^X to the σ-field $\mathscr{H}$ generated by $\bigvee \bar{\mathscr{F}}_{S_n -}$ and the set $\{\zeta > S_\infty\}$ in such a way that $P^X\{\zeta > S_\infty\} = 0$. This finishes the construction of P^X because P is contained in the P^X-completion of $\mathscr{H}$: the sets $\bar{A} = A \times (t, \infty]$ with $A \in \mathscr{F}_t^0$ satisfy

$$\bar{A} \cap \{\zeta \leqq S_\infty\} = \bigcup_n \bar{B}_n \cap \{\zeta \leqq S_\infty\}$$

where $\bar{B}_n := \bar{A} \cap \{S_n > t\} \in \bar{\mathscr{F}}_{S_n -}$. We have still to verify (1.6) but this is a special case of the following lemma. The uniqueness is clear since the sets arising in (1.6) generate $\mathscr{P}$ on $\{\zeta > 0\}$, and since $P^X[\zeta > 0] = 1$ by (1.6) and the right continuity of $E[X_t]$.

(1.8) **Lemma.** *If T is a stopping time and $\bar{A} \in \bar{\mathscr{F}}_T$ then*

$$P^X[\bar{A} \cap \{\zeta > T\}] = \frac{1}{E[X_0]} E[X_T; A \cap \{T < \infty\}] \tag{1.9}$$

where $A \in \mathscr{F}_T$ is associated to $\bar{A}$ via (1.4).

Proof. We may assume $E[X_0]=1$. Since $\zeta \leqq \lim S_n(P^X)$ and $\bar{A} \cap \{S_n > T\} \in \bar{\mathscr{F}}_{S_n-}$ we obtain

$$\begin{aligned} P^X[\bar{A} \cap \{\zeta > T\}] &= \lim_n P^X[\bar{A} \cap \{S_n > T\} \cap \{\zeta > T\}] \\ &= \lim_n E[M_{S_n} D_T;\, A \cap \{S_n > T\}] \\ &= \lim_n E[M_{S_n \wedge T} D_T;\, A \cap \{S_n > T\}] \\ &= \lim_n E[X_T;\, A \cap \{T < S_n\}] \\ &= E[X_T;\, A \cap \{T < \infty\}] \end{aligned}$$

applying optional stopping to the uniformly integrable martingale $(M_{S_n \wedge t})_{t \geqq 0}$ in the third step and noting $S_n \uparrow \infty (P)$ in the fourth.

(1.10) *Remark.* The lemma shows in particular that (X_t) *remains strictly positive up to its lifetime:* if $T_X := \inf\{t > 0 | X_t = 0\}$ then we have

$$P^X[T_X < \zeta] = \frac{1}{E[X_0]} E[X_{T_X};\, T_X < \infty] = 0$$

by right continuity of (X_t).

2. Classification of X in Terms of the Lifetime

We are going to translate the usual classification of supermartingales into a classification of the lifetime ζ. Let us first recall some definitions. X *is of class* (D) if the family $\{X_T | T$ *stopping time*$\}$ is uniformly integrable. If X is a *potential* i.e. $\lim_{t \uparrow \infty} E[X_t] = 0$ then X is of class (D) if and only if $\lim_n E[X_{R_n}] = 0$ where

$$R_n := \inf\{t > 0 | X_t > n\}$$

(cf. [6] VI T 20). X is called *regular* if $\lim E[X_{T_n}] = E[X_{\lim T_n}]$ for any increasing sequence of uniformly bounded stopping times. Any martingale is regular. Note also that any regular potential is of class (D): for any $c < \infty$ we have $\lim E[X_{R_n}] \leqq \lim E[X_{R_n \wedge c}]$ by optional stopping, and since $R_n \uparrow \infty (P)$ regularity implies $\lim E[X_{R_n \wedge c}] = E[X_c]$ which can be made arbitrarily small.

Suppose now that $T, T_1, T_2, \ldots$ are stopping times and that the sequence (T_n) increases. Let us say that T is *previsible* (resp. *totally unprevisible*) *by* (T_n) if

$$P^X[T_n \uparrow T,\ T_n < T \text{ for any } n] = 1 \quad (\text{resp. } = 0).$$

T is called *totally inaccessible* if T is totally unprevisible by any sequence (T_n).

In order to simplify the notation we assume from now on $E[X_0] = 1$.

(2.1) **Proposition.**

(i) X *is a martingale* $\Leftrightarrow \zeta = \infty\ (P^X)$

(ii) X *is a potential* $\Leftrightarrow \zeta < \infty\ (P^X)$

(iii) X *is a local martingale* $\Leftrightarrow \zeta$ *is previsible by* $(n \wedge R_n)\ (P^X)$.

If X is a potential, then we have:

(iv) *X is a local martingale* $\Leftrightarrow$ *ζ is previsible by (R_n) (P^X)*

(v) *X belongs to class (D)* $\Leftrightarrow$ *ζ is totally unprevisible by (R_n) (P^X)*

(vi) *X is regular* $\Leftrightarrow$ *ζ is totally inaccessible (P^X).*

Proof. Since $P^X\{\zeta>t\}=E[X_t]$, we have (i) and (ii). Suppose now that X is a local martingale. The stopping times S_n used in the construction of P^X reduce to

$$S_n=n\wedge\inf\left\{t>0\,|\,X_t\notin\left[\frac{1}{n},n\right]\right\},$$

and since (X_{S_n}) is a martingale, we have $S_n<\zeta$ (P^X) by (1.9) and therefore $S_n\uparrow\zeta$ (P^X) because $S_\infty\geqq\zeta$ (P^X). We can now replace S_n by $n\wedge R_n$ if we know that (X_t) is bounded away from 0 up to time ζ (P^X); this will be shown in (4.3).

Conversely, let us assume that ζ is previsible by $(n\wedge R_n)$ (P^X). Then we have, for $t\geqq s$ and $A\in\mathscr{F}_s$,

$$E[X_{n\wedge R_n\wedge t};A]=P^X[A\times(s,\infty]\cap\{\zeta>n\wedge R_n\wedge t\}]=P^X[A\times(s,\infty]].$$

Comparing for $t>s$ and $t=s$, we see that $(X_{n\wedge R_n\wedge t})_{t\geqq 0}$ is a martingale, and since $n\wedge R_n\uparrow\infty$ (P), (X_t) is a local martingale.

For the rest of the proof we assume that X is a potential. As $\zeta<\infty$ (P^X), we can replace $n\wedge R_n$ by R_n in (iii) and obtain (iv). To prove (v) and (vi), let us observe first that ζ is totally unprevisible by a sequence of stopping times increasing to T if and only if

$$\bigcap_n\{T_n<\zeta\}\subseteq\{T<\zeta\}\ (P^X) \tag{2.2}$$

X is in class (D) if and only if $\lim E[X_{R_n}]=0$, and this is equivalent to

$$P^X\Big[\bigcap_n\{R_n<\zeta\}\Big]=\lim_n P^X\{R_n<\zeta\}=\lim_n E[X_{R_n}]=0.$$

Since $\lim R_n\geqq\lim S_n\geqq\zeta$ (P^X), (2.2) yields (v). As to (vi), let (T_n) be a sequence of stopping times increasing to T. For any $a>0$ we have

$$P^X\Big[\bigcap_n\{T_n\wedge a<\zeta\}\Big]=\lim_n E[X_{T_n\wedge a}],\qquad P^X[\{T\wedge a<\zeta\}]=E[X_{T\wedge a}],$$

and so (2.2) implies that X is regular as soon as ζ is totally inaccessible. But the converse is also true since a is arbitrary and $\zeta<\infty$ (P^X).

3. Decomposition of P^X

In this section we decompose the measure P^X in a series of terms which may be viewed as the different components of the terminal behaviour of X; this interpretation will be made more precise in Section 5. We continue to assume $E[X_0]=1$.

Let us first consider the rather transparent case where X is a martingale. P^X is then concentrated on $\Omega\times\{\infty\}$, and thus it may be identified with its projection on $(\Omega,\mathscr{F})$:

$$P^X[A]:=P^X[A\times\{\infty\}]=\lim_{t\uparrow\infty}P^X[A\times(t,\infty]]$$
$$=\lim_{t\uparrow\infty}E[X_t;A]\qquad\Big(A\in\bigcup_{s\geqq 0}\mathscr{F}_s\Big).$$

It is now easy to see that X is uniformly integrable if and only if P^X, considered as a measure on $\mathscr{F}$, is absolutely continuous with respect to P, and that the density is given by the boundary function $X_\infty := \lim_{t \uparrow \infty} X_t$.

In order to discuss the general case let us first relate P^X to the measure Q^X induced by the increasing part of X. Recall the additive decomposition $X = L - I$ (Doob-Meyer) which we used in the proof of (1.5). $I = (I_t)_{t \geq 0}$ induces a random measure $\beta(\omega, ds)$ on $[0, \infty]$ via $\beta(\omega, (s, t]) := I_t(\omega) - I_s(\omega)$ and $\beta(\omega, \{\infty\}) := 0$, and if we take the product of P with the kernel β then we obtain a σ-finite measure Q^X on the product field $\bar{\mathscr{F}}$:

$$Q^X[\bar{A}] := E[\beta(\omega, \bar{A}_\omega)] \tag{3.1}$$

where $\bar{A}_\omega$ is the ω-section of $\bar{A} \in \bar{\mathscr{F}}$. Let $Q^X|\mathscr{P}$ denote the restriction of Q^X to $\mathscr{P}$, and let us define $R_\infty := \sup R_n$ where (R_n) is the sequence used in (2.1). The following lemma shows in particular that P^X and $Q^X|\mathscr{P}$ are essentially equal if X is a potential of class (D) because then we have $\zeta < R_\infty$ (P^X) by (2.1). This establishes the connection between our results and the construction of Doleans in [2].

(3.2) **Lemma.** *P^X and $Q^X|\mathscr{P}$ coincide on $\{\zeta < R_\infty\}$.*

Proof. Let $U_n := n \wedge R_n$ $(n \geq 0)$; then $U_n < R_\infty$ by (1.1). For $s > 0$ and $A \in \mathscr{F}_s$ we have

$$Q^X[A \times (s, \infty] \cap \{\zeta < R_\infty\}] = \sum_{n \geq 0} Q^X[A \times (s, \infty] \cap \{U_n < \zeta \leq U_{n+1}\}]$$

and, since $(L_{U_{n+1} \wedge t})$ is a uniformly integrable martingale,

$$\begin{aligned} Q^X[A \times (s, \infty] \cap \{U_n < \zeta \leq U_{n+1}\}] &= E[I_{U_{n+1}} - I_{U_n}; A \cap \{U_n \geq s\}] \\ &\quad + E[I_{U_{n+1}} - I_s; A \cap \{U_n < s\}] \\ &= E[X_{U_n} - X_{U_{n+1}}; A \cap \{U_n \geq s\}] + E[X_s - X_{U_{n+1}}; A \cap \{U_n < s\}] \\ &= P^X[A \times (s, \infty] \cap \{U_n < \zeta \leq U_{n+1}\}]; \end{aligned}$$

this implies (3.2).

We are now ready to prove the following decomposition theorem.

(3.3) **Theorem.** *There is*

(i) *a sequence of previsible stopping times T_n $(n = 1, 2, \ldots, \infty)$ and for each $n = 1, 2, \ldots, \infty$ a measure $N^X(n, \cdot)$ on $\mathscr{F}_{T_n-}$*

(ii) *a strictly increasing time change $(T_t)_{t \geq 0}$* ([6] VII, D 13) *and for each $t \in (0, \infty)$ a measure $S^X(t, \cdot)$ on $\mathscr{F}_{T_t-}$*

such that

$$P^X[\bar{A}] = \sum_{1 \leq n \leq \infty} N^X(n, \bar{A}_{T_n}) + \int_0^\infty S^X(t, \bar{A}_{T_t})\, dt \qquad (\bar{A} \in \mathscr{P}). \tag{3.4}$$

Proof. Let us first note that for $\bar{A} \in \mathscr{P}$ and any stopping time T the section $\bar{A}_T$ is in $\mathscr{F}_{T-}$, and that $A \times \{T\} \in \mathscr{P}$ for $A \in \mathscr{F}_{T-}$ ([7] Appendice 1). We define now $T_\infty = R_\infty$ and $N^X(\infty, A) := P^X[A \times \{R_\infty\}]$. On $\mathscr{P} \cap \{\zeta < R_\infty\}$ the measure P^X

coincides with Q^X by (3.2). By [7] 306 we have

$$I_t = I_t^c + \sum_{n \geq 1} c_n I_{[T_n, \infty)}$$

where (I_t^c) is the continuous part of I and (T_n) is a sequence of previsible stopping times. Let $(T_t)_{t \geq 0}$ be the time change associated to (I_t^c) (see [6] VII D 13, T 12); the continuity of (I_t^c) implies that (T_t) is strictly increasing. Then we have

$$Q^X[\bar{A} \cap \{\zeta < R_\infty\}] = \sum c_n E[\bar{A}_{T_n}; T_n < R_\infty] + \int_0^\infty E[\bar{A}_{T_t}; T_t < R_\infty]\, dt,$$

and if we define $N^X(n, A) := c_n P[A \cap \{T_n < R_\infty\}]$ on $\mathscr{F}_{T_n-}$ and $S^X(t, A) := E[A; T_t < R_\infty]$ on $\mathscr{F}_{T_t-}$, we obtain (3.4).

4. Relative Supermartingales

Let X be a supermartingale as above with $E[X_0] = 1$. Defining $\mathscr{P}_t^0 := \bar{\mathscr{F}}_t^0 \cap \mathscr{P}$ we obtain an increasing family of σ-fields on the probability space $(\bar{\Omega}, \mathscr{P}, P^X)$, and we can consider the supermartingales relative to this system. These *relative supermartingales* are a useful tool in the boundary theory of Section 5.

Let us associate to any process $Z = (Z_t^0)_{t \geq 0}$ on Ω a process $\bar{Z} = (\bar{Z}_t^0)_{t \geq 0}$ on $\bar{\Omega}$ by defining

(4.1) $$\bar{Z}_t^0(\bar{\omega}) := Z_t^0(\omega)\, I_{(t, \infty]}(s)$$

for $\bar{\omega} = (\omega, s) \in \bar{\Omega}$ (I_A denotes the characteristic function of the set A). If Z is adapted to $(\mathscr{F}_t^0)$ then $\bar{Z}$ is adapted to $(\mathscr{P}_t^0)$. Conversely, for any process $\bar{Z}$ which is adapted to $(\mathscr{P}_t^0)$ we can find a process Z adapted to $(\mathscr{F}_t^0)$ such that (4.1) holds at least on $\{\zeta > t\}$ just as in (1.3).

We assume now that $Y = (Y_t^0)_{t \geq 0}$ is adapted to $(\mathscr{F}_t^0)_{t \geq 0}$ and define the *ratio process* $Z = Y/X$ by

$$Z_t^0 := \frac{Y_t^0}{X_t^0} I_{\{X_t^0 \neq 0\}};$$

$\bar{Z} = (\bar{Z}_t^0)$ is the corresponding process on $\bar{\Omega}$. The following proposition shows that ratios of supermartingales on Ω reduce to relative supermartingales on $\bar{\Omega}$:

(4.2) **Proposition.** *If Y is a supermartingale then $\bar{Z}$ is a supermartingale relative to P^X. Conversely, if $\bar{Z}$ is a supermartingale relative to P^X and Z is associated to $\bar{Z}$ via* (4.1) *then $Y := ZX$ (i.e. $Y = (Y_t^0)$ with $Y_t^0 := Z_t^0 X_t^0$) is a supermartingale with respect to P.*

Proof. If $\bar{A} \in \mathscr{P}_s^0$ then $\bar{A} \cap \{\zeta > s\} = A \times (s, \infty]$ for a suitable set $A \in \mathscr{F}_s^0$, and for any $t \geq s$ we have

$$\begin{aligned} E^X[\bar{Z}_t^0; \bar{A}] &= E^X[\bar{Z}_t^0; \bar{A} \cap \{\zeta > t\}] = E^X[\bar{Z}_t^0; \bar{A} \cap \{\zeta > s\}] \\ &= E[Y_t^0 I_{\{X_t^0 \neq 0\}}; A]. \end{aligned}$$

If we compare the result for $t = s$ and $t > s$ and note $\{X_t^0 \neq 0\} \subseteq \{X_s^0 \neq 0\}$ (P) then we obtain the first statement. Conversely, if $\bar{Z}$ is a supermartingale relative to P^X

then $$E[Z_t X_t; A] = E^X[\bar{Z}_t; A \times (t, \infty]] \quad (A \in \mathscr{F}_s,\ s \leqq t)$$

and this implies the second part.

As a first application let us show that *the paths of X are bounded away from* 0 (P^X). This means in particular that the *exit values* $\lim_{t \uparrow \zeta} X_t$ are strictly positive (P^X), and this was needed in the proof of (2.1).

(4.3) **Corollary.** $\inf_{t<\zeta} \bar{X}_t > 0$ (P^X).

Proof. Consider the ratio process $Z = 1/X$. Z is a supermartingale relative to P^X. Therefore its paths are bounded (P^X), and this translates into (4.3) because $T_X = \zeta$ (P^X) by (1.10).

5. Exit Measures and Extremal Structure

In this section we fix a system of σ-fields $\mathscr{E}_t = \mathscr{F}_t^0$ ($t \geqq 0$) which is *invariant under conditioning*. By this we mean that

$$E[\varphi | \mathscr{F}_s^0] = E[\varphi | \mathscr{E}_s] \quad (s \leqq t) \tag{5.1}$$

whenever $t \geqq 0$ and φ is an integrable function which is $\mathscr{E}_t$-measurable. We denote by $\bar{\mathscr{E}}_t$ the σ-field on $\bar{\Omega}$ generated by the sets $A \times (t, \infty]$ with $A \in \mathscr{E}_t$ and call

$$\bar{\mathscr{E}} := \{\bar{A} \in \mathscr{P} | \bar{A} \cap \{\zeta > t\} \in \bigvee_{u>t} \bar{\mathscr{E}}_u \ (t \geqq 0)\}$$

the *exit field*. $\bar{\mathscr{E}} \cap \{\zeta = t\}$ may be identified with the tail field $\mathscr{E}_{t-} := \bigcap_{s<t} \bigvee_{s<u<t} \mathscr{E}_u$, and in particular we can identify $\bar{\mathscr{E}} \cap \{\zeta = \infty\}$ with the *boundary field* $\mathscr{E}_\infty$.

We are now going to look at the convex cone S of those supermartingales $X = (X_t^0)_{t \geqq 0}$ with $E[X_0^0] \neq 0$ (defined as above) which are adapted to $(\mathscr{E}_t)_{t \geqq 0}$:

$$X_t^0 \textit{ is } \mathscr{E}_t\textit{-measurable} \quad (t \geqq 0).$$

Let us call the restriction μ^X of the measure P^X to the exit field $\bar{\mathscr{E}}$ the *exit measure* of $X \in S$. If X is a martingale then we can view μ^X as a measure on $\mathscr{E}_\infty$ (cf. Section 3) and call it the *boundary measure* of X.

(5.2) **Theorem.** *Any $X \in S$ is determined by its exit measure (up to a constant factor).*

Proof. Suppose that X and Y are in S, that $E[X_0] = E[Y_0] = 1$, and that the exit measures coincide. We are going to show that $P^X = P^Y$ on $\mathscr{P}$, and this implies $X = Y$ in the sense that X is a modification of Y.

We may and do assume that P^Y is absolutely continuous with respect to P^X (if not, replace X by $X' = \frac{1}{2}(X + Y)$ and note that $Y = X'$ implies $Y = X$). Let Z_∞ be a $\mathscr{P}$-measurable density function on $\bar{\Omega}$. The assumption $P^X = P^Y$ on $\bar{\mathscr{E}}$ implies $E[Z_\infty | \bar{\mathscr{E}}] = 1$ (P^Y), and it is enough to show that in fact we have $Z_\infty = 1$ (P^Y).

Let S_n be the set of dyadic rationals $k 2^{-n}$ ($0 \leqq k \leqq n 2^n$), $J_{ns} := (s, s + 2^{-n}]$ for $n > s \in S_n$ and $J_{nn} := (n, \infty]$, and denote by $\mathscr{P}^n$ the σ-field generated by the sets $A \times J_{ns}$ ($A \in \mathscr{F}_s^0$, $s \in S_n$). The sequence $(\mathscr{P}^n)_{n \geqq 1}$ increases to $\mathscr{P}$, and by the martingale convergence theorem we obtain

$$Z_n := E^X[Z_\infty | \mathscr{P}^n] \to Z_\infty \ (P^X).$$

We are now going to choose versions Z_n such that $\underline{\lim}\, Z_n$ is $\bar{\mathscr{E}}$-measurable; we can then identify Z_∞ with $E^X[Z_\infty|\bar{\mathscr{E}}]=1$ (P^X), and this proves $P^X=P^Y$ on $\mathscr{P}$.

For any $n \geqq 1$ we define

$$\Delta_{ns}\, X := \begin{cases} X_s^0 - E[X_{s+2^{-n}}^0 | \mathscr{F}_s^0] & (n > s \in S_n) \\ X_s^0 & (s = n) \end{cases}$$

and, in the same way, $\Delta_{ns}\, Y$. By hypothesis (5.1) we may assume that $\Delta_{ns}\, X$ and $\Delta_{ns}\, Y$ are $\mathscr{E}_s$-measurable. We define now

$$\varphi_{ns} := \frac{\Delta_{ns}\, Y}{\Delta_{ns}\, X} I_{\{\Delta_{ns} X \neq 0\}},$$

note that

(5.3) $$P^X[\{\Delta_{ns}\, X = 0\} \times J_{ns}] = E[X_s - X_{s+2^{-n}};\ \{\Delta_{ns}\, X = 0\}] = 0$$

for $n > s \in S_n$ and similarly for $s = n$, and define

$$Z_n(\bar{\omega}) := \sum_{s \in S_n} \varphi_{ns}(\omega)\, I_{J_{ns}}(t)$$

for $\bar{\omega} = (\omega, t) \in \bar{\Omega}$. Z_n is $\mathscr{P}^n$-measurable. For $s \in S_n$ and $A \in \mathscr{F}_s^0$ we have

$$\begin{aligned} E^X[Z_n;\, A \times J_{ns}] &= E^X[Z_n;\, (A \cap \{\Delta_{ns}\, X \neq 0\}) \times J_{ns}] \\ &= E[\varphi_{ns}\, \Delta_{ns}\, X;\, A \cap \{\Delta_{ns}\, X \neq 0\}] \\ &= E[\Delta_{ns}\, Y;\ A \cap \{\Delta_{ns}\, X \neq 0\}]. \end{aligned}$$

On the other hand, since P^Y is absolutely continuous with respect to P^X, we can use (5.3) to write

$$\begin{aligned} P^Y[A \times J_{ns}] &= P^Y[(A \cap \{\Delta_{ns}\, X \neq 0\}) \times J_{ns}] \\ &= E[\Delta_{ns}\, Y;\ A \cap \{\Delta_{ns}\, X \neq 0\}] \end{aligned}$$

and this shows that Z_n is in fact a version of $E^X[Z_\infty|\mathscr{P}^n]$. Furthermore we have

$$(\underline{\lim}_n\, Z_n)\, I_{\{\zeta > t\}} = \underline{\lim}_n \sum_{\substack{s \in S_n \\ s > t}} \varphi_{ns}\, I_{J_{ns}}$$

where the right side is $\bigvee_{u > t} \bar{\mathscr{E}}_u$-measurable, and this finishes the proof.

Combining (3.3) and (5.2) we can characterize any $X \in S$ by its *terminal behaviour* in the following sense:

(5.4) **Corollary.** *Any $X \in S$ is determined by*

(i) *a sequence $(T_n)_{n=1,2,\ldots}$ of previsible stopping times and a corresponding sequence of measures $\nu^X(n, \cdot)$ on $\mathscr{E}_{T_n-}$ (the "exit behaviour at time T_n")*

(ii) *a strictly increasing time change $(T_t)_{t \geqq 0}$ and a corresponding system of measures $\sigma^X(t, \cdot)$ on $\mathscr{E}_{T_t-}$ (the "exit behaviour at time T_t")*

where we have $\mathscr{E}_{T-} = \{A \in \mathscr{F} | A \times \{T\} \in \bar{\mathscr{E}}\}$ for any stopping time T.

In order to establish the relation between exit measures and the Choquet integral representation in S we need the following $0-1$ *law* for the extremal rays in S. Let us call $X \in S$ *extremal* if $\frac{1}{E[X_0]} X$ is an extremal point of the convex set $S_1 := \{X \in S | E[X_0] = 1\}$, and let us denote the set of these extremal points by ∂S_1.

(5.5) **Theorem.** *X is extremal if and only if the exit measure μ^X is $0-1$.*

Proof. 1) Let us assume that μ^X is $0-1$ on $\bar{\mathscr{E}}$, and that X is a convex combination of Y and Z in S_1. Then it is easy to check that P^X is a convex combination of P^Y and P^Z. But this implies $P^Y = P^Z$ on $\bar{\mathscr{E}}$ by assumption. Theorem (5.2) now shows that Y and Z are both proportional to X, and this means that X is extremal.

2) We assume now that X is extremal and take a set $\bar{A} \in \bar{\mathscr{E}}$ such that $P^X[\bar{A}] \neq 0$. The process $\bar{Z}_t^0 := E^X[I_{\bar{A}} | \mathscr{P}_t^0]$ $(t \geqq 0)$ is a martingale relative to P^X which is bounded by 1, and we may choose an $\bar{\mathscr{E}}_t$-measurable version as shown in 3) below. By (4.2) the process $Y = ZX$ is a supermartingale. Recalling the construction of $Z = (Z_t^0)$ it is easy to see that we can choose Z_t^0 $\mathscr{E}_t$-measurable, and this shows $Y \in S$. Moreover Y is majorized by X, and since X is extremal we have $Y = cX$ for some constant $c > 0$. Using (1.6) we obtain that P^X coincides with the normalized restriction of P^X to $\bar{A}$. But this implies $P^X[\bar{A}] = 1$.

3) We have still to verify that $\bar{Z}_t^0 := E^X[I_{\bar{A}} | \mathscr{P}_t^0]$ has an $\bar{\mathscr{E}}_t$-measurable version whenever $\bar{A} \in \bar{\mathscr{E}}$. Since $\bar{A} \cap \{\zeta \leqq t\} \in \mathscr{P}_t^0$ and $\bar{A} \cap \{\zeta > t\} \in \bigvee_{s>t} \bar{\mathscr{E}}_s =: \mathscr{H}$ we can assume $\bar{A} \in \mathscr{H}$. The class $\mathscr{C}$ of sets $\bar{A} \in \mathscr{H}$ which do have the desired property is closed under disjoint unions and proper differences, and it contains the generating sets $A_u \times (u, \infty]$ with $u > t$ and $A_u \in \mathscr{E}_u$:

$$E^X[A_u \times (u, \infty] | \mathscr{P}_t^0](\omega, s) = E[X_u I_{A_u} | \mathscr{F}_t^0](\omega) I_{(t, \infty]}(s)$$

by the construction of P^X, and this function may be chosen $\bar{\mathscr{E}}_t$-measurable by hypothesis (5.1). It follows now by induction that $\mathscr{C}$ contains all the finite intersections of such sets, and this implies $\mathscr{C} = \mathscr{H}$.

We can now identify the exit measure μ^X as the Choquet measure of X whenever the latter exists. Suppose that X is *barycenter* of a measure $\tilde{\mu}^X$ defined on a fixed σ-field $\partial\mathscr{S}_1$ on ∂S_1. To be precise: we assume that $\varphi(Z) := E[Z_t; A]$ is a $\partial\mathscr{S}_1$-measurable function on ∂S_1 and that

$$E[X_t; A] = \int E[Z_t; A]\, \tilde{\mu}^X(dZ) \qquad (A \in \mathscr{F}_t,\ t \geqq 0). \tag{5.6}$$

Let us also assume $E[X_0] = 1$. Then we can conclude:

(5.7) **Corollary.** *The measure algebras of $(\mu^X, \bar{\mathscr{E}})$ and $(\tilde{\mu}^X, \partial\mathscr{S}_1)$ are isomorphic.*

Proof. (5.6) implies

$$P^X[A \times (t, \infty]] = \int P^Z[A \times (t, \infty]]\, \tilde{\mu}^X(dZ) \qquad (t \geqq 0,\ A \in \mathscr{F}_t).$$

Hence P^X is the product of $\tilde{\mu}^X$ with the kernel $Z \to P^Z$ from $\partial\mathscr{S}_1$ to $\mathscr{P}$. For $\bar{A} \in \bar{\mathscr{E}}$ we obtain by (5.5)

$$\mu^X[\bar{A}] = \int P^Z[\bar{A}]\, \tilde{\mu}^X(dZ) = \tilde{\mu}^X[\tilde{A}]$$

where $\tilde{A} := \{Z | P^Z[\bar{A}] = 1\}$, and this establishes the isomorphy.

6. Appendix on Standard Systems

Let T be a partially ordered non-void index set and let $(\mathscr{F}_t^0)_{t\in T}$ be an increasing family of σ-fields on Ω. We call $(\mathscr{F}_t^0)_{t\in T}$ a *standard system* if

(i) each measurable space $(\Omega, \mathscr{F}_t^0)$ is a standard Borel space (cf. [9] p. 133) i.e. $\mathscr{F}_t^0$ is σ-isomorphic to the σ-field of Borel sets on some complete separable metric space

(ii) for any increasing sequence (t_i) in T and for any decreasing sequence $A_i \subseteq \Omega$ such that A_i is an atom of $\mathscr{F}_{t_i}$ we have $\bigcap_i A_i \neq \emptyset$.

In Section 1 we used the following properties of standard systems.

(6.1) *Remark.* Let $(\mathscr{F}_t^0)_{t\geqq 0}$ be a standard system with $T=[0, \infty)$, and denote by $(\mathscr{F}_t)_{t\geqq 0}$ its right continuous modification.

1) If $(T_n)_{n\geqq 1}$ is an increasing sequence of $(\mathscr{F}_t)$-stopping times then $(\mathscr{F}_{T_n-})_{n\geqq 1}$ (cf. Section 1) is a standard system.

2) If $(\mathscr{H}_t^0)_{t\geqq 0}$ is a standard system on Ω' then $(\mathscr{F}_t^0 \times \mathscr{H}_t^0)_{t\geqq 0}$ is a standard system on $\Omega \times \Omega'$.

Proof. 2) is clear. As to 1) note that $\mathscr{F}_{T_n-}$ is countably generated and contained in $\mathscr{F} := \bigvee_{t\geqq 0} \mathscr{F}_t^0$. $\mathscr{F}$ is standard Borel by [9] V Th 4.1, and then the same is true for $\mathscr{F}_{T_n-}$ by [9] V Th 2.4. If $A_n \in \mathscr{F}_{T_n-}$ $(n=1, 2, \ldots)$ is a decreasing sequence of atoms then we have $A_n = A_n \cap \{T_n = t_n\} \in \mathscr{F}_{t_n}$ for some increasing sequence (t_n), and this implies $\bigcap_n A_n \neq \emptyset$ by (ii).

For the convenience of the reader we quote the extension theorem used in Section 1. Let $(\mathscr{F}_n)_{n\geqq 1}$ be an increasing sequence of σ-fields on Ω satisfying (i), and suppose we have a consistent sequence of probability measures μ_n on $\mathscr{F}_n$ $(n=1,2,\ldots)$.

(6.2) **Theorem** (cf. [9] V Th 3.2 and 4.1). *If condition* (ii) *holds then* $(\mu_n)_{n\geqq 1}$ *admits an extension to a probability measure on* $\bigvee_{n\geqq 1} \mathscr{F}_n$. *In the general case there is a measure* μ *on the inverse limit of the measurable spaces* $(\Omega, \mathscr{F}_n)$ *such that the projection of* μ *on* $\mathscr{F}_n$ *coincides with* μ_n.

(6.3) *Examples.* 1) Take $\Omega=[0, 1]$ with Lebesgue measure P on the σ-field $\mathscr{F}$ of Borel sets, and let $\mathscr{F}_t^0$ be the σ-field generated by the dyadic intervals $[(k-1)\,2^{-n}, k\,2^{-n})$ $(1 \leqq k < 2^{-n},\ n \leqq t)$. Then $(\mathscr{F}_t^0)_{t\geqq 0}$ is a standard system. In this simple case our results are fairly obvious. In particular (taking $\mathscr{E}_t = \mathscr{F}_t^0$): positive martingales correspond to positive measures on $\mathscr{F}$, and extremal martingales correspond to point measures.

2) Let E be a *state space* (e.g. locally compact with countable base), let Δ be an additional absorbing point, and define Ω as the space of right continuous paths $\omega: [0, \infty) \to E \cup \{\Delta\}$ which stay at Δ once they get there i.e. after the *absorption time* $\alpha(\omega) := \inf\{t>0 \mid \alpha(t)=\Delta\}$, and which have limits from the left on $(0, \alpha(\omega))$. Define $\xi_t(\omega) = \omega(t)$ and let $\mathscr{F}_t^0$ be the σ-field generated by the functions ξ_s with $s \leqq t$. Then $(\mathscr{F}_t^0)_{t\geqq 0}$ is a standard system: Meyer has shown (i) (cf. [1] p. 100 and in particular line 1), and (ii) is easy to check. A thorough discussion of this crucial example together with a direct construction of the measure P^X can be found in [8].

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