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Uniformity in Weak Convergence

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Section 1. Introduction

Consider probability measures on the σ-field $\mathcal{S}$ of Borel subsets of a metric space S. If P_n and P are such probability measures, P_n is said to converge weakly to P (written $P_n \Rightarrow P$) if $\int f\, dP_n \to \int f\, dP$ for each bounded, continuous real function f on S, or, equivalently, if $P_n(A) \to P(A)$ for each P-continuity set, a P-continuity set being an element A of $\mathcal{S}$ whose boundary ∂A satisfies $P(\partial A) = 0$. Also, it can be shown that, if $P_n \Rightarrow P$, the convergence $\int f\, dP_n \to \int f\, dP$ holds for each bounded, real, measurable function that is continuous almost everywhere P (see [7] or [2]).

Let $\mathcal{B}(S)$ denote the class of all bounded, real, measurable functions defined on S. If $\mathcal{F}$ is a subclass of $\mathcal{B}(S)$, we shall say that $\mathcal{F}$ is a *P-uniformity class* if

$$\lim_{n\to\infty} \sup_{f\in} \left| \int f\, dP_n - \int f\, dP \right| = 0 \tag{1}$$

holds for every sequence $\{P_n\}$ that converges weakly to P. (Of course, even if $\mathcal{F}$ is not a P-uniformity class, (1) will hold for special sequences $\{P_n\}$ such as $P_n \equiv P$.)

We shall say that $\mathcal{F}$ is a *P-continuity class* if every function in $\mathcal{F}$ is continuous except on a set of P-measure 0.

If $\mathfrak{A}$ is a subclass of the σ-field $\mathcal{S}$ of Borel sets, we call $\mathfrak{A}$ a *P-uniformity class* [*P-continuity class*] if the class of indicator functions of sets in $\mathfrak{A}$ is a P-uniformity class [P-continuity class]. Thus $\mathfrak{A} \subset \mathcal{S}$ is a P-uniformity class if

$$\lim_{n\to\infty} \sup_{A\in\mathfrak{A}} \left| P_n A - PA \right| = 0 \tag{2}$$

holds for every sequence $\{P_n\}$ that converges weakly to P, and $\mathfrak{A}$ is a P-continuity class if $P(\partial A) = 0$ for all sets A in $\mathfrak{A}$.

We shall find necessary and sufficient conditions for P-uniformity and then derive some effective criteria for the case of a subclass $\mathfrak{A}$ of $\mathcal{S}$.

Throughout what follows, S is assumed separable. Let ϱ denote the metric on S and denote by $S(x, \delta)$ the open sphere with center x and radius δ. For $\delta > 0$ and $A \subset S$, define the *δ-neighbourhood* of A by

$$A^\delta = \{x : \varrho(x, A) < \delta\}, \tag{3}$$

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and define the δ-*boundary* of A by

(4) $$\partial_\delta A = \{x : \varrho(x, A) < \delta,\ \varrho(x, A^c) < \delta\},$$

where A^c denotes the complement of A. Note that $\partial_\delta A$ is the union of the two disjoint sets $A^\delta \setminus A$ and $(A^c)^\delta \setminus (A^c)$. The ordinary boundary ∂A satisfies

(5) $$\partial A = A^- \setminus A^0 = \bigcap_{\delta > 0} \partial_\delta A = \bigcap_{\delta > 0} (\partial A)^\delta,$$

where A^- and A^0 denote the closure and interior of A.

If $\mathscr{F}$ is a subclass of $\mathscr{B}(S)$ and B a subset of S then we define the *oscillation* of $\mathscr{F}$ on B by

(6) $$\omega_{\mathscr{F}}(B) = \sup\{|f(x) - f(y)| : f \in \mathscr{F}, x, y \in B\}.$$

In the case where $\mathscr{F}$ consists of a single function f we use the notation $\omega_f(B)$ or $\omega_f B$.

Theorem 1. *If $\mathscr{F}$ is a subclass of $\mathscr{B}(S)$ then a necessary and sufficient condition that $\mathscr{F}$ be a P-uniformity class is that*

(7) $$\omega_{\mathscr{F}}(S) < \infty$$

and that

(8) $$\lim_{\delta \to 0} \sup_{f \in \mathscr{F}} P\{x : \omega_f S(x, \delta) > \varepsilon\} = 0$$

for all positive ε.

Since the P-uniformity is not affected if we subtract constants from the functions in $\mathscr{F}$, the condition (7) is essentially equivalent to the condition that $\mathscr{F}$ be uniformly bounded. Notice that the set $\{x : \omega_f S(x, \delta) > \varepsilon\}$ occuring in (8) is open, and hence belongs to $\mathscr{S}$. Theorem 1 clearly extends to complex-valued functions and to functions mapping S into euclidean k-space. It also follows from Theorem 1 that a P-uniformity class is a P-continuity class. Another important consequence is

Theorem 2. *If $\mathfrak{A}$ is a subclass of $\mathscr{S}$ then a necessary and sufficient condition that $\mathfrak{A}$ be a P-uniformity class is that*

(9) $$\lim_{\delta \to 0} \sup_{A \in \mathfrak{A}} P(\partial_\delta A) = 0.$$

In Sections 2 and 3 we prove Theorem 2. Then in Section 4 follows the proof of Theorem 1. We could have proved the more general Theorem 1 first; however, we hope that what we gain in clarity justifies the slight duplication.

The remaining sections concentrate on the study of P-uniformity for a subclass $\mathfrak{A}$ of $\mathscr{S}$. Note that although (9) depends on the metric ϱ, the notion of P-uniformity class is purely topological — in the sense that if $\mathfrak{A}$ is a P-uniformity class then it remains so if ϱ is replaced by an equivalent metric. It follows by Theorem 2, therefore, that $\mathfrak{A}$ is a P-uniformity class if and only if (9) holds for some metric equivalent to ϱ and if and only if (9) holds for all metrics equivalent to ϱ.

Since clearly

(10) $$(\partial A)^\delta \subset \partial_\delta A,$$

the condition (9) implies

(11) $$\lim_{\delta \to 0} \sup_{A \in \mathfrak{A}} P((\partial A)^\delta) = 0.$$

Theorem 3. *If S is locally connected, then* (9) *and* (11) *are equivalent, and each is necessary and sufficient that $\mathfrak{A}$ be a P-uniformity class.*

We have noted that (9) implies (11) even without local connectivity; we shall show by counterexample in Section 9 that (11) need not imply (9) if S is not locally connected. If every sphere in S is connected (a condition stronger than local connectivity), then, since a connected set that meets both A and A^c must meet ∂A, we have

$$\partial_\delta A \subset (\partial A)^\delta, \tag{12}$$

from which it follows that (11) implies (9). We defer to Section 5 the proof that (11) implies (9) in the general locally connected space.

If S is a Banach space, then every sphere in S is connected (even arcwise connected); in particular, S is locally connected. Thus our results are applicable to k-dimensional Euclidean space R^k and to the space C of continuous functions on [0,1], with the uniform topology. Another space of interest in the applications is the space D of functions with discontinuities only of the first kind; it can be shown that, with Skorohod's topology J_1 [*10*], every sphere in D is (arcwise) connected.

Since $\partial\partial A = \partial A$ if A is closed, we have the following corollary to Theorem 3.

Corollary. *If S is locally connected and if $\mathfrak{A}$ consists exclusively of closed sets, then $\mathfrak{A}$ is a P-uniformity class if and only if $\partial\mathfrak{A} = \{\partial A : A \in \mathfrak{A}\}$ is a P-uniformity class.*

The further study of P-uniformity is based on a topological device. Let $\mathfrak{M}$ be the class of closed, bounded, nonempty subsets of S; under Hausdorff's metric

$$\Delta(M_1, M_2) = \inf\{\delta : \delta > 0,\ M_1 \subset M_2^\delta,\ M_2 \subset M_1^\delta\}, \tag{13}$$

$\mathfrak{M}$ is a topological space [*1*]. It will be convenient to include in $\mathfrak{M}$ the empty set, regarded as an isolated point in the topology (notice that (13) is infinite if M_1 or M_2 is empty).

Theorem 4 shows that the compact subsets of $\mathfrak{M}$ play an important role in the theory. The space $\mathfrak{M}$ is compact if S is compact; if $\mathfrak{M}_0$ consists of subsets of a fixed compact set in S, then $\mathfrak{M}_0$ is compact if and only if it is closed; if S is complete then $\mathfrak{M}$ is complete (these results can be found in [*1*]).

Theorem 4. *If S is locally connected and if $\mathfrak{A}$ is a P-continuity class, then each of the following three conditions is sufficient for $\mathfrak{A}$ to be a P-uniformity class.*

(i) *The class*

$$\partial\mathfrak{A} = \{\partial A : A \in \mathfrak{A}\} \tag{14}$$

is a compact subset of $\mathfrak{M}$.

(ii) *There exists a sequence $\{B_k\}$ of bounded sets such that $\lim_k P(B_k^0) = 1$ and such that, for each k, the class*

$$\partial(B_k \cap \mathfrak{A}) = \{\partial(B_k \cap A) : A \in \mathfrak{A}\} \tag{15}$$

is a compact subset of $\mathfrak{M}$.

(iii) *There exists a sequence $\{B_k\}$ of closed, bounded sets such that $\lim_k P(B_k^0) = 1$ and such that, for each k, the class*

$$B_k \cap \partial\mathfrak{A} = \{B_k \cap \partial A : A \in \mathfrak{A}\} \tag{16}$$

is a compact subset of $\mathfrak{M}$.

In Section 6 we shall show (without using the hypothesis of local connectivity) that each of these three conditions, together with the assumption that $\mathfrak{A}$ is a P-continuity class, implies (11); Theorem 4 will then follow by Theorem 3 and the hypothesis of local connectivity.

The important condition in Theorem 4 is (i). The more general conditions (ii) and (iii) do not involve really different ideas; they are introduced because the classes (15) and (16) are always subsets of $\mathfrak{M}$ (compact or not), which need not be true of (14). Notice that if $\mathfrak{A}$ is a P-continuity class, then the elements of (14) and (16) all have P-measure 0 and that the same is true of (15) if $P(\partial B_k) = 0$. Finally, in connection with the compactness requirements, notice that a compact subset of $\mathfrak{M}$ remains compact if the empty subset of S (as an element of $\mathfrak{M}$) is adjoined to it or removed from it.

The conditions (ii) and (iii) of Theorem 4 are often rather restrictive. It is easy to find sets B_k such that $\lim_k P(B_k^0) = 1$ but usually the sets B_k will then not be compact (this is indeed the situation in any infinite dimensional Banach space) and it becomes difficult, unless $\mathfrak{A}$ is rather small, to ensure that the classes in (15) and (16) are compact. Note that Theorem 4 breakes down if we replace the condition $\lim_k P(B_k^0) = 1$ by $\lim_k P(B_k) = 1$ (it is easy to construct a counterexample with P a unit mass). Section 7 is devoted to the proof of the following theorem, which to some extent overcomes the difficulties.

Theorem 5. *Let S be complete and locally connected and let $\mathfrak{A}$ be a P-continuity class consisting entirely of closed sets. Then $\mathfrak{A}$ is a P-uniformity class if to every compact set K there exists another compact set K^* such that $K \subset K^*$ and such that $K^* \cap \partial\mathfrak{A}$ is a compact subset of $\mathfrak{M}$.*

The compactness of $\mathfrak{A}$ itself is sometimes more easily checked than that of $\partial\mathfrak{A}$. This suffices if the elements of $\mathfrak{A}$ are convex:

Theorem 6. *Let S be a (separable) Banach space, real or complex, and let $\mathfrak{A}$ be a class of closed, convex P-continuity sets. Then $\mathfrak{A}$ is a P-uniformity class if $\mathfrak{A}$ is itself a compact subset of $\mathfrak{M}$, or if for each closed sphere B, the class*

$$B \cap \mathfrak{A} = \{B \cap A : A \in \mathfrak{A}\} \tag{17}$$

is a compact subset of $\mathfrak{M}$.

We shall prove this result in Section 8 by reducing it to Theorem 4.

It follows from Theorem 5 and Mazur's theorem (see V.2.6 of [*3*]) that the class $\mathfrak{A}$ in Theorem 6 is a P-uniformity class if $K \cap \partial\mathfrak{A}$ is a compact subset of $\mathfrak{M}$ for all compact and convex sets K. It is not known whether it suffices to assume that $K \cap \mathfrak{A}$ is a compact subset of $\mathfrak{M}$ for all compact and convex sets K.

Section 9 contains various illustrations and applications of all these theorems. Most of the applications generalise results of Ranga Rao [*8*].

Section 2. Proof of Sufficiency in Theorem 2

We shall need the following lemma.

Lemma 1. *For each positive δ there exists a P-uniformity class $\mathcal{U}_\delta$ such that for each subset A of S there exist in $\mathcal{U}_\delta$ sets V and W satisfying $W \subset A \subset V$ and $V \setminus W \subset \partial_\delta A$.*

Proof. Choose about each x in S an open sphere S_x satisfying $P(\partial S_x) = 0$ and diam $S_x < \delta$. Since S is separable, it follows by Lindelöf's theorem [6] that some sequence $\{S_{x_1}, S_{x_2}, \ldots\}$ of the sets S_x covers S. If

$$U_i = S_{x_i} \cap \bigcap_{j<i} S_{x_j}^c,$$

then $\{U_1, U_2, \ldots\}$ is a finite or countable partition of S into P-continuity sets of diameter less than δ. Let $\mathscr{U}_\delta$ denote the σ-field generated by the U_i; $\mathscr{U}_\delta$ consists of the unions of the U_i.

If $P_n \Rightarrow P$, then $P_n(U_i) \to P(U_i)$ for each i, so that, by Scheffé's theorem [9],

$$\sup_{V \in \mathscr{U}_\delta} |P_n V - P V| \leqq \sum_i |P_n(U_i) - P(U_i)| \to 0$$

as $n \to \infty$. This proves that $\mathscr{U}_\delta$ is a P-uniformity class. From the fact that each U_i has diameter less than δ, it follows that there exists a set V in $\mathscr{U}_\delta$ such that $A \subset V \subset A^\delta$, which in turn implies $V \backslash A \subset \partial_\delta A$. The existence of a W in $\mathscr{U}_\delta$ with $W \subset A$ and $A \backslash W \subset \partial_\delta A$ follows by applying this argument to the set A^c. It follows that $V \backslash W \subset \partial_\delta A$ which proves the lemma.

Suppose now that (9) holds. Let $\{P_n\}$ be a sequence that converges weakly to P. Given a positive η, first choose a positive δ such that

$$\sup_{A \in \mathfrak{A}} P(\partial_\delta A) < \eta. \tag{18}$$

Let $\mathscr{U}_\delta$ satisfy the conditions of Lemma 1. Choose N so that

$$\sup_{V \in \mathscr{U}_\delta} |P_n V - P V| < \eta \tag{19}$$

holds for all $n \geqq N$. From this and Lemma 1 it follows that

$$|P_n A - PA| < \eta + P(\partial_\delta A)$$

for all $n \geqq N$ and all $A \in \mathscr{S}$. It follows by (18) that

$$\sup_{A \in \mathfrak{A}} |P_n A - PA| < 2\eta$$

for all $n \geqq N$. Since η was arbitrary, $\mathfrak{A}$ is a P-uniformity class.

Section 3. Proof of Necessity in Theorem 2

We need two lemmas.

Lemma 2. *For each positive δ and each set A in S, there exists a finite or countable, pairwise disjoint class $\{N_i\}$ of Borel sets such that $\partial_\delta A \subset \bigcup_i N_i$, such that diam $N_i < 6\delta$ for each i, and such that each N_i meets both A and A^c.*

Proof. If $\partial_\delta A = 0$, these conditions are formally fulfilled by an empty class of sets N_i. We assume $\partial_\delta A \neq 0$.

Since S is separable, it follows by Lindelöf's theorem that for some sequence $(y_1, y_2, \ldots)$ of points in $\partial_\delta A$, the spheres $S(y_j, \delta)$ cover $\partial_\delta A$. Each $S(y_j, \delta)$ meets both A and A^c.

Let $x_1 = y_1$; let x_2 be the first y_j beyond x_1 distant at least 2δ from x_1; let x_3 be the first y_j beyond x_2 distant at least 2δ from x_1 and from x_2. Continue in this way (the process may terminate). The spheres $S(x_i, \delta)$ are disjoint, and, since each y_j is within 2δ of some x_i: $\partial_\delta A \subset \bigcup_i S(x_i, 3\delta)$.

Let $B = \bigcup_i S(x_i, \delta)$ and

$$M_i = S(x_i, 3\delta) \cap B^c \cap \bigcap_{k<i} S(x_k, 3\delta)^c .$$

If $N_i = S(x_i, \delta) \cup M_i$, then the collection $\{N_i\}$ has all the properties required of it.

Lemma 3. *Let (r_i, s_i) be (finitely or infinitely many) pairs of real numbers. If $\sum_i (r_i + s_i) \geqq \varepsilon$, then either $\sum_{r_i \geqq s_i} r_i \geqq \varepsilon/4$ or $\sum_{s_i > r_i} s_i \geqq \varepsilon/4$.*

Proof. Put $t = \sum_i (r_i + s_i)$. If $\sum_{r_i \geqq s_i} r_i < t/4$ and $\sum_{s_i > r_i} s_i < t/4$, then

$$t = \sum_{r_i \geqq s_i} r_i + \sum_{s_i > r_i} s_i + \sum_{s_i > r_i} r_i + \sum_{r_i \geqq s_i} s_i < \tfrac{1}{2} t + \sum_{s_i > r_i} r_i + \sum_{r_i \geqq s_i} s_i \leqq \tfrac{1}{2} t + $$
$$+ \min\{\sum_i r_i, \sum_i s_i\} \leqq t ,$$

a contradiction.

Now suppose that (9) fails. We shall show that $\mathfrak{A}$ is not a P-uniformity class. Since (9) fails, there exists a positive ε such that for all positive δ,

$$P(\partial_\delta A_\delta) \geqq \varepsilon \tag{20}$$

holds for some A_δ in $\mathfrak{A}$. We shall use $\partial_\delta A_\delta$ to construct a probability measure P_δ very close to P.

By Lemma 2, there exist pairwise disjoint Borel sets $N_{\delta i}$ such that $\partial_\delta A_\delta \subset \subset \bigcup_i N_{\delta i}$, such that diam $N_{\delta i} < 6\delta$, and such that $N_{\delta i}$ meets A_δ in some point $x_{\delta i}$ and meets A_δ^c in some point $y_{\delta i}$. Put $r_{\delta i} = P(N_{\delta i} \cap A_\delta)$ and $s_{\delta i} = P(N_{\delta i} \cap A_\delta^c)$. Then

$$\sum_i r_{\delta i} + \sum_i s_{\delta i} = \sum_i P(N_{\delta i}) \geqq P(\partial_\delta A_\delta) \geqq \varepsilon$$

by (20). By Lemma 3, therefore, either

$$\sum_{r_{\delta i} \geqq s_{\delta i}} r_{\delta i} \geqq \varepsilon/4 \tag{21}$$

or

$$\sum_{s_{\delta i} > r_{\delta i}} s_{\delta i} \geqq \varepsilon/4 . \tag{22}$$

If (21) holds (for a particular δ), define P_δ to coincide with P outside $\bigcup_i N_{\delta i}$ and to consist in each $N_{\delta i}$ of point masses at $x_{\delta i}$ and $y_{\delta i}$ in accordance with the requirements

$$P_\delta(x_{\delta i}) = \begin{cases} 0 & \text{if } r_{\delta i} \geqq s_{\delta i} \\ r_{\delta i} & \text{if } s_{\delta i} > r_{\delta i} , \end{cases} \quad P_\delta(y_{\delta i}) = \begin{cases} r_{\delta i} + s_{\delta i} & \text{if } r_{\delta i} \geqq s_{\delta i} \\ s_{\delta i} & \text{if } s_{\delta i} > r_{\delta i} . \end{cases} \tag{23}$$

If (22) holds instead of (21), define P_δ in the same way, but with (23) replaced by

$$P_\delta(x_{\delta i}) = \begin{cases} r_{\delta i} & \text{if } r_{\delta i} \geqq s_{\delta i} \\ r_{\delta i} + s_{\delta i} & \text{if } s_{\delta i} > r_{\delta i} , \end{cases} \quad P_\delta(y_{\delta i}) = \begin{cases} s_{\delta i} & \text{if } r_{\delta i} \geqq s_{\delta i} \\ 0 & \text{if } s_{\delta i} > r_{\delta i} . \end{cases} \tag{24}$$

In either case, P_δ agrees with P outside $\bigcup_i N_i$, $P_\delta(N_{\delta i}) = P(N_{\delta i})$ for each i, and

(25) $$|P(A_\delta) - P_\delta(A_\delta)| \geqq \varepsilon/4\,.$$

Since $\operatorname{diam} N_{\delta i} < 6\delta$, it follows from this that $\lim_{\delta\to 0}\int f\,dP_\delta = \int f\,dP$ holds for every bounded, uniformly continuous real function f. Therefore (see Theorem 2.1 of [2]) $P_\delta \Rightarrow P$ as $\delta \to 0$. Because of (25), $\mathfrak{A}$ is not a P-uniformity class.

Section 4. Proof of Theorem 1

In this section we shall find it convenient to define the δ, ε-*boundary* of a function f in $\mathscr{B}(S)$ by

(26) $$\partial_{\delta,\varepsilon}(f) = \{x : \omega_f S(x,\delta) > \varepsilon\}\,.$$

The condition (8) in Theorem 1 can then be written as

(27) $$\lim_{\delta\to 0}\ \sup_{f\in\mathscr{F}} P(\partial_{\delta,\varepsilon}(f)) = 0\,.$$

First, let us prove the sufficiency in Theorem 1. This is done in a way analogous to the sufficiency proof in Section 2. The fact corresponding to Lemma 1 is obtained as follows. Let δ be positive and C a finite positive constant. As before let $\{U_i\}$ be a finite or countably infinite decomposition of S into P-continuity sets of diameter less than δ. Let $\mathscr{F}_{\delta,C}$ denote the class of functions of the form $\sum \alpha_i \chi_{U_i}$ with all the α's real numbers bounded in modulus by C; here χ_{U_i} denotes the characteristic function of U_i. Then it is easy to see that $\mathscr{F}_{\delta,C}$ is a P-uniformity class. If f is any function in $\mathscr{B}(S)$ such that $|f(x)| \leqq C$ for all x in S then there exist two functions h and g, both in $\mathscr{F}_{\delta,C}$ such that $h \leqq f \leqq g$ and such that $\int (g-h)\,dP \leqq \varepsilon + 2C\,P(\partial_{\delta,\varepsilon}(f))$ holds for all positive ε.

Assume now that $\mathscr{F}$ satisfies (7) and (8). By subtracting suitable constants from the functions in $\mathscr{F}$, we see, by condition (7), that we may assume that $|f(x)| \leqq C$ holds for all x in S and all f in $\mathscr{F}$, with C a finite constant. By the remarks above, it is easy to complete the proof.

Now let us prove the necessity of the boundedness condition (7). Assume that (7) does not hold. Then for every positive integer n, there exists a function f_n in $\mathscr{F}$ with $\omega_{f_n}(S) > n$. Let

$$\alpha_n = \inf\{f_n(x) : x\in S\} \quad\text{and}\quad \beta_n = \sup\{f_n(x) : x \in S\}\,.$$

Then $\beta_n - \alpha_n > n$. Divide the closed interval $[\alpha_n, \beta_n]$ in n disjoint intervals of equal length. For one of these intervals, say for I_n, $Pf_n^{-1}(I_n) \geqq 1/n$. Clearly, there exists a point x_n in S such that

(28) $$|f_n(x_n) - t| \geqq (n-1)/2$$

for all t in I_n^-. Let Q_n be the signed measure that agrees with $-(n\,P(f_n^{-1}(I_n)))^{-1}\cdot P$ on the set $f_n^{-1}(I_n)$ and vanishes outside $f_n^{-1}(I_n)$ except at the point x_n, where Q_n has the mass $1/n$. It is easy to see that $P_n = P + Q_n$ is a probability measure. Also,

$$\left|\int f_n\,dP_n - \int f_n\,dP\right| = \left|\int f_n\,dQ_n\right| = 1/n\,|f_n(x_n) - t|$$

for some t in I_n^-. Hence, by (28),

$$\left|\int f_n\,dP_n - \int f_n\,dP\right| \geqq 1/2 - 1/2n\,,$$

and since $\{P_n\}$ clearly converges weakly to P, $\mathscr{F}$ cannot be a P-uniformity class. This proves necessity of condition (7).

Lastly we want to prove the necessity of (27). This proof is very much analogous to the proof of necessity in Theorem 2. Instead of Lemma 2, we now use the fact that for each f in $\mathscr{B}(S)$, and for each pair δ, ε of positive numbers there exists a finite or countable, pairwise disjoint class $\{N_i\}$ of Borel sets such that $\partial_{\delta,\varepsilon}(f) \subset \subset \bigcup_i N_i$, such that diam $N_i < 6\delta$ for each i, and such that $\omega_f(N_i) > \varepsilon$ for each i. This result is proved in the same way as Lemma 2.

Now suppose that (27) fails. Then there exists a pair ε, η of positive numbers such that for all positive δ there exists a function f_δ in $\mathscr{F}$ with $P(\partial_{\delta,\varepsilon}(f_\delta)) \geqq \eta$. Construct the class $\{N_{\delta,i}\}$ corresponding to the set $\partial_{\delta,\varepsilon}(f_\delta)$. Then

$$\sum_i P(N_{\delta,i})\, \omega_{f_\delta}(N_{\delta,i}) \geqq \eta\, \varepsilon$$

and we can use Lemma 3 with

$$r_{\delta,i} = P(N_{\delta,i}) \cdot \sup\{f_\delta(x) : x \in N_{\delta,i}\} - \int_{N_{\delta,i}} f_\delta\, dP$$

and

$$s_{\delta,i} = \int_{N_{\delta,i}} f_\delta\, dP - P(N_{\delta,i}) \cdot \inf\{f_\delta(x) : x \in N_{\delta,i}\}.$$

The proof continues along the same lines as the proof in Section 3; the details are left to the reader.

Section 5. Proof of Theorem 3

Because of (10), we need only show that if S is locally connected, then (11) implies (9). Given a positive ε, choose a positive η such that

$$\sup_{A \in \mathfrak{A}} P((\partial A)^\eta) < \varepsilon. \tag{29}$$

By local connectedness and separability, there exist finitely or countably many connected sets E_i such that $S \subset \bigcup_i E_i^0$ and such that diam $E_i < \eta$. Let

$$D_{i\delta} = \{x : x \in E_i^0,\ \varrho(x, E_i^c) \geqq \delta\}.$$

For fixed i, $D_{i\delta} \uparrow E_i^0$ as $\delta \downarrow 0$.

Choose an integer i_0 so large that $P(S - \bigcup_{i \leqq i_0} E_i^0) < \varepsilon$, and then choose a positive δ so small that $P(E_i^0 - D_{i\delta}) < \varepsilon / i_0$ for $i \leqq i_0$. Then

$$P(S - \bigcup_i D_{i\delta}) < 2\varepsilon. \tag{30}$$

Let us prove that

$$D_{i\delta} \cap \partial_\delta A \subset (\partial A)^\eta \tag{31}$$

for every i and every A. If $x \in \partial_\delta A$, then $\varrho(x, y) < \delta$ and $\varrho(x, z) < \delta$ for some y in A and z in A^c. If $x \in D_{i\delta}$, then y and z lie in E_i, so that E_i meets both A and A^c. Since E_i is connected, it meets ∂A in some point w. Since x and w both lie in E_i, and since diam $E_i < \eta$, we have $\varrho(x, \partial A) \leqq \varrho(x, w) < \eta$, or $x \in (\partial A)^\eta$. This proves (31).

From (30) and (31) we conclude that $P(\partial_\delta A) \leqq 2\varepsilon + P((\partial A)^\eta)$. If $A \in \mathfrak{A}$, then it follows further by (29) that $P(\partial_\delta A) < 3\varepsilon$ holds for the δ chosen, and hence holds for all smaller δ. Thus (9) holds, which completes the proof of Theorem 3.

Section 6. Proof of Theorem 4

The essential point of Theorem 4 is contained in the following lemma, which does not require the assumption of local connectedness.

Lemma 4. *Suppose that $\mathfrak{M}_0$ is a compact subset of $\mathfrak{M}$. Then, for any probability measure P,*

$$\lim_{\delta\to 0} \sup_{M\in\mathfrak{M}_0} P(M^\delta) = \sup_{M\in\mathfrak{M}_0} P(M). \tag{32}$$

In particular, if $P(M) = 0$ for each M in $\mathfrak{M}_0$, then

$$\lim_{\delta\to 0} \sup_{M\in\mathfrak{M}_0} P(M^\delta) = 0. \tag{33}$$

Proof. Let η be a fixed positive number. For each M in $\mathfrak{M}_0$ choose δ_M such that $P(M^{2\delta_M}) < P(M) + \eta$. Since $\mathfrak{M}_0$ is compact there exist finitely many sets $M_1, \ldots, M_\nu$ in $\mathfrak{M}_0$ such that, for all M in $\mathfrak{M}_0$, $\Delta(M, M_i) < \delta_{M_i} = \delta_i$ for some $i = 1, \ldots, \nu$. Put $\delta = \min\{\delta_{M_1}, \ldots, \delta_{M_\nu}\}$. Then it is easy to see that for each M in $\mathfrak{M}_0$, $M^\delta \subset M_i^{2\delta_i}$ for some $i = 1, \ldots, \nu$. It follows that

$$\sup_{M\in\mathfrak{M}_0} P(M^\delta) \leqq \sup_{M\in\mathfrak{M}_0} P(M) + \eta.$$

Suppose $\mathfrak{A}$ is a P-continuity class. It follows immediately from Lemma 4 that condition (i) of Theorem 4 implies (11).

Consider condition (ii) of Theorem 4. Given η, choose k so that $P(B_k^0) > 1 - \eta$. Since

$$\partial A \subset [\partial(B_k \cap A)] \cup [S - B_k]^-, \tag{34}$$

we have

$$(\partial A)^\delta \subset [\partial(B_k \cap A)]^\delta \cup [S - B_k]^\delta, \tag{35}$$

so that

$$P((\partial A)^\delta) \leqq P([\partial(B_k \cap A)]^\delta) + P([S - B_k]^\delta). \tag{36}$$

Since $\partial(B_k \cap A) \subset \partial(B_k) \cup \partial(A) \subset [S - B_k^0] \cup \partial(A)$ we have

$$P(\partial(B_k \cap A)) < \eta \tag{37}$$

for all $A \in \mathfrak{A}$. Hence, for small enough δ, we have, by Lemma 4, $P([\partial(B_k \cap A)]^\delta) < 2\eta$ for all $A \in \mathfrak{A}$; and for small enough δ we have

$$P([S - B_k]^\delta) < P([S - B_k]^-) + \eta = P(S - B_k^0) + \eta < 2\eta.$$

Therefore, by (36), (11) holds.

The same sort of argument shows that condition (iii) of Theorem 4 implies (11). We need only replace (34) by

$$\partial A \subset [B_k \cap \partial A] \cup [S - B_k] \tag{38}$$

and continue as before.

Each of the three sets of hypotheses in Theorem 4 thus implies (11) (and this is true even if S is not locally connected). As remarked after the statement of the theorem, this suffices for its proof.

Section 7. Proof of Theorem 5

By the corollary to Theorem 3, it suffices to show that $\partial\mathfrak{A}$ is a P-uniformity class. Assuming $P_n \Rightarrow P$, we must show that

$$\lim_{n\to\infty} \sup_{A\in\mathfrak{A}} P_n(\partial A) = 0. \tag{39}$$

The proof of (39) does not use the local connectivity of S. Let η be positive. Since $P_n \Rightarrow P$ and since S is separable and complete there exists a compact set K such that $P_n(K) > 1 - \eta$ for all n ([7], Lemma I.3). Choose K^* according to the conditions in Theorem 5. Then

$$P_n(\partial A) \leqq \eta + P_n(K^* \cap \partial A) \tag{40}$$

for all n. By an argument familiar to us from the proof of Theorem 4, there exists a $\delta > 0$ such that

$$P((K^* \cap \partial A)^\delta) < \eta \tag{41}$$

for all $A \in \mathfrak{A}$.

Now let $\mathscr{U}_\delta$ be a P-uniformity class as constructed in Lemma 1. Choose N so that

$$\sup_{V\in\mathscr{U}_\delta} |P_n V - P V| < \eta \tag{42}$$

holds for all $n \geqq N$. Let A be any set in $\mathfrak{A}$. By Lemma 1 we can choose V in $\mathscr{U}_\delta$ so that $K^* \cap \partial A \subset V \subset (K^* \cap \partial A)^\delta$. Then

$$P_n(K^* \cap \partial A) \leqq P_n V \leqq |P_n V - P V| + P((K^* \cap \partial A)^\delta).$$

By (40), (41), and (42) it follows that

$$\sup_{A\in\mathfrak{A}} P_n(\partial A) < 3\eta$$

holds for all $n \geqq N$. This proves that $\partial\mathfrak{A}$ is a P-uniformity class.

Section 8. Proof of Theorem 6

Let $\mathfrak{A}$ be a P-continuity class of closed convex sets in the (separable) Banach space S. Suppose at first that $\mathfrak{A}$ is a compact subset of $\mathfrak{M}$. Since S is locally connected, Theorem 4 applies. Condition (i) of that theorem is fulfilled if we can show that for each compact subset $\mathfrak{A}$ of $\mathfrak{M}$ consisting entirely of convex sets, $\partial\mathfrak{A}$ is also a compact subset of $\mathfrak{M}$. This will easily follow if we can show that for each pair of bounded, closed, non-empty convex sets A and B we have $\Delta(A, B) \geqq \Delta(\partial A, \partial B)$; actually we shall prove that for such sets equality holds:

$$\Delta(A, B) = \Delta(\partial A, \partial B). \tag{43}$$

The inequality $\Delta(A, B) \leqq \Delta(\partial A, \partial B)$ follows easily from the fact that, since $A[B]$ is bounded, each point of $A[B]$ is a convex combination of two points of $\partial A[\partial B]$.

To prove the reverse inequality we shall show that if $A \subset B^\lambda$ and if $B \subset A^\lambda$ then the relations $\partial A \subset \partial B^{\lambda+\delta}$ and $\partial B \subset \partial A^{\lambda+\delta}$ hold for every positive δ; obviously we need only prove the first relation. Suppose $x \in \partial A$. Since $A \backslash B \subset \partial B^\lambda$, we may assume that $x \in B$. Consider a point y such that $y \notin A$ and $y \in S(x, \delta/2)$. There exists (see V.2.12 of [3]) a linear functional f of norm 1 and a real α such that $\operatorname{Re} f(y) = \alpha$ and such that $\operatorname{Re} f(z) \leqq \alpha$ for all $z \in A$. Choose θ so that $\lambda < \theta < \lambda + \delta/2$, and consider $y + \theta w$, where w is an element of S of norm 1 satisfying $f(w) > \lambda/\theta$. Since $\operatorname{Re} f(y + \theta w) > \alpha + \lambda$, $y + \theta w$ cannot lie in A^{λ}. Hence $y + \theta w$ cannot lie in B. Since $x \in B$, there must be a point z of ∂B on the segment from x to $y + \theta w$. We have $z \in S(x, \lambda + \delta)$, and the inclusion $\partial A \subset \partial B^{\lambda+\delta}$ follows.

The equation (43), together with the fact that the bounded closed convex subsets of S constitute a closed subset of $\mathfrak{M}$, implies that a class $\mathfrak{A}$ consisting of bounded, closed, convex subsets of S is a closed subset of $\mathfrak{M}$ if and only if $\partial\mathfrak{A}$ is a closed subset of $\mathfrak{M}$; also $\mathfrak{A}$ is a compact subset of $\mathfrak{M}$ if and only if $\partial\mathfrak{A}$ is a compact subset of $\mathfrak{M}$. Thus condition (i) of Theorem 4 holds if $\mathfrak{A}$ is compact in $\mathfrak{M}$. If the class $B \cap \mathfrak{A}$ is compact in $\mathfrak{M}$ for each closed sphere B, then it is easy to see that condition (ii) of Theorem 4 holds.

It is perhaps worth while to point out that, if the class $\mathfrak{A}$ in Theorem 6 consists of convex sets with non-empty interior, then we need not assume that the sets in $\mathfrak{A}$ are closed, since then $\mathfrak{A}$ will be a P-uniformity class if and only if the class

(44) $$\mathfrak{A}^- = \{A^- : A \in \mathfrak{A}\}$$

is a P-uniformity class. To see this note that for convex sets A with $A^0 \neq 0$ we have $\partial(A) = \partial(A^-)$.

Section 9. Examples and Applications

The first two examples illustrate various points of the theory; the remaining ones are applications to concrete cases of interest.

Example 1. Let S be the space of sequences $x = (x_1, x_2, \ldots)$ of 0's and 1's, metrized by $\varrho(x, y) = \sum_i |x_i - y_i| \, 2^{-i}$ (product topology); S is completely disconnected. Let $\mathfrak{A}$ be the class of sets of the form

(45) $$A = \{x : x_1 = u_1, \ldots, x_k = u_k\},$$

with $(u_1, \ldots, u_k)$ varying over all finite sequences of 0's and 1's. Since $\partial A = 0$ for each A in $\mathfrak{A}$, (11) holds no matter what P is.

It is not hard to show that the δ-boundary of (45) is given by

(46) $$\partial_\delta A = \begin{cases} 0 & \text{if } 2^{-k} \geqq \delta, \\ \left\{y : \sum_{i=1}^{k} |y_i - u_i| \, 2^{-i} < \delta\right\} & \text{if } 2^{-k} < \delta, \end{cases}$$

and that therefore, by Theorem 2, $\mathfrak{A}$ is a P-uniformity class if and only if for all η there exists a δ such that

(47) $$P\left\{y : \sum_{i=1}^{k} |y_i - u_i| \, 2^{-i} < \delta\right\} < \eta$$

holds for all sequences $u_1, \ldots, u_k$ of length $k > -\log_2 \delta$.

If P has a mass of η at some point z, then (47) is violated with $u_i = z_i$, $i \leqq k$, so that $\mathfrak{A}$ is not a P-uniformity class. Let $A_k(y)$ be that set of the form (45) that contains y. If P has no point masses, then for each y, $P(A_k(y))$ converges monotonically to 0 as $k \to \infty$. Since $P(A_k(y))$ is continuous in y for each k, and since S is compact, the convergence is uniform. It follows via the condition (47) that $\mathfrak{A}$ is a P-uniformity class. (To see this, note that

$$\left\{y : \sum_{i=1}^{k} |y_i - u_i|\, 2^{-i} < \delta\right\} \subset A_{k'}(u) \quad \text{with} \quad k' = [-\log_2 \delta],$$

the integer part of $-\log_2 \delta$). Thus $\mathfrak{A}$ is a P-uniformity class if and only if P has no point masses.

Since $\partial\mathfrak{A}$ consists of the empty set alone, we see that if S is not connected, then (11) need not imply (9), the sufficiency condition in Theorem 3 fails, and Theorem 4 fails.

Example 2. If S is a countable, discrete space, then $\mathscr{S}$ itself is a P-uniformity class for every P: By the remark following the statement of Theorem 2, we may work with any metric that generates the topology; if we take the distance between distinct points to be 1, then $\partial_\delta A = 0$ for all A if $\delta < 1$, so that (9) always holds.

Example 3. Let X be a (real or complex) separable Banach space and denote by $\mathscr{B}(S, X)$ the class of all bounded measurable functions mapping S into X. Since X is separable, $\mathscr{B}(S, X)$ is well defined (see III. 6.11 of [3]). We can then define P-uniformity etc. for a subclass $\mathscr{F}$ of $\mathscr{B}(S, X)$. By Λ we denote the class of all continuous linear functionals on X with norm at most 1. $\Lambda\mathscr{F}$ is then the class of scalar valued functions on S of the form $\varphi(f)$ with φ in Λ and f in $\mathscr{F}$. Since

$$\sup_{f\in\mathscr{F}} \left\| \int f\, dP_n - \int f\, dP \right\| = \sup_{\varphi(f)\in\Lambda\mathscr{F}} \left| \int \varphi(f)\, dP_n - \int \varphi(f)\, dP \right|, \tag{48}$$

we find that $\mathscr{F}$ is a P-uniformity class if and only if $\Lambda\mathscr{F}$ is a P-uniformity class. Therefore by Theorem 1 and the relation $\omega_{\Lambda\mathscr{F}}(S) = \omega_{\mathscr{F}}(S)$ it follows that the two conditions

$$\omega_{\mathscr{F}}(S) < \infty \tag{49}$$

and

$$\lim_{\delta\to 0} \sup_{\varphi(f)\in\Lambda\mathscr{F}} P(\partial_{\delta,\varepsilon}(\varphi(f))) = 0 \tag{50}$$

for all $\varepsilon > 0$ are necessary and sufficient that $\mathscr{F}$ be a P-uniformity class. Obviously (50) is implied by

$$\lim_{\delta\to 0} \sup_{f\in\mathscr{F}} P(\partial_{\delta,\varepsilon}(f)) = 0. \tag{51}$$

Thus (49) and (51) for all $\varepsilon > 0$ are sufficient for $\mathscr{F}$ to be a P-uniformity class. This result is not surprising since the sufficiency proof in Section 4 of Theorem 1 applies equally well to Banach-space-valued functions. The sufficiency of (49) and (51) tells us that the weak convergence $P_n \Rightarrow P$ implies that $\int f\, dP_n \to \int f\, dP$ for each f in $\mathscr{B}(S, X)$ for which f is continuous except on a set of P measure 0. We do not know if it is still true that P-uniformity implies P-continuity.

Example 4. If P is a unit mass at x_0 and if $\mathscr{F} \subset \mathscr{B}(S)$ is a P-uniformity class, then it is easy to see that $\mathscr{F}$ is equicontinuous at x_0; that is, for each positive ε there exists a positive δ such that $\omega_{\mathscr{F}} S(x_0, \delta) < \varepsilon$. Thus a necessary condition that $\mathscr{F} \subset \mathscr{B}(S)$ be a P-uniformity class for *all* P is that $\mathscr{F}$ be equicontinuous on S and $\omega_{\mathscr{F}}(S) < \infty$. We shall now show that these conditions are also sufficient. The only thing to show is that (8) holds for all P. Let the probability measure P and the pair of positive numbers ε, η be given. Choose, for each x in S, a positive δ_x such that $\omega_{\mathscr{F}} S(x, 2\delta_x) \leqq \varepsilon$, or, what is the same, such that $\omega_f S(x, 2\delta_x) \leqq \varepsilon$ for all f in $\mathscr{F}$. It follows (in the notation of Section 4) that $S(x, \delta_x) \subset (\partial_{\delta_x,\varepsilon}(f))^c$ for all f in $\mathscr{F}$. By Lindelöf's theorem it follows that $S = \bigcup_i S(x_i, \delta_{x_i})$ for some sequence of points $\{x_i\}$. Now choose N so large that $P\left(\bigcup_{i=1}^{N} S(x_i, \delta_{x_i})\right) > 1 - \eta$ and put $\delta = \min\{\delta_{x_1}, \ldots, \delta_{x_N}\}$. It is then easy to see that $\bigcup_{i=1}^{N} S(x_i, \delta_{x_i}) \subset (\partial_{\delta,\varepsilon}(f))^c$ for all f in $\mathscr{F}$. Hence $P(\partial_{\delta,\varepsilon}(f)) < \eta$ for all f in $\mathscr{F}$. This argument shows that (8) holds for all P and, since (7) holds by hypothesis, we have proved the desired result. The paper [*8*] by Ranga Rao also contains this result; in fact Ranga Rao gives a nice direct proof.

Example 5. In this example we shall study mappings preserving weak convergence. Let S and S' be metric spaces, h a measurable mapping from S into S', and P a probability measure on S. We shall say that the pair (h, P) *preserves weak convergence* if the sequence $\{P_n h^{-1}\}$ converges weakly to Ph^{-1} in S' for every sequence $\{P_n\}$ converging weakly to P in S. As usual we shall assume that the space S is separable.

From the definition of weak convergence we find, by transforming integrals over S' into integrals over S, that $P_n h^{-1} \Rightarrow Ph^{-1}$ if and only if $\int g(h)\, dP_n \to \to \int g(h) dP$ holds for every bounded, continuous function g mapping S' into the reals. It therefore follows by our results on P-uniformity (applied to a family $\mathscr{F}$ consisting of a single function) that (h, P) preserves weak convergence if and only if the composite function $g(h)$ is a P-continuity function for every bounded, continuous g mapping S' into the reals.

Thus a sufficient condition for (h, P) to preserve weak convergence is that h be a P-continuity function. In fact, it is fairly easy to prove directly that this result holds even without separability of S.

We shall now prove the necessity, assuming the space S' is separable; that is, we shall prove that h must be a P-continuity function if (h, P) preserves weak convergence and if S' is separable. In case S' is the real line R, this result follows from the above remarks. It is then easy to prove the result for $S' = R^\infty$, a countable product of copies of R, and the final step from R^∞ to the general separable metric space S' follows from the fact that any such space is homeomorphic to a subset of R^∞.

Lastly we shall prove that necessity holds without any assumptions on S', provided that no uncountable cardinal $\varkappa$ satisfies $2^\varkappa = 2^{\aleph_0}$ (a condition strictly weaker than the continuum hypothesis, see Theorem 3 of [*11*]). We shall prove this by showing that the image $h(S)$ of h is separable. If this were not so then there would exist a positive ε and a family $\{y_\alpha\}_{\alpha \in I}$ of points in $h(S)$ such that the cardinality $\varkappa$ of the index set I is greater than $\aleph_0$ and such that $\varrho(y_\alpha, y_\beta) > \varepsilon$ for $\alpha \neq \beta$. Any union of y_α's is closed, and hence a Borel set, and it follows that $\bigcup\{h^{-1}(y_\alpha) : \alpha \in I^*\}$ is a Borel set for every subset I^* of I. We have thus found

$2^\varkappa$ distinct Borel sets in S. Since there are at most $2^{\aleph_0}$-Borel sets in S and since $\varkappa > \aleph_0$ we arrive at a contradiction.

Example 6. We shall examine the content of Theorem 6 in an Euclidean space. In this example, $\mathfrak{A}$ will denote the class of all measurable, convex subsets of R^k. In Euclidean space the boundary of a convex set coincides with the boundary of its closure. Thus, by Theorem 3, a subclass $\mathfrak{A}_0$ of $\mathfrak{A}$ is a P-uniformity class if and only if the class

$$\mathfrak{A}_0^- = \{A^- : A \in \mathfrak{A}_0\} \tag{52}$$

is a P-uniformity class. Combining this with Theorem 6 we obtain:

A subclass $\mathfrak{A}_0$ of $\mathfrak{A}$ is a P-uniformity class if $\mathfrak{A}_0$ is a P-continuity class and if the class

$$B \cap \mathfrak{A}_0^- = \{B \cap A^- : A \in \mathfrak{A}_0\} \tag{53}$$

is a compact subset of $\mathfrak{M}$ for each closed sphere B.

Since a closed sphere is compact, $B \cap \mathfrak{A}_0^-$ is a compact subset of $\mathfrak{M}$ if and only if it is a closed subset of $\mathfrak{M}$. In particular, we see that the class $\mathfrak{A}$ itself is a P-uniformity class if it is a P-continuity class. This result was proved in a different way by RANGA RAO [8]. (The fact that $B \cap \mathfrak{A}^-$ is compact is the familiar selection theorem of BLASCHKE [4]). Taking $\mathfrak{A}_0$ to consist of sets of the form $\{x : x_i \leqq a_i, i = 1, \ldots, k\}$ with a ranging over a closed subset of R^k, we obtain the classical fact that if k-dimensional distribution functions F_n converge to a k-dimensional distribution function F at continuity points of F, then the convergence is uniform over each closed set of continuity points. Taking $\mathfrak{A}_0$ to consist of the convex polyhedra with at most m faces, we obtain another result due to RANGA RAO [8]. Or we may take $\mathfrak{A}_0$ to consist of spheres—examples may be multiplied at will.

Example 7. In the plane R^2, consider the class of rectifiable arcs of length at most l, and let $\mathfrak{A}$ be the class of measurable planar sets with such curves as boundaries.

Let ∂A be a rectifiable arc of length at most l. For each positive δ, there exists along the curve a succession of points $z_0, z_1, \ldots, z_k$ such that z_0 and z_k are the endpoints, such that $\varrho(z_{i-1}, z_i) \leqq \delta$, $1 \leqq i \leqq k$, and such that $k - 1 \leqq l/\delta$. Since the $k + 1$ spheres $S(z_i, 2\delta)$ cover $(\partial A)^\delta$, the Lebesgue measure of $(\partial A)^\delta$ is at most $4\pi\delta^2((l/\delta) + 2)$. It follows that if P is absolutely continuous with respect to Lebesgue measure, then (11) holds, so that $\mathfrak{A}$ is a P-uniformity class. (Since the convex subsets of a bounded planar set have perimeters of bounded length, these ideas afford another approach to Example 6 in case k is 2 and P is absolutely continuous with respect to Lebesgue measure.)

This result fails if there is no upper bound to the lengths of the arcs ∂A. Take P arbitrary and take B to be a disc with $P(B) > 0$. For every δ there exists a (long) rectifiable arc ∂A that comes within δ of each point of B, so that $(\partial A)^\delta \supset B$. If $\partial\mathfrak{A}$ contains all rectifiable arcs, therefore, the condition (11) (equivalent to (9), since R^k is locally connected) cannot hold.

Example 8. Let S be the space C of continuous functions $x = x(t)$ on $[0, 1]$, with the uniform topology. Let V be a subset of C with the following property: For $r > 0$ and $x \in C$, let $\tau_r(x)$ be that element of C whose value at t is $x(t)$ if

$|x(t)| \leqq r$, r if $x(t) > r$, and $-r$ if $x(t) < -r$. Let $V_r = \{\tau_r(x) : x \in V\}$. We demand that V_r be a compact subset of C for each $r > 0$, which we may express by saying that V is *truncation-compact*. (Note that if V is compact, then it is truncation-compact. Also, if V is truncation-compact, then $V \cap B$ is compact for each closed sphere B, although the converse to this is false.)

Let $\mathfrak{A}$ consist of all sets of the form

$$A = \{x : f(t) \leqq x(t) \leqq g(t), \quad 0 \leqq t \leqq 1\}, \tag{54}$$

with f and g ranging over a fixed set V. We shall prove the following result.

The class $\mathfrak{A}$ *is a* P*-uniformity class if* (i) P *is absolutely continuous with respect to Wiener measure,* (ii) V *is truncation-compact, and* (iii) *for each* f *in* V, *either* $f(0) \neq 0$ *or else there exist positive* ε *and* δ *(depending on* f*) such that*

$$|f(t)| \leqq (1-\varepsilon)(2t \log\log t^{-1})^{1/2} \tag{55}$$

for $0 < t < \delta$.

If B_r is the closed sphere of radius r about the origin, then $B_r \cap \mathfrak{A}$ consists of the sets (54) with f and g in V_r. Since V_r is compact, it follows easily that $B_r \cap \mathfrak{A}$ is a compact subset of $\mathfrak{M}$. Now the elements of $\mathfrak{A}$ are convex, and for each closed sphere B we have $B \subset B_r$ for large r. Therefore Theorem 6 applies: If $\mathfrak{A}$ is a P-continuity class, then it is a P-uniformity class. (We have thus far used only the assumption that V_r is truncation-compact.)

There remains the question of when $\mathfrak{A}$ is a P-continuity class. If P is absolutely continuous with respect to Wiener measure W, then $\mathfrak{A}$ is a P-continuity class if it is a W-continuity class; we shall deduce from assumption (iii) above that $\mathfrak{A}$ is a W-continuity class.

For f in C, let $H_f[H_f']$ consist of those x in C for which $x(t) \leqq f(t)$ $[x(t) \geqq f(t)]$ holds for all $t > 0$ and for which $x(t) = f(t)$ holds for some $t > 0$; let $J_f[J_f']$ consist of those x in C for which $x(t) < f(t)$ $[x(t) > f(t)]$ holds for all $t > 0$ and for which $x(0) = f(0)$. The boundary of (54) satisfies

$$\partial A \subset H_f' \cup J_f' \cup H_g \cup J_g, \tag{56}$$

and hence $\mathfrak{A}$ will be a W-continuity class if

$$W(H_f) = W(H_f') = W(J_f) = W(J_f') = 0 \tag{57}$$

for all f in V.

Let us prove $W(H_f) = 0$. For $\delta > 0$, let $H_{f,\delta}$ be the set of x for which $x(t) \leqq f(t)$ for all $t \geqq \delta$ and for which $x(t) = f(t)$ for some $t \geqq \delta$. Since $H_f \subset \bigcup_{\delta > 0} H_{f,\delta}$, it will suffice to show that $W(H_{f,\delta}) = 0$ for all positive δ. But

$$\begin{aligned} W(H_{f,\delta}) &= (2\pi\delta)^{-1/2} \int_{-\infty}^{f(\delta)} W\{H_{f,\delta} \,\|\, x(\delta) = u\} \exp(-u^2/2\delta)\, du \\ &= (2\pi\delta)^{-1/2} \int_{-\infty}^{f(\delta)} W\{H_{f-u,\delta} \,\|\, x(\delta) = 0\} \exp(-u^2/2\delta)\, du, \end{aligned}$$

and if this integral were positive, then $W\{H_{f-u,\delta} \| x_\delta = 0\}$ would be positive for uncountably many values of u, in copious contradiction to the fact that the $H_{f-u,\delta}$ for distinct u are disjoint. This proves $W(H_f) = 0$, and $W(H_f') = 0$ follows by the symmetric argument.

Therefore $\mathfrak{A}$ is a W-continuity class if

$$W(J_f) = W(J_f') = 0 \tag{58}$$

for all f in V. If $f(0) \neq 0$, then certainly (58) holds. Suppose that $f(0) = 0$, and consider $W(J_f)$. By Khinchine's local law of the iterated logarithm (see p. 33, of [5]), it follows that, for x in a set of Wiener measure 1, $x(t)$ exceds the right side of (55) for values of t arbitrarily close to 0, so that $W(J_f) = 0$ if $f \in V$. The symmetric analysis applies to $W(J_f)$.

This completes the proof of the italicized statement above, which generalizes Theorem 5.1 of RANGA RAO [8]. The result can be strengthened a little by using Kolmogorov's test (see [5]) in place of (55). Also, the local law of the iterated logarithm can be used in the opposite direction to construct an A of the form (54) with $W(\partial A) > 0$: take $-f(t) = g(t) = (3t \log \log t^{-1})^{1/2}$.

By the remark in Section 1 following the statement of Theorem 6, it seems plausible that the result even holds if we relax the condition (ii) that V be truncation-compact to the condition that V be truncation-closed.

In closing, we may note those results in the paper that do not require the overall hypothesis of separability. Separability is not required to prove that (9) implies (11), to prove (via (12)) that (11) implies (9) if all the spheres are connected, or to prove Lemma 4.

The proof of (43) is due in part to D. G. KENDALL; C. A. ROGERS suggested investigating $\partial_\delta A$ in place of $(\partial A)^\delta$ at an early stage of the work when Theorem 1 was not available to us; and D. H. FREMLIN pointed out to us the cardinality argument involved in Example 5.

The second author intends to publish another paper on uniformity shortly.

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