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Finite Elastic Deformations of an Infinite Plate Perforated by Two Circular Holes under Biaxial Tension*

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Summary: Based on the theory of finite deformations and by the use of a given reference frame in undeformed body, stresses of an incompressible isotropic infinite plate with two circular holes subjected to biaxial tension at infinity are investigated. The material is assumed to be in the state of finite plane strain. The method of successive approximation is used in connection with the complex variable method of plane elasticity. Numerical results due to the second-order approximation theory are given for stresses, and compared with those predicted by classical theory of elasticity.

Übersicht: Es werden die Spannungen in einer unendlich ausgedehnten Platte mit zwei kreisförmigen Löchern nach der Theorie endlicher Verformungen und unter Verwendung eines im unverformten Körper festgelegten Bezugsgerüstes untersucht. Die Platte besteht aus inkompressiblem isotropen Material und wird einer im Unendlichen zweiachsialen Verzerrung ausgesetzt. Der Verzerrungszustand wird als eben angenommen. Bei den Untersuchungen werden die komplexen Methoden der ebenen Elastizitätstheorie in Verbindung mit schrittweisen Näherungen angewendet. Aufgrund einer Annäherung zweiter Ordnung werden die Spannungen numerisch ausgerechnet und mit den Werten verglichen, die von der klassischen Elastizitätstheorie vorausgesagt werden.

1. Introduction. In the theoretical problem of large elastic deformations of incompressible isotropic hyperelastic bodies, basic equations become nonlinear and the problem is rather difficult to be solved. A general theory of finite deformations has been previously treated by *Cauchy*, *Brillouin* and *Murnaghan* and greatly developed by *Green* and *Zerna* [1]. Moreover, a number of problems has been solved completely by *Rivlin* [2] and by *Green* and *Shield* [3], whereas there are many practical problems in which the magnitude of deformation is much larger than that considered in classical elasticity but remains reasonably small. For such problems the successive approximation approach based on the general theory of large elastic deformation is quite useful, although it may be more desirable to obtain the closed solution without any restriction, imposed either upon the magnitude of the deformation or on the form of the strain energy function [2, 3]. Recently *Adkins*, *Green* and *Shield* [4] have developed a general method of successive approximation for the problem of plane strain for incompressible isotropic materials. In evolving this method it is assumed that the stresses and displacements are expressed as power series of a perturbation parameter ε, the choice of which depends on the problem under consideration. The first order term of this expansion corresponds to the classical elasticity and a solution can be obtained by means of *Muskhelishvili*'s complex variable technique [5]. The second order term can also be expressed in terms of similar complex potential functions.

This theory has been applied to treat the finite deformation problems and a few problems have been solved by this approach [4, 6]. These studies are worked by the use of coordinates in the deformed body. The solution in terms of a given reference frame in undeformed body may also be interesting for comparing the second approximation solution with that given by classical elasticity.

In the present paper the application of the method of successive approximation are performed to obtain the solution of the plane strain problem in terms of a system of coordinates in the undeformed body. Stresses in an infinite plate with two circular holes are discussed. It is assumed that the plate is subjected to biaxial tension and holes take given shapes before deformation. The basic formulas and notations developed by *Adkins*, *Green* and *Shield* in their theory of finite elasticity [4] will be used throughout this paper.

2. Basic Equations. The cartesian coordinates of points in the undeformed and deformed states are denoted by ξ_α, x_α ($\alpha = 1, 2$), $\xi_3 = x_3$ respectively and the complex coordinate system in the

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undeformed body is defined by tensor transformations as follows (Greek indices take, henceforth, the values 1, 2):

$$\zeta^1 = \frac{\partial \zeta}{\partial \xi_1}\,\xi_1 + \frac{\partial \zeta}{\partial \xi_2}\,\xi_2 = \xi_1 + i\,\xi_2 = \zeta\,, \qquad \zeta^2 = \frac{\partial \bar{\zeta}}{\partial \xi_1}\,\xi_1 + \frac{\partial \bar{\zeta}}{\partial \xi_2}\,\xi_2 = \xi_1 - i\,\xi_2 = \bar{\zeta}\,. \tag{1}$$

Moreover, we choose the moving system of coordinates θ_α to coincide with the system of coordinates ζ^α so that

$$\theta_1 = \zeta^1 = \zeta\,, \quad \theta_2 = \zeta^2 = \bar{\zeta}\,. \tag{2}$$

Now, let u, v denote the components of displacement along the ξ_α-axes of the undeformed body respectively and define the displacement function $D(\zeta, \bar{\zeta})$ as follows:

$$\left.\begin{aligned} x_1 + i\,x_2 &= \zeta + D\,, & x_1 - i\,x_2 &= \bar{\zeta} + \bar{D}\,, \\ D &= u + i\,v\,, & \bar{D} &= u - i\,v\,. \end{aligned}\right\} \tag{3}$$

From the conditions of incompressibility of the material and state of plane strain, invariants of strains can be written as

$$I_3 = 1\,, \qquad I_1 = I_2 = I\,,$$

and therefore

$$\frac{\partial D}{\partial \zeta} + \frac{\partial \bar{D}}{\partial \bar{\zeta}} + \frac{\partial D}{\partial \zeta}\frac{\partial \bar{D}}{\partial \bar{\zeta}} - \frac{\partial D}{\partial \bar{\zeta}}\frac{\partial \bar{D}}{\partial \zeta} = 0\,. \tag{4}$$

Denote $a_{\alpha\beta}$ and $A_{\alpha\beta}$ respectively the covariant metric tensors associated with curvilinear coordinates θ_α in a plane $\xi_3 = 0$ of the undeformed body and in a plane $x_3 = 0$ of the deformed body:

$$a_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad A_{\alpha\beta} = \begin{pmatrix} \frac{\partial \bar{D}}{\partial \zeta}\left(\frac{\partial D}{\partial \zeta} + 1\right) & \frac{1}{2} + \frac{\partial D}{\partial \bar{\zeta}}\frac{\partial \bar{D}}{\partial \zeta} \\ \frac{1}{2} + \frac{\partial D}{\partial \bar{\zeta}}\frac{\partial \bar{D}}{\partial \zeta} & \frac{\partial D}{\partial \bar{\zeta}}\left(\frac{\partial \bar{D}}{\partial \bar{\zeta}} + 1\right) \end{pmatrix}. \tag{5}$$

By the use of (5), first and second strain invariants are given as follows:

$$I_1 = I_2 = I = 3 + 4\,\frac{\partial D}{\partial \bar{\zeta}}\frac{\partial \bar{D}}{\partial \zeta}\,. \tag{6}$$

The equations of equilibrium for plane strain in the absence of body forces are identically satisfied if the stress tensor $\tau^{\alpha\beta}$ is expressed in terms of *Airy*'s stress function $\Phi(\theta_1, \theta_2)$ by

$$\tau^{\alpha\beta} = (1/\sqrt{I_3})\,\varepsilon^{\alpha\gamma}\,\varepsilon^{\beta\varrho}\,\Phi\|_{\gamma\varrho}\,, \tag{7}$$

where

$$\varepsilon^{\alpha\gamma}\sqrt{A} = e^{\alpha\gamma 3} \qquad (e^{\alpha\gamma 3}\text{: } \textit{Eddington}\text{'s epsilon})\,.$$

Here A is the determinant of the covariant metric tensor $A_{\alpha\beta}$ of the deformed body and the symbol $\|$ denotes covariant differentiation with respect to the deformed body. Then we obtain the governing equation for stress function $\Phi(\zeta, \bar{\zeta})$ as follows:

$$2\,A_{12}\frac{\partial^2 \Phi}{\partial \zeta^2} - 2\,A_{11}\frac{\partial^2 \Phi}{\partial \zeta \partial \bar{\zeta}} - \frac{\partial A_{11}}{\partial \bar{\zeta}}\frac{\partial \Phi}{\partial \zeta} - \frac{\partial A_{11}}{\partial \zeta}\frac{\partial \Phi}{\partial \bar{\zeta}} + A_{11}\,H = 0\,, \tag{8}$$

where

$$H = 2\,(dW(I)/dI)\,. \tag{9}$$

Now we take strain energy function $W(I)$ proposed by *Mooney* [7]

$$W(I_1, I_2) = C_1\,(I_1 - 3) + C_2\,(I_2 - 3) = (C_1 + C_2)\,(I - 3)\;(= W(I))\,, \tag{10}$$

where C_1, C_2 are material constants.

From (7) the complex stress components $T^{\alpha\beta}$ referred to coordinates in the undeformed body are given as follows:

$$T^{11} = \bar{T}^{22} = -\,4\,\Phi\|_{22}\,, \qquad T^{12} = 4\,\Phi\|_{12}\,. \tag{11}$$

If the resultant force P across any arc of a curve in the deformed body has components (X, Y) along the ξ_1- and ξ_2-axes respectively, we obtain the equation in the appropriate form

$$P = X + i\,Y = 2\,i\left\{\frac{\partial \Phi}{\partial \bar{\zeta}} + \frac{\partial D}{\partial \zeta}\frac{\partial \Phi}{\partial \bar{\zeta}} - \frac{\partial D}{\partial \bar{\zeta}}\frac{\partial \Phi}{\partial \zeta}\right\}. \tag{12}$$

The basic equations of classical elasticity are derived from these equations by simply ignoring the squares and products of displacements and their spatial derivatives. In the second approximation theory, we expand $D(\zeta\ \bar{\zeta})$ in the power series of a small quantity ε as follows [8]:

$$D = \varepsilon({}^0D) + \varepsilon^2({}^1D) + \cdots . \tag{13}$$

Then from (6) and (9)

$$I = 3 + 4\,\varepsilon^2 \left\{ \frac{\partial^0 D}{\partial \bar{\zeta}} \frac{\partial^0 \bar{D}}{\partial \zeta} + \cdots \right\}, \qquad H = {}^0H + 4\,\varepsilon^2\,({}^2H) \frac{\partial^0 D}{\partial \bar{\zeta}} \frac{\partial^0 \bar{D}}{\partial \zeta} + \cdots, \tag{14}$$

where

$${}^0H = 2 \left(\frac{dW}{dI} \right)_{I=3}, \qquad {}^2H = 2 \left(\frac{d^2W}{dI^2} \right)_{I=3}.$$

For *Mooney* materials we take

$${}^0H = 2\,(C_1 + C_2) = E/3\,, \qquad {}^2H = 0\,,$$

where E is the *Young*'s modulus.

Now, we expand $\Phi(\zeta, \bar{\zeta})$ also in the power series of ε as

$$\Phi = {}^0H\,\varepsilon\,\{{}^0\Phi + \varepsilon({}^1\Phi) + \cdots\}\,. \tag{15}$$

By substituting (13), (14) and (15) in the incompressibility conditions (4) and in (8) respectively, and equating to zero all the coefficients of ε, we obtain

$$\frac{\partial^0 D}{\partial \zeta} + \frac{\partial^0 \bar{D}}{\partial \bar{\zeta}} = 0\,, \qquad \frac{\partial^1 D}{\partial \zeta} + \frac{\partial^1 \bar{D}}{\partial \bar{\zeta}} + \frac{\partial^0 D}{\partial \zeta} \frac{\partial^0 \bar{D}}{\partial \bar{\zeta}} - \frac{\partial^0 D}{\partial \bar{\zeta}} \frac{\partial^0 \bar{D}}{\partial \zeta} = 0\,, \tag{16a), (16b}$$

$$\frac{\partial^{20}\Phi}{\partial \zeta^2} + \frac{\partial^0 \bar{D}}{\partial \zeta} = 0\,, \tag{17a}$$

$$\frac{\partial^{21}\Phi}{\partial \zeta^2} + \frac{\partial^1 \bar{D}}{\partial \zeta} = 2\,\frac{\partial^0 \bar{D}}{\partial \zeta} \frac{\partial^2 \Phi}{\partial \zeta\,\partial \bar{\zeta}} + \frac{\partial^{20} \bar{D}}{\partial \zeta^2} \frac{\partial^0 \Phi}{\partial \bar{\zeta}} - \frac{\partial^{20} \bar{D}}{\partial \zeta\,\partial \bar{\zeta}} \frac{\partial^0 \Phi}{\partial \zeta} - \frac{\partial^0 D}{\partial \zeta} \frac{\partial^0 \bar{D}}{\partial \zeta}\,. \tag{17b}$$

The solutions of (16a), (17a) can be expressed in complex potential functions $\varphi(\zeta)$, $\psi(\zeta)$ given by *Muskhelishvili* as follows:

$$\left.\begin{aligned} {}^0\Phi(\zeta, \bar{\zeta}) &= \zeta\,\bar{\varphi}(\bar{\zeta}) + \bar{\zeta}\,\varphi(\zeta) + \int^{\zeta} \psi(\zeta)\,d\zeta + \int^{\bar{\zeta}} \bar{\psi}(\bar{\zeta})\,d\bar{\zeta}\,, \\ {}^0D(\zeta, \bar{\zeta}) &= \varphi(\zeta) - \zeta\,\bar{\varphi}'(\bar{\zeta}) - \bar{\psi}(\bar{\zeta})\,. \end{aligned}\right\} \tag{18}$$

By taking account of (18) the solutions of (16b), (17b) can also be expressed in similar complex potential functions $\Delta(\zeta)$, $\lambda(\zeta)$ as follows:

$$\left.\begin{aligned} \frac{\partial^1\Phi(\zeta, \bar{\zeta})}{\partial \bar{\zeta}} &= \Delta(\zeta) + \zeta\,\bar{\Delta}'(\bar{\zeta}) + \bar{\lambda}(\bar{\zeta}) - \frac{5}{4}\,\zeta\,\{\bar{\varphi}'(\bar{\zeta})\}^2 + \frac{7}{4}\,\varphi(\zeta)\,\bar{\varphi}'(\bar{\zeta}) - \frac{1}{4}\,\Gamma(\zeta, \bar{\zeta})\,, \\ {}^1D(\zeta, \bar{\zeta}) &= \Delta(\zeta) - \zeta\,\bar{\Delta}'(\bar{\zeta}) - \bar{\lambda}(\bar{\zeta}) - \frac{1}{2}\,\bar{\varphi}'(\bar{\zeta})\,\bar{\psi}(\bar{\zeta}) - \frac{5}{4}\,\varphi(\zeta)\,\bar{\varphi}'(\bar{\zeta}) + \\ &\quad + \frac{1}{4}\,\Lambda(\zeta, \bar{\zeta}) - \frac{1}{2} \int^{\zeta} \{\varphi'(\zeta)\}^2\,d\zeta + \int^{\bar{\zeta}} \bar{\varphi}''(\bar{\zeta})\,\bar{\psi}(\bar{\zeta})\,d\bar{\zeta} + \frac{5}{4}\,\zeta\,\{\bar{\varphi}'(\bar{\zeta})\}^2\,, \end{aligned}\right\} \tag{19}$$

where

$$\left.\begin{aligned} \Gamma(\zeta, \bar{\zeta}) &= 6\,\{\zeta\,\bar{\varphi}''(\bar{\zeta}) + \bar{\psi}'(\bar{\zeta})\}\,\{\bar{\zeta}\,\varphi'(\zeta) + \psi(\zeta) + \bar{\varphi}(\bar{\zeta})\} + \\ &\quad + \{6\,\varphi'(\zeta) + \bar{\varphi}'(\bar{\zeta})\}\,\{\zeta\,\bar{\varphi}'(\bar{\zeta}) + \bar{\psi}(\bar{\zeta}) + \varphi(\zeta)\}\,, \\ \Lambda(\zeta, \bar{\zeta}) &= 2\,\{\zeta\,\bar{\varphi}''(\bar{\zeta}) + \bar{\psi}'(\bar{\zeta})\}\,\{\bar{\zeta}\,\varphi'(\zeta) + \psi(\zeta) + \bar{\varphi}(\bar{\zeta})\} - \\ &\quad - \{2\,\varphi'(\zeta) - 3\,\bar{\varphi}'(\bar{\zeta})\}\,\{\zeta\,\bar{\varphi}'(\bar{\zeta}) + \bar{\psi}(\bar{\zeta}) + \varphi(\zeta)\}\,. \end{aligned}\right\} \tag{20}$$

Furthermore, it is assumed that the resultant force P can also be expanded in the power series of ε as follows:

$$P = {}^0H\,\varepsilon\,\{{}^0P + \varepsilon({}^1P) + \cdots\}\,. \tag{21}$$

Then from (12) and (21) we have

$$ {}^0P = 2\,i\left\{\frac{\partial^0\Phi}{\partial\bar{\zeta}}\right\}, \qquad {}^1P = 2\,i\left\{\frac{\partial^1\Phi}{\partial\bar{\zeta}} + \frac{\partial^0 D}{\partial\zeta}\,\frac{\partial^0\Phi}{\partial\bar{\zeta}} - \frac{\partial^0 D}{\partial\bar{\zeta}}\,\frac{\partial^0\Phi}{\partial\zeta}\right\}. \tag{22} $$

On the basis of this mathematical preparation, we proceed to investigate the finite plane strain problem of infinite plate with two circular holes.

3. Infinite Plate with Two Circular Holes. The theory developed in the previous section are applied to the problem of an infinite plate having two circular holes subjected to biaxial tension. Attention will be confined to terms of the first and second orders only.

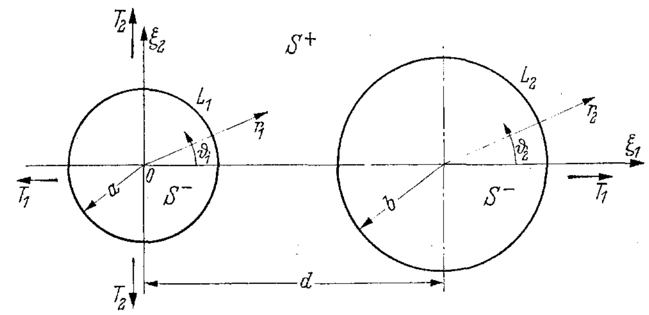

Fig. 1. Coordinates in undeformed body.

a) Boundary conditions. We take the coordinates system in the undeformed body as shown in Fig. 1 and introduce the following nondimensional quantities

$$ x = \zeta/d\,, \qquad \lambda = a/d\,, \qquad \nu = b/d\,. \tag{23} $$

Let T_1, T_2 denote the principal stresses at infinity, we have

$$ T^{11} = \overline{T}^{22} = T_1 - T_2\,, \qquad T^{12} = T_1 + T_2 \tag{24} $$

at infinity. We take here as the perturbation parameter

$$ \varepsilon = \frac{T_1 - T_2}{4\,{}^0H} = \frac{3\,T}{4\,m\,E}, \tag{25} $$

where

$$ T = T_1 + T_2\,, \qquad m = \frac{T_1 + T_2}{T_1 - T_2}, \tag{26} $$

and also put $h = (T_1 - T_2)/(T_1 - T_2) = 1$. Therefore, from (11), (15), (24), (25) and (26), the conditions at infinity yield

$$ \left.\begin{aligned} &\frac{\partial^{2\,0}\Phi}{\partial\bar{x}^2} = \frac{\partial^{2\,0}\Phi}{\partial x^2} = -\,h + 0\left(\frac{1}{|x|^2}\right),\\ &\frac{\partial^{2\,0}\Phi}{\partial x\,\partial\bar{x}} = m + 0\left(\frac{1}{|x|^2}\right),\\ &\frac{\partial^{2\,1}\Phi}{\partial\bar{x}^2} = \frac{\partial^{2\,1}\Phi}{\partial x^2} = \frac{\partial^{2\,1}\Phi}{\partial x\,\partial\bar{x}} = 0\left(\frac{1}{|x|^2}\right) \end{aligned}\right\}\ \text{for large } |x|\,. \tag{27} $$

From (22) the boundary condition which should hold on the free boundary of a hole for first and second order terms respectively we find

$$ \frac{\partial^0\Phi}{\partial\bar{x}} = \frac{\partial^0\Phi}{\partial x} = \text{const.}\,, \tag{28a} $$

$$ \frac{\partial^1\Phi}{\partial\bar{x}} + \frac{\partial^0 D}{\partial x}\,\frac{\partial^0\Phi}{\partial\bar{x}} - \frac{\partial^0 D}{\partial\bar{x}}\,\frac{\partial^0\Phi}{\partial x} = \text{const.}\,, \tag{28b} $$

where constants in the right hand side of (28) should be determined from some freedom in the choice of complex stress functions corresponding to the same state of stress.

b) Evaluation of complex potential functions. It is easily shown that the first two equations of (27) will be satisfied if

$$ \varphi(x) = \frac{m}{2}\,x + \varphi_0(x)\,, \qquad \psi(x) = -\,h\,x + \psi_0(x)\,. \tag{29} $$

The functions $\varphi_0(x)$ and $\psi_0(x)$ are holomorphic in the region S^+ including the point at infinity and can be determined by the boundary condition. Then, $\varphi_0(x)$ and $\psi_0(x)$ can be expanded in series and therefore the required functions $\varphi(x)$ and $\psi(x)$ can be written in following forms

$$\varphi(x) = \frac{m}{2}x + \sum_{k=0}^{\infty}\frac{c_k}{x^k} + \sum_{k=1}^{\infty}\frac{e_k}{(x-1)^k}, \quad \psi(x) = -hx + \sum_{k=0}^{\infty}\frac{f_k}{x^k} + \sum_{k=1}^{\infty}\frac{g_k}{(x-1)^k}. \tag{30}$$

where $c_0, f_0, c_k, e_k, f_k, g_k$ ($k = 1, 2, \ldots$) are unknown real coefficients of expansion.

From (18) and (28a) the boundary conditions on the peripheries of holes become

$$\begin{aligned} &\varphi(x) + x\,\overline{\varphi}'(\overline{x}) + \overline{\psi}(\overline{x}) = \text{const.} = C_j' \qquad (j = 1, 2)\,, \\ &j = 1 \quad \text{on } L_1 \quad (x = \lambda\,\sigma_1): \qquad C_1' = 0\,, \\ &j = 2 \quad \text{on } L_2 \quad (x = 1 + \nu\,\sigma_2): \quad C_2' \neq 0\,, \end{aligned} \tag{31}$$

where $\sigma_j = e^{i\vartheta_j}$ (ϑ_j = polar angle; $j = 1, 2$) and C_2' is unknown constant. With the aid of (30), these equations are transformed to

$$\sum_{k=0}^{\infty}\frac{c_k}{(\lambda\sigma_1)^k} + \sum_{k=1}^{\infty}\frac{e_k}{(\lambda\sigma_1 - 1)^k} - \lambda\sigma_1\sum_{k=1}^{\infty}k\frac{c_k}{(\lambda\sigma_1^{-1})^{k+1}} - \lambda\sigma_1\sum_{k=1}^{\infty}k\frac{e_k}{(\lambda\sigma_1^{-1}-1)^{k+1}} +$$
$$+ \sum_{k=0}^{\infty}\frac{f_k}{(\lambda\sigma_1^{-1})^k} + \sum_{k=1}^{\infty}\frac{g_k}{(\lambda\sigma_1^{-1}-1)^k} + m\lambda\sigma_1 - h\lambda\sigma_1^{-1} = 0\,, \qquad \text{on } L_1\,, \tag{32a}$$

and

$$\sum_{k=0}^{\infty}\frac{c_k}{(1+\nu\sigma_2)^k} + \sum_{k=1}^{\infty}\frac{e_k}{(\nu\sigma_2)^k} - (1+\nu\sigma_2)\sum_{k=1}^{\infty}k\frac{c_k}{(1+\nu\sigma_2^{-1})^{k+1}} - (1+\nu\sigma_2)\sum_{k=1}^{\infty}k\frac{e_k}{(\nu\sigma_2^{-1})^{k+1}} +$$
$$+ \sum_{k=0}^{\infty}\frac{f_k}{(1+\nu\sigma_2^{-1})^k} + \sum_{k=1}^{\infty}\frac{g_k}{(\nu\sigma_2^{-1})^k} + m(1+\nu\sigma_2) - h(1+\nu\sigma_2^{-1}) = C_2'\,, \qquad \text{on } L_2\,. \tag{32b}$$

Equating the coefficient of each power of σ_1 on both sides of (32a) and taking account of the single-valuedness of displacement, we obtain

$$\left.\begin{aligned} c_1 &= h\lambda^2 - \sum_{k=1}^{\infty}(-1)^k g_k\,{}^1\gamma_k\lambda^2 - \sum_{k=1}^{\infty}(-1)^k e_k\,{}^2s_{k+1}\lambda^4\,, \\ f_1 &= -m\lambda^2 - \sum_{k=1}^{\infty}(-1)^k e^k\,{}^1\gamma_k\lambda^2 - \sum_{k=1}^{\infty}(-1)^k e_k\,{}^0s_{k+1}\lambda^2\,, \\ c_n &= -\sum_{k=1}^{\infty}(-1)^k g_k\,{}^n\gamma_k\lambda^{2n} - \sum_{k=1}^{\infty}(-1)^k e^k\,{}^{n+1}s_{k+1}\lambda^{2n+2}\,, \qquad n \geq 2\,, \\ f_n &= -\sum_{k=1}^{\infty}(-1)^k e_k\,{}^n\gamma_k\lambda^{2n} + (n-2)\,c_{n-2}\lambda^2\,, \qquad n \geq 2\,, \\ c_0 &= f_0 = \frac{1}{2}\left[-\sum_{k=1}^{\infty}(-1)^k e_k\,{}^0\gamma_k - \sum_{k=1}^{\infty}(-1)^k e_k\,{}^1s_{k+1}\lambda^2 - \sum_{k=1}^{\infty}(-1)^k g_k\,{}^0\gamma_k\right], \end{aligned}\right\} \tag{33a}$$

where ${}^n\gamma_k$ are binominal coefficients, i.e. ${}^n\gamma_k = \binom{k+n-1}{n}$ and ${}^ns_{k+1} = k\,{}^n\gamma_{k+1}$.

By applying a similar process for (32b), we have

$$\left.\begin{aligned} e_1 &= h\nu^2 - \sum_{k=1}^{\infty}c_k\,{}^1s_{k+1}\nu^2 + \sum_{k=1}^{\infty}c_k\,{}^2s_{k+1}\nu^4 + \sum_{k=1}^{\infty}f_k\,{}^1\gamma_k\nu^2\,, \\ g_1 &= -m\nu^2 + \sum_{k=1}^{\infty}c_k\,{}^1\gamma_k\nu^2 + \sum_{k=1}^{\infty}c_k\,{}^0s_{k+1}\nu^2\,, \\ e_n &= \sum_{k=1}^{\infty}(-1)^n c_k\,{}^ns_{k+1}\nu^{2n} - \sum_{k=1}^{\infty}(-1)^n c_k\,{}^{n+1}s_{k+1}\nu^{2n+2} - \sum_{k=1}^{\infty}(-1)^n f_k\,{}^n\gamma_k\nu^{2n}\,, \qquad n \geq 2\,, \\ g_n &= (n-1)\,e_{n-1} + (n-2)\,e_{n-2}\nu^2 - \sum_{k=1}^{\infty}(-1)^n c_k\,{}^n\gamma_k\nu^{2n}\,, \qquad n \geq 2\,, \\ C_2' &= m - h + \sum_{k=1}^{\infty}c_k\,{}^0\gamma_k - \sum_{k=1}^{\infty}c_k\,{}^0s_{k+1} + \sum_{k=1}^{\infty}c_k\,{}^1s_{k+1}\nu^2 + \sum_{k=1}^{\infty}f_k\,{}^0\gamma_k + c_0 + f_0\,. \end{aligned}\right\} \tag{33b}$$

The set of equations (33) is the simultaneous linear equations for determining the unknown coefficients of expansion of (30), when the values of m, h, λ and ν are given. To solve this set of equations, it is rather convenient to use the iteration method in which the set is reduced to a finite one by taking only the first l equations involving the first l unknowns of each set of expansion coefficients. Thus we shall seek a solution in the form

$$c_n = \lim_{p\to\infty} c_n^{(p)} \text{ etc.}. \tag{34}$$

Zeroth iterations ($p = 0$) of c_n and f_n only are to be substituted into (33). It may be noted that the solution is valid only in the case where each of limiting values in (34) is convergent. The convergence seems rather difficult of proof, although it seems unlikely from the physical consideration that there will be divergence when $\lambda = a/d < \varepsilon_1$, $\nu = b/d < \varepsilon_2$, and $\varepsilon_1 + \varepsilon_2 < 1$.

Next, we shall determine the complex potential functions $\Delta(x)$ and $\lambda(x)$ of the second order term. For this purpose we see that the equations in the last line of (27) will be satisfied if

$$\frac{\partial^1\Phi}{\partial\bar{x}} = \Delta(x) + x\,\bar{\Delta}'(\bar{x}) + \bar{\lambda}(\bar{x}) - \frac{5}{4}x\,\{\bar{\varphi}'(\bar{x})\}^2 + \frac{7}{4}\varphi(x)\,\bar{\varphi}'(\bar{x}) - \frac{1}{4}\,\Gamma(x,\bar{x}) = 0\left(\frac{1}{|x|}\right) \tag{35}$$

for large $|x|$. Functions $\Delta(x)$ and $\lambda(x)$ can now be expressed as follows:

$$\Delta(x) = B_1 x + \Delta_0(x)\,, \qquad \lambda(x) = B_2 x + \lambda_0(x)\,, \tag{36}$$

where $\Delta_0(x)$ and $\lambda_0(x)$ are holomorphic in the region S^+, so that these functions can be expressed in series. The quantities B_1 and B_2 are constants determined from the singularity of $\Delta(x)$ and $\lambda(x)$ at infinity respectively. Further, for large $|x|$:

$$\left.\begin{aligned} \Gamma(x,\bar{x}) &= -\frac{19}{2}h\,m\,\bar{x} + \frac{19}{2}m^2 x + 0\left(\frac{1}{|x|}\right), \\ \frac{5}{4}x\,\{\bar{\varphi}'(\bar{x})\}^2 - \frac{7}{4}\varphi(x)\,\bar{\varphi}'(\bar{x}) &= -\frac{m^2}{8}x + 0\left(\frac{1}{|x|}\right). \end{aligned}\right\} \tag{37}$$

Substituting (36), (37) into (35) we find the constants B_1, B_2 and finally

$$\left.\begin{aligned} \Delta(x) &= \frac{3(m^2+2h^2)}{8}x + \sum_{k=0}^{\infty}\frac{C_k}{x^k} + \sum_{k=1}^{\infty}\frac{E_k}{(x-1)^k}, \\ \lambda(x) &= -\frac{19\,h\,m}{8}x + \sum_{k=0}^{\infty}\frac{F_k}{x^k} + \sum_{k=1}^{\infty}\frac{G_k}{(x-1)^k}, \end{aligned}\right\} \tag{38}$$

where $C_0, F_0, C_k, E_k, F_k, G_k$ $(k = 1, 2, \ldots)$ are real coefficients not yet determined. These constants will be given by the boundary conditions at the edges of holes for second order term, that is

$$\frac{\partial^1\Phi}{\partial\bar{x}} + \frac{\partial^0 D}{\partial x}\,\frac{\partial^0\Phi}{\partial\bar{x}} - \frac{\partial^0 D}{\partial\bar{x}}\,\frac{\partial^0\Phi}{\partial x} = \text{const.} \tag{39}$$

Substituting (18), (19) and (38) into (39) and with the aid of the first order solutions, we obtain

$$\begin{aligned} &\frac{3(m^2+2h^2)}{4}x + \sum_{k=0}^{\infty}\frac{C_k}{x^k} + \sum_{k=1}^{\infty}\frac{E_k}{(x-1)^k} - x\sum_{k=1}^{\infty}k\frac{C_k}{\bar{x}^{k+1}} - x\sum_{k=1}^{\infty}k\frac{E_k}{(\bar{x}-1)^{k+1}} - \\ &- \frac{19\,h\,m}{8}\bar{x} + \sum_{k=0}^{\infty}\frac{F_k}{\bar{x}^k} + \sum_{k=1}^{\infty}\frac{G_k}{(\bar{x}-1)^k} - \frac{5}{4}x\left\{\frac{m}{2} - \sum_{k=1}^{\infty}k\frac{c_k}{\bar{x}^{k+1}} - \sum_{k=1}^{\infty}k\frac{e_k}{(\bar{x}-1)^{k+1}}\right\}^2 + \\ &+ \frac{7}{4}\left\{\frac{m}{2}x + \sum_{k=0}^{\infty}\frac{c_k}{x^k} + \sum_{k=1}^{\infty}\frac{e_k}{(x-1)^k}\right\}\left\{\frac{m}{2} - \sum_{k=1}^{\infty}k\frac{c_k}{\bar{x}^{k+1}} - \sum_{k=1}^{\infty}k\frac{e_k}{(\bar{x}-1)^{k+1}}\right\} - \\ &- \frac{C_j'}{4}\left[2x\left\{\sum_{k=1}^{\infty}k(k+1)\frac{c_k}{\bar{x}^{k+2}} + \sum_{k=1}^{\infty}k(k+1)\frac{e_k}{(\bar{x}-1)^{k+2}} - 2h + \frac{7}{2}m - 2\sum_{k=1}^{\infty}k\frac{f_k}{\bar{x}^{k+1}} - \right.\right. \\ &- 2\sum_{k=1}^{\infty}k\frac{g_k}{(\bar{x}-1)^{k+1}} - 2\sum_{k=1}^{\infty}k\frac{c_k}{x^{k+1}} - 2\sum_{k=1}^{\infty}k\frac{e_k}{(x-1)^{k+1}} - 5\sum_{k=1}^{\infty}k\frac{c_k}{\bar{x}^{k+1}} - \\ &\left.- 5\sum_{k=1}^{\infty}k\frac{e_k}{(\bar{x}-1)^{k+1}}\right] = C_j'' \qquad (j = 1, 2)\,, \end{aligned} \tag{40}$$

$$\begin{aligned} j &= 1 \quad \text{on } L_1 \quad (x = \lambda\,\sigma_1)\;: \quad C_1'' = 0\,, \\ j &= 2 \quad \text{on } L_2 \quad (x = 1 + \nu\,\sigma_2)\colon \quad C_2'' \neq 0\,, \end{aligned}$$

where C_2'' is unknown constant. Comparing the coefficients of various powers of σ_j on both sides of (40), and taking account of the single-valuedness of displacement, we get the set of equations as follows: From the boundary condition holding on L_1

$$\left.\begin{aligned}
C_1 &= \frac{19\,h\,m}{8}\lambda^2 - \sum_{k=1}^{\infty}(-1)^k G_k\,{}^1\gamma_k\lambda^2 - \sum_{k=1}^{\infty}(-1)^k E_k\,{}^2s_{k+1}\lambda^4 + \Theta_1^{(1)}\,,\\
F_1 &= -\frac{3\,(m^2+2\,h^2)}{4}\lambda^2 - \sum_{k=1}^{\infty}(-1)^k E_k\,{}^1\gamma_k\lambda^2 - \sum_{k=1}^{\infty}(-1)^k E_k\,{}^0s_{k+1}\lambda^2 + \Theta_1^{(3)}\,,\\
C_n &= -\sum_{k=1}^{\infty}(-1)^k G_k\,{}^n\gamma_k\lambda^{2n} - \sum_{k=1}^{\infty}(-1)^k E_k\,{}^{n+1}s_{k+1}\lambda^{2n+2} + \Theta_n^{(1)},\quad n\ge 2\,,\\
F_n &= -\sum_{k=1}^{\infty}(-1)^k E_k\,{}^n\gamma_k\lambda^{2n} + (n-2)\,C_{n-2}\lambda^2 + \Theta_n^{(3)}\,,\quad n\ge 2\,,\\
\begin{matrix}C_0\\F_0\end{matrix}\Big\} &= \frac{1}{2}\left[-\sum_{k=1}^{\infty}(-1)^k E_k\,{}^0\gamma_k - \sum_{k=1}^{\infty}(-1)^k E_k\,{}^1s_{k+1}\lambda^2 - \sum_{k=1}^{\infty}(-1)^k G_k\,{}^0\gamma_k + \begin{Bmatrix}\Theta_0^{(1)}\\ \Theta_0^{(3)}\end{Bmatrix}\right].
\end{aligned}\right\}\quad (41\,\mathrm{a})$$

Also the boundary condition given on L_2 yields

$$\left.\begin{aligned}
E_1 &= \frac{19\,h\,m}{8}\nu^2 - \sum_{k=1}^{\infty}C_k\,{}^1s_{k+1}\nu^2 + \sum_{k=1}^{\infty}C_k\,{}^2s_{k+1}\nu^4 + \sum_{k=1}^{\infty}F_k\,{}^1\gamma_k\nu^2 + \Theta_1^{(2)}\,,\\
G_1 &= -\frac{3\,(m^2+2\,h^2)}{4}\nu^2 + \sum_{k=1}^{\infty}C_k\,{}^1\gamma_k\nu^2 + \sum_{k=1}^{\infty}C_k\,{}^0s_{k+1}\nu^2 + \Theta_1^{(4)}\,,\\
E_n &= \sum_{k=1}^{\infty}(-1)^n C_k\,{}^ns_{k+1}\nu^{2n} - \sum_{k=1}^{\infty}(-1)^n C_k\,{}^{n+1}s_{k+1}\nu^{2n+2} -\\
&\quad - \sum_{k=1}^{\infty}(-1)^n F_k\,{}^n\gamma_k\nu^{2n} + \Theta_n^{(2)}\,,\quad n\ge 2\,,\\
G_n &= (n-1)\,E_{n-1} + (n-2)\,E_{n-2}\nu^2 - \sum_{k=1}^{\infty}(-1)^n C_k\,{}^n\gamma_k\nu^{2n} + \Theta_n^{(4)}\,,\quad n\ge 2\,,
\end{aligned}\right\}\quad (41\,\mathrm{b})$$

where $\Theta_0^{(1)}$, $\Theta_0^{(3)}$, $\Theta_n^{(i)}$ $(i = 1, 2, 3, 4;\ n = 1, 2, \ldots)$ are the known constants given by the constants found in the functions $\varphi(x)$ and $\psi(x)$ only. The unknown coefficients included in (38) are determined from (41) by the method of iteration. We shall, however, omit detailed discussions about this method of solution of these infinite systems of equations, as these are quite similar to those applied to the first order solution.

4. Stresses. If we take only the first and second order terms of the series expansion of *Airy*'s stress function, i.e.

$$\Phi(\zeta,\bar\zeta) \cong \frac{T}{4\,m}\left({}^0\Phi + \frac{3\,T}{4\,m\,E}\,{}^1\Phi\right),$$

then from (11) we find the complex stress components $T^{\alpha\beta}$ and the corresponding stress components $t^{\alpha\beta}$ referred to ξ_α-axes as follows:

$$T^{11} = \overline{T}^{22} = t^{11} - t^{22} + 2\,i\,t^{12} \cong -\frac{T}{m}\left\{\frac{\partial^{20}\Phi}{\partial\bar\zeta^2} + \frac{3\,T}{4\,m\,E}\left(\frac{\partial^{21}\Phi}{\partial\bar\zeta^2} - \frac{\partial^{20}D}{\partial\bar\zeta^2}\,\frac{\partial^0\Phi}{\partial\zeta} + \frac{\partial^{20}D}{\partial\bar\zeta\,\partial\zeta}\,\frac{\partial^0\Phi}{\partial\bar\zeta}\right)\right\},$$

$$T^{12} = t^{11} + t^{22} \cong \frac{T}{m}\left\{\frac{\partial^{20}\Phi}{\partial\zeta\,\partial\bar\zeta} + \frac{3\,T}{4\,m\,E}\left(\frac{\partial^{21}\Phi}{\partial\zeta\,\partial\bar\zeta} - \frac{\partial^{20}D}{\partial\zeta\,\partial\bar\zeta}\,\frac{\partial^0\Phi}{\partial\zeta} - \frac{\partial^{20}\bar D}{\partial\bar\zeta\,\partial\zeta}\,\frac{\partial^0\Phi}{\partial\bar\zeta}\right)\right\}.$$

Thus the stress tensor is easily obtained by the use of (18), (19) and the complex potential functions, found in the previous section.

We are interested in finding the stresses at the edge of the hole and σ_{ξ_1} $(= t^{11})$ at the points $(d/2, \xi_2)$ etc.. To this end, the following expressions will be utilized.

$$\frac{\partial^2 {}^0\Phi}{\partial x\, \partial \bar{x}} = 2\,\mathrm{Re}\,\{\varphi'(x)\}\,,$$

$$\frac{\partial^2 {}^0\Phi}{\partial \bar{x}^2} = x\, \bar{\varphi}''(\bar{x}) + \bar{\psi}'(\bar{x})\,,$$

$$\frac{\partial^2 {}^1\Phi}{\partial x\, \partial \bar{x}} = 2\,\mathrm{Re}\,\{\Lambda'(x)\} - 3\,\mathrm{Re}\,[\varphi''(x)\,\{\varphi(x) + x\,\bar{\varphi}'(\bar{x}) + \bar{\psi}(\bar{x})\}] - $$
$$- \frac{3}{2}\,[\mathrm{Re}^2\,\{x\,\bar{\varphi}''(\bar{x}) + \bar{\psi}'(\bar{x})\} + \mathrm{Im}^2\,\{x\,\bar{\varphi}''(\bar{x}) + \bar{\psi}'(\bar{x})\} + 2\,\mathrm{Re}\,\{\varphi'^2(x)\}]\,,$$

$$\frac{\partial^2 {}^1\Phi}{\partial \bar{x}^2} = x\bar{\Lambda}''(\bar{x}) + \bar{\lambda}'(\bar{x}) - \frac{5}{2}\,x\,\bar{\varphi}'(\bar{x})\,\bar{\varphi}''(\bar{x}) + \frac{7}{4}\,\varphi(x)\,\bar{\varphi}''(\bar{x}) -$$
$$- \frac{3}{2}\,[\{x\,\bar{\varphi}'''(\bar{x}) + \bar{\psi}''(\bar{x})\}\{\bar{x}\,\varphi'(x) + \psi(x) + \bar{\varphi}(\bar{x})\} +$$
$$+ \{x\,\bar{\varphi}''(\bar{x}) + \bar{\psi}'(\bar{x})\}\,\{\varphi'(x) + \bar{\varphi}'(\bar{x})\}] - \frac{1}{4}\,[\bar{\varphi}''(\bar{x})\,\{x\,\bar{\varphi}'(\bar{x}) + \bar{\psi}(\bar{x}) + \varphi(x)\} +$$
$$+ \{6\,\varphi'(x) + \bar{\varphi}'(\bar{x})\}\,\{x\,\bar{\varphi}''(\bar{x}) + \bar{\psi}'(\bar{x})\}]\,,$$

where Re denotes real part, while Im denotes imaginary part.

5. Numerical Examples. Numerical calculations by the method mentioned above were worked out for the hoop stresses along the rim of hole etc. for a range of parameters: λ, ν, m and T/E. Most of the numerical calculations were performed on the NEAC 2200 at the Computing Center Tohoku University.

Fig. 2 shows the hoop stresses on the peripheries of holes in the case of simple tension parallel to the ξ_1-axis of magnitude $T_1/E = 0.06$, $m = 1$, $\lambda = \nu = 0.3$. The hoop stress given by the present analysis has the maximum value at the same point where the first order solution has the maximum one. The largest difference of the magnitude of circumferential stresses between second order elasticity and linear one occurs in the vicinity of the point where the stress concentrates and attains to about 19% of the maximum one due to the linear elasticity.

In Fig. 3 the distribution of hoop stress along the circumference of hole and of stress σ_{ξ_1} along the ξ_1-axis between two holes is shown for the case of simple tension parallel to the ξ_2-axis of magnitude $T_2/E = 0.06$, $m = -1$, $\lambda = \nu = 0.3$. The maximum hoop stress occurs at the point $\xi_1/d = 0.3$,

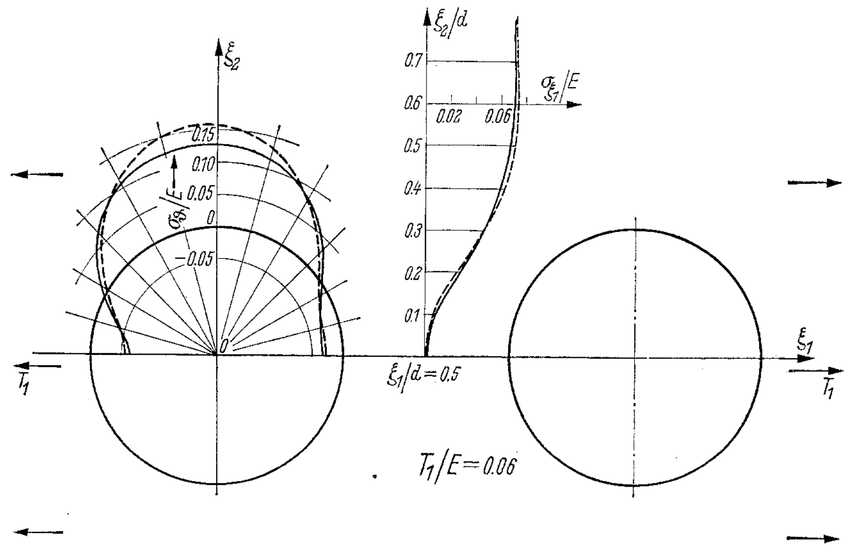

Fig. 2. Hoop stresses (σ_ϑ) along the rim of holes under simple tension parallel to the ξ_1-axis: $m = 1$, $T_1/E = 0.06$, $\lambda = \nu = 0.3$; ——— finite elasticity, — — — linear elasticity.

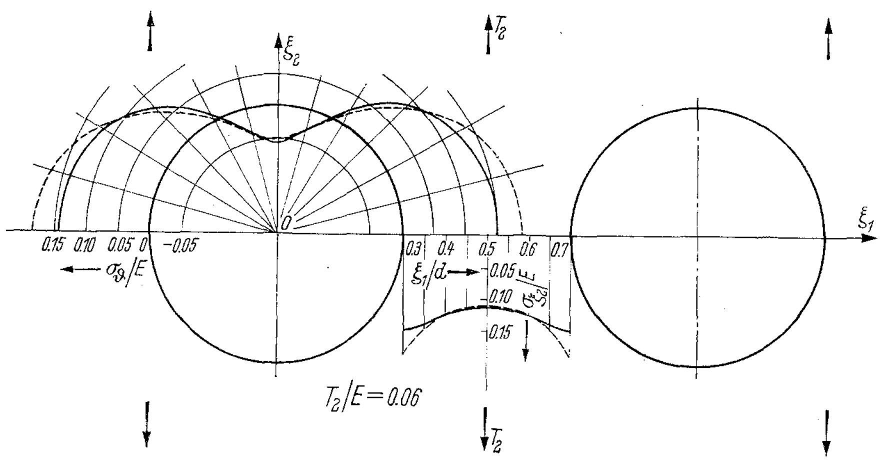

Fig. 3. Hoop stresses (σ_ϑ) along the rim of holes under simple tension parallel to the ξ_2-axis: $m = -1$, $T_2/E = 0.06$, $\lambda = \nu = 0.3$; ——finite elasticity, — — — linear elasticity.

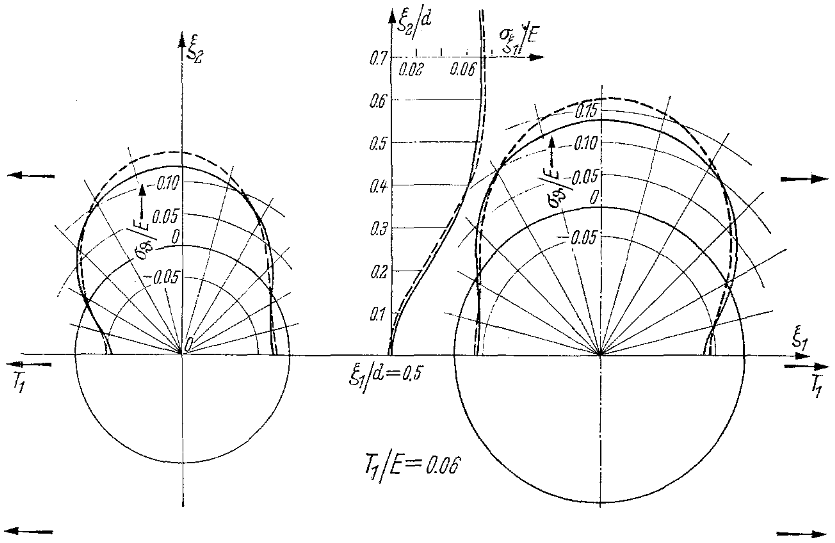

Fig. 4. Hoop stresses (σ_ϑ) along the circular hole rims under simple tension parallel to the ξ_1-axis: $m = 1$, $T_1/E = 0.06$, $\lambda = 0.25$, $\nu = 0.35$; —— finite elasticity, — — — linear elasticity.

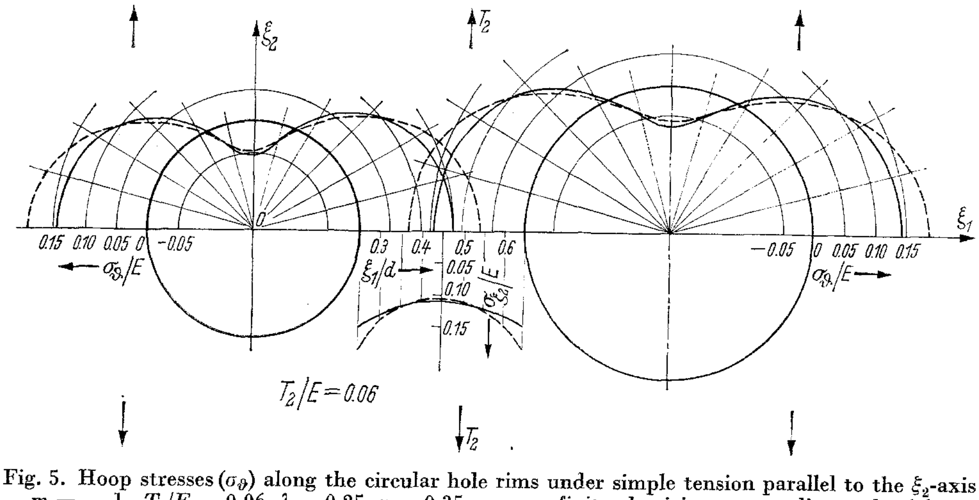

Fig. 5. Hoop stresses (σ_ϑ) along the circular hole rims under simple tension parallel to the ξ_2-axis: $m = -1$, $T_2/E = 0.06$, $\lambda = 0.25$, $\nu = 0.35$; —— finite elasticity, ——— linear elasticity.

0.7 on the ξ_1-axis. It can be observed that in this case linear theory overestimates about 21% at maximum for the magnitude of hoop stresses on the edges of holes in comparison with that due to the second order theory. Here it might be noted that the shapes of holes in the deformed state are different from each other for each theory, because we choose the coordinate system $(\zeta, \bar{\zeta})$ related to the undeformed body.

Fig. 4 shows the interference between two unequal holes for the distribution of circumferential stresses along the rim of holes in the case of simple tension of magnitude $T_1/E = 0.06$, $m = 1$, $\lambda = 0.25$, $\nu = 0.35$.

Fig. 5 shows the hoop stress distribution on the hole peripheries for the same configuration with Fig. 4 but different method of loading, i.e. $m = -1$, $T_2/E = 0.06$, $\lambda = 0.25$, $\nu = 0.35$. The maximum circumferential stress due to the second order theory occurs at the point $\xi_1/d = 0.25$ on the ξ_1-axis. The largest difference of the circumferential stress between two cases is given at the point $\xi_1/d = -0.25$ on the ξ_1-axis and attains to about 23% of the maximum one due to the first order solution.

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