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On the Strong Comparison Theorems for Solutions of Stochastic Differential Equations

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Introduction

Comparison problems of solutions to stochastic differential equations (SDE) have been discussed first by Skorohod [13] with the idea of applying them to the uniqueness problem of solutions to SDE.

In the last ten years, several authors have produced comparison theorems or applications there of to some control problems, to pathwise uniqueness and to the explosion or recurrence problems of solutions of SDE, as well as to the study of diffusion process on a Riemannian manifold. These include Anderson [1], Yamada [16], Bonami-Karoui-Roynette-Reinhard [2], Debiard-Gaveau-Mazet [3], Ikeda-Watanabe [6, 7], Doss [4], Doss-Lenglart [5], Malliavin [11], O'Brien [12], Kesten-Ogura [9] and Takeuchi [14].

A common feature of these comparison theorems, except that of [5][1], is that two solutions of SDE, $x_1(t)$ and $x_2(t)$ are compared in the form; $x_1(t) \leqq x_2(t)$ a.s. That is to say comparison in the weak sense.

The purpose of the present paper is to give some non-contact or strong comparison theorems for solutions of SDE. In § 1, we will discuss the non-contact property of solutions of the same one-dimensional SDE. In § 2, the same problem will be treated in the multi-dimensional case. As an application, we will show in § 3 that solutions of one dimensional SDE can be interpreted as homeomorphisms on R under local Lipschitz condition for coefficients.

In § 4, we will discuss a strong comparison theorem for solutions of two SDE for which drift coefficients are strongly ordered but with the same diffusion coefficient.

§ 1. Non-contact Property of Solution of SDE; One Dimensional Case

In this section, we will discuss the non-contact or strong comparison problem of solutions of one SDE but with different initial conditions in one-dimensional case.

[1] Doss-Lenglart [5] gave a strong comparison theorem under the C^2-continuity of diffusion coefficient

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Theorem 1.1. *Suppose we are given the following;*

(i) *a real continuous function $\sigma(t,x)$ defined on $[0,\infty)\times R$ such that*

$$|\sigma(t,x)-\sigma(t,y)|\leqq\rho(|x-y|),\qquad x,y\in R,\ t\geqq 0, \tag{1.1}$$

where $\rho(u)$ is a continuous increasing function defined on $[0,\infty)$ such that $\rho(0)=0$ and

$$\int_0^1\int_x^1\frac{dy}{\rho^2(y)}dx=\int_0^1\frac{y}{\rho^2(y)}dy=+\infty;\quad and \tag{1.2}$$

(ii) *a real continuous function $b(t,x)$ defined on $[0,\infty)\times R$ such that*

$$|b(t,x)-b(t,y)|\leqq\kappa(|x-y|),\qquad x,y\in R,\ t\geqq 0, \tag{1.3}$$

where $\kappa(u)$ is a continuous increasing function defined on $[0,\infty)$ such that $\kappa(0)=0$ and

$$\lim_{x\downarrow 0}\left[\sup_{x\leqq y\leqq 1}\kappa(y)\int_y^1\frac{du}{\rho^2(u)}\Big/\int_x^1\int_y^1\frac{du}{\rho^2(u)}dy\right]=0. \tag{1.4}$$

Let $(\Omega,\mathfrak{F},P;\mathfrak{F}_t)$ be a probability space with right continuous increasing family $(\mathfrak{F}_t)_{t\geqq 0}$ of sub-σ-fields of $\mathfrak{F}$, each containing all P-null sets and suppose we are given the following processes and random variables defined on it;

(i) *two $\mathfrak{F}_0$ measurable random variables ξ and η;*

(ii) *two $\mathfrak{F}_t$ adapted continuous processes $x(t,\xi)$ and $x(t,\eta)$;*

(iii) *a one-dimensional $\mathfrak{F}_t$-Brownian motion $B(t)$ such that $B(0)=0$.*

We assume that they satisfy the following conditions;

$$\begin{aligned}x(t,\xi)=\xi&+\int_0^t\sigma(s,x(s,\xi))\,dB(s)\\&+\int_0^t b(s,x(s,\xi))\,ds,\quad a.s.\ on\quad 0\leqq t<\zeta_1,\end{aligned} \tag{1.5}$$

$$\begin{aligned}x(t,\eta)=\eta&+\int_0^t\sigma(s,x(s,\eta))\,dB(s)\\&+\int_0^t b(s,x(s,\eta))\,ds,\quad a.s.\ on\quad 0\leqq t<\zeta_2,\end{aligned} \tag{1.5'}$$

where the stochastic integral is understood in the sense of Ito integral,

$$\zeta_1=\sup\{t;\ \sup_{s\in[0,t]}|x(s,\xi)|<+\infty\}$$

and

$$\zeta_2=\sup\{t;\ \sup_{s\in[0,t]}|x(s,\eta)|<+\infty\}.$$

Then, the relation

$$\xi(\omega)<\eta(\omega),\quad a.s. \tag{1.6}$$

implies

$$P(x(t,\xi)<x(t,\eta), 0\leqq t<\zeta)=1 \tag{1.7}$$

where $\zeta=\zeta_1\wedge\zeta_2$[2].

Proof. We will divide the proof into two steps.

(1°) In this step, we shall prepare several notations and discuss their properties.

Set $\Omega_L=\left\{\omega;\frac{1}{L}\leqq\eta(\omega)-\xi(\omega)\leqq L\right\}$, for $1<L<+\infty$. Then we observe that Ω_L belongs to $\mathfrak{F}_0$ and

$$P(\Omega_L)\uparrow 1 \quad \text{as } L \text{ tends to } +\infty. \tag{1.8}$$

Put $\tilde{\sigma}_L=\sup\{t;\ \sup_{s\in[0,t]}|x(s,\xi)|<L \text{ and } \sup_{s\in[0,t]}|x(s,\eta)|<L\}$. Then, by the definition of ζ, we have $\tilde{\sigma}_L\uparrow\zeta$, a.s., as L tends to $+\infty$.

Let $\sigma_{\frac{1}{m}}=\inf\left\{0<t<\zeta;\ x(t,\eta)-x(t,\xi)=\frac{1}{m}\right\}$ $(\inf\phi=\zeta)$, and $\tau=\inf\{0<t<\zeta;\ x(t,\eta)-x(t,\xi)=0\}$ $(\inf\phi=\zeta)$.

Since, $x(t,\eta)-x(t,\xi)$ is continuous in t and $x(0,\eta)-x(0,\xi)=\eta-\xi>0$, we have

$$\sigma_{\frac{1}{m}}\uparrow\tau \text{ as } m \text{ tends to } +\infty \quad \text{a.s. on} \quad \{\tau<\zeta\}. \tag{1.9}$$

(2°) In this step, we will show (1.7).

Define a C^2-function ϕ on $(0,\infty)$ by

$$\phi(x)=\int_x^1\int_y^1\frac{du}{\rho^2(u)}\,dy.$$

Letting $\tilde{t}=t\wedge\sigma_{\frac{1}{m}}\wedge\tilde{\sigma}_L$ for a fixed positive number t, we observe that $\tilde{t}$ is a $\mathfrak{F}_t$-stopping time and $\tilde{t}<\zeta$ a.s.

Applying Ito's formula, we have

$$\begin{aligned}
\phi(x(\tilde{t},\eta)-x(\tilde{t},\xi))&=\phi(\eta-\xi)\\
&\quad+\int_0^{\tilde{t}}\phi'(x(s,\eta)-x(s,\xi))\{\sigma(s,x(s,\eta))-\sigma(s,x(s,\xi))\}\,dB(s)\\
&\quad+\int_0^{\tilde{t}}\phi'(x(s,\eta)-x(s,\xi))\{b(s,x(s,\eta))-b(s,x(s,\xi))\}\,ds\\
&\quad+\tfrac{1}{2}\int_0^{\tilde{t}}\phi''(x(s,\eta)-x(s,\xi))\{\sigma(s,x(s,\eta))-\sigma(s,x(s,\xi))\}^2\,ds\\
&=I_1+I_2+I_3+I_4,\ \text{say.}
\end{aligned} \tag{1.10}$$

Since $\phi(x)$ decreases on $(0,1]$ and increases on $[1,\infty)$, we get for I_1

$$E[I_1;\Omega_L]\leqq\phi\left(\frac{1}{L}\right)+\phi(L), \qquad \text{for a fixed } L>1. \tag{1.11}$$

[2] $a\wedge b$ stands for the minimum of a and b

Noticing that I_2 is a martingale with zero mean, we have

$$E[I_2;\Omega_L]=0. \tag{1.12}$$

By condition (1.3) and the fact that $|\phi'(x)|=\left|\int_x^1 \frac{du}{\rho^2(u)}\right|$ increases on $[1,\infty)$, for I_3 we get

$$\begin{aligned}E[|I_3|;\Omega_L]&\leqq E\left[\int_0^{\tilde t}|\phi'(x(s,\eta)-x(s,\xi))|\,\kappa(x(s,\eta)-x(s,\xi))\,ds\right]\\&=t\{\phi'(2L)\,\kappa(2L)+\sup_{\frac{1}{m}\leqq x\leqq 1}|\phi'(x)\,\kappa(x)|\}.\end{aligned} \tag{1.13}$$

By condition (1.1) and the fact that $\phi''(x)=\frac{1}{\rho^2(x)}$, we have for I_4

$$\begin{aligned}E[I_4;\Omega_L]&\leqq E\left[\int_0^{\tilde t}\phi''(x(s,\eta)-x(s,\xi))\,\rho^2(x(s,\eta)-x(s,\xi))\,ds\right]\\&=E\left[\int_0^{\tilde t}\frac{\rho^2(x(s,\eta)-x(s,\xi))}{\rho^2(x(s,\eta)-x(s,\xi))}\,ds\right]\leqq t.\end{aligned} \tag{1.14}$$

Thus, the above inequalities (1.11), (1.13), (1.14) and the equality (1.12) imply that there exists a constant $K(L,t)$ which only depends on L and t such that

$$E[\phi(x(\tilde t,\eta)-x(\tilde t,\xi));\Omega_L]\leqq K(L,t)+t\sup_{\frac{1}{m}\leqq x\leqq 1}|\phi'(x)\,\kappa(x)|. \tag{1.15}$$

On the other hand, by the fact that $\phi(x)\geqq 0$ on $(0,\infty)$, we observe that

$$E[\phi(x(\tilde t,\eta)-x(\tilde t,\xi));\Omega_L]\geqq E\left[\phi\left(\frac{1}{m}\right);\sigma_{\frac{1}{m}}<t\wedge\tilde\sigma_L,\Omega_L\right].$$

Combining this with (1.15), we have

$$\begin{aligned}&P(\sigma_{\frac{1}{m}}<t\wedge\tilde\sigma_L,\Omega_L)\\&\qquad\leqq\frac{1}{\phi\left(\frac{1}{m}\right)}\{K(L,t)+t\sup_{\frac{1}{m}\leqq x\leqq 1}|\phi'(x)\,\kappa(x)|\}\\&\qquad=\frac{K(L,t)}{\phi\left(\frac{1}{m}\right)}+t\left\{\sup_{\frac{1}{m}\leqq x\leqq 1}\kappa(x)\int_x^1\frac{du}{\rho^2(u)}\Big/\int_{\frac{1}{m}}^1\int_x^1\frac{du}{\rho^2(u)}\,dx\right\}.\end{aligned} \tag{1.16}$$

By condition (1.2), we know that $\phi\left(\frac{1}{m}\right)\uparrow+\infty$ as m tends to $+\infty$. Thus, by (1.4), (1.9) and (1.16), we observe that

$$P(\tau<t\wedge\tilde\sigma_L,\Omega_L)=0.$$

Letting L and t tend to $+\infty$, we obtain that $P(\tau<\zeta)=0$.

This implies immediately that

$$P(x(t,\xi)<x(t,\eta), \quad 0\leqq t<\zeta)=1. \quad \text{Q.E.D.}$$

Remark 1.1. (a) The functions $\rho(u)=Ku$, $\kappa(u)=Ku$ satisfy (1.2) and (1.4); that is to say, if σ and b satisfy the Lipschitz condition, then (1.7) holds.

(b) The functions $\rho(u)=Ku\left(\log\frac{2}{u}\right)^{\frac{1}{2}}$, $\kappa(u)=Ku\log\frac{2}{u}$ satisfy the conditions (1.2) and (1.4).

(c) The condition (1.2) is best possible for our conclusion in the following sense. If (1.2) fails, we can find the solutions $x(t,\xi)$ and $x(t,\eta)$ which satisfy all the conditions in Theorem 1.1 except for (1.2) such that (1.7) fails. Indeed, let $\sigma(t,x)=\rho(x)$ and $b(t,x)\equiv 0$ where $\rho(u)$ is a continuous function on R with $\rho(0)=0$, non increasing on $(-\infty,0)$, non decreasing on $(0,\infty)$, positive and locally Lipschitz continuous on $R-\{0\}$ and satisfies

$$\int_0^1 \rho^{-2}(u)\,du=\int_{-1}^0 \rho^{-2}(u)\,du=+\infty.$$

Let also $\xi\equiv 0$ and $\eta\equiv x_0$ for an $x_0>0$. Then the SDE's (1.5) and (1.5′) have unique solutions $x(t,0)$ and $x(t,x_0)$ respectively (c.f. [15]), and all the conditions in Theorem 1.1 except for (1.2) are fulfilled. But $x(t,0)\equiv 0$ by the uniqueness, and $x(t,x_0)$ is a realization of the diffusion process corresponding to the generator $\frac{1}{2}\rho^2(x)\frac{d^2}{dx^2}$ starting at x_0. Hence (1.2) is equivalent to that the state 0 is non-exit (inaccessible) in Feller's sense for this diffusion (c.f. [8]) and so it is equivalent to (1.7).

§2. Non-contact Property of Solutions of Multi-Dimensional SDE

In this section, we will discuss non-contact property of solutions of multi-dimensional SDE of which coefficients are locally Lipschitz continuous.

Let $\sigma(t,x)=(\sigma^i_j(t,x))$ $i=1,\dots,n$, $j=1,\dots,r$, and $b(t,x)=(b^i(t,x))$ $i=1,\dots,n$ be defined on $[0,\infty)\times R^n$ continuous in (t,x) such that $\sigma(t,x)$ is an $n\times r$ matrix and $b(t,x)$ is an n-vector.

We assume for σ and b the following;

$$\begin{gathered}|\sigma^i_j(t,x)-\sigma^i_j(t,y)|\leqq K_{T,L}|x-y|,\\ 1\leqq i\leqq n,\ 1\leqq j\leqq r,\ |x|\leqq L,\ |y|\leqq L,\ t\in[0,T],\\ |b^i(t,x)-b^i(t,y)|\leqq K_{T,L}|x-y|,\\ 1\leqq i\leqq n,\ |x|\leqq L,\ |y|\leqq |L|,\ t\in[0,T],\end{gathered} \tag{2.1}$$

where the positive constant $K_{T,L}$ only depends on L and T.

We consider the following SDE;

$$dx(t)=\sigma(t,x(t))\,dB(t)+b(t,x(t))\,dt, \tag{2.2}$$

or in component wise

$$dx^i(t)=\sum_{j=1}^{r}\sigma_j^i(t,x(t))\,dB^j(t)+b^i(t,x(t))\,dt,\qquad i=1,\dots,n. \tag{2.2'}$$

By a solution of the Eq. (2.2), we mean a family of stochastic processes

$$\{x(t)=(x^1(t),\dots,x^n(t)),\ B(t)=(B^1(t),\dots,B^r(t))\}$$

defined on a usual probability space $(\Omega,\mathfrak{F},P;\mathfrak{F}_t)$ with an increasing family of sub-σ-fields such that

(i) $x(t)$ is continuous in t on $t<\tilde{\zeta}$,

(ii) $x(t)$ is adapted to $\mathfrak{F}_t$,

(iii) $B(t)$ is an r-dimensional $\mathfrak{F}_t$-Brownian motion such that $B(0)=0$,

(iv) $x(t)$ satisfies

$$x^i(t)=x^i(0)+\sum_{j=1}^{r}\int_0^t\sigma_j^i(s,x(s))\,dB^j(s)+\int_0^t b^i(s,x(s))\,ds,$$
$$\text{a.s. on } 0\leqq t<\tilde{\zeta}\qquad i=1,\dots,n,$$

where the stochastic integral is understood in the sense of Ito integral and

$$\tilde{\zeta}=\sup\{t;\ \sup_{s\in[0,t]}|x(s)|<+\infty\}.$$

Let $x(t,\xi)$ and $x(t,\eta)$ be solutions of the equation (2.2) such that $x(0)=\xi$ and $x(0)=\eta$ respectively.

Put $\zeta=\sup\{t;\ \sup_{s\in[0,t]}|x(s,\xi)|<+\infty$ and $\sup_{s\in[0,t]}|x(s,\eta)|<+\infty\}$. Then, we have the following theorem.

Theorem 2.1. *Under the condition* (2.1), *the relation* $|\eta(\omega)-\xi(\omega)|>0$, *a.s. implies*

$$P(|x(t,\eta)-x(t,\xi)|>0,\ 0\leqq t<\zeta)=1. \tag{2.3}$$

Proof. As in the proof of Theorem 1.1, we begin the proof with introducing several notations and their simple properties.

Put $\Omega_L=\left\{\omega,\frac{1}{L}\leqq|\eta(\omega)-\xi(\omega)|\leqq L\right\}$, for $L>1$. Then, we observe that Ω_L belongs to $\mathfrak{F}_0$ and that $P(\Omega_L)\uparrow 1$ as L tends to $+\infty$.

Set $\tilde{\sigma}_L=\sup\{t;\ \sup_{s\in[0,t]}|x(s,\xi)|<L$ and $\sup_{s\in[0,t]}|x(s,\eta)|<L\}$. Then, we see that $\tilde{\sigma}_L$ tends to ζ as L tends to $+\infty$.

Letting $\sigma_{\frac{1}{m}}=\inf\left\{0<t<\zeta;\ |x(t,\eta)-x(t,\xi)|=\frac{1}{m}\right\}$ $(\inf\phi=\zeta)$, and $\tau=\inf\{0<t<\zeta;\ |x(t,\eta)-x(t,\xi)|=0\}$ $(\inf\phi=\zeta)$, we have

$$\sigma_{\frac{1}{m}}\uparrow\tau\qquad\text{a.s.}\quad\text{on }\{\tau<\zeta\}. \tag{2.4}$$

Now, we introduce a C^2-function $\phi(u)$ on $(0,\infty)$ by

$$\phi(u)=\int_u^1\int_s^1\frac{dv}{v^2}\,ds=\begin{cases}-u-\log u+1, & 0<u\leqq 1,\\ u-\log u-1, & 1<u<+\infty.\end{cases}$$

Put $\tilde{t}=t\wedge\sigma_{\frac{1}{m}}\wedge\tilde{\sigma}_L$ for a fixed positive number t.

Then, by Ito's formula, we have

$$\begin{aligned}
\phi(|x(\tilde{t},\eta)-x(\tilde{t},\xi)|)&=\phi(|\eta(\omega)-\xi(\omega)|)+\text{a martingale}\\
&+\int_0^{\tilde{t}}\sum_{i=1}^n\phi'(|x(s,\eta)-x(s,\xi)|)\\
&\cdot\frac{x^i(s,\eta)-x^i(s,\xi)}{|x(s,\eta)-x(s,\xi)|}\{b^i(s,x(s,\eta))-b^i(s,x(s,\xi))\}\,ds\\
&+\frac{1}{2}\int_0^{\tilde{t}}\sum_{i,j=1}^n\phi'(|x(s,\eta)-x(s,\xi)|)\\
&\cdot\frac{|x(s,\eta)-x(s,\xi)|^2\delta_{ij}-(x^i(s,\eta)-x^i(s,\xi))(x^j(s,\eta)-x^j(s,\xi))}{|x(s,\eta)-x(s,\xi)|^3}\\
&\times\left\{\sum_{k=1}^r(\sigma_k^i(s,x(s,\eta))-\sigma_k^i(s,x(s,\xi)))(\sigma_k^j(s,x(s,\eta))-\sigma_k^j(s,x(s,\xi)))\right\}ds\\
&+\frac{1}{2}\int_0^{\tilde{t}}\sum_{i,j=1}^n\phi''(|x(s,\eta)-x(s,\xi)|)\frac{(x^i(s,\eta)-x^i(s,\xi))(x^j(s,\eta)-x^j(s,\xi))}{|x(s,\eta)-x(s,\xi)|^2}\\
&\times\left\{\sum_{k=1}^r(\sigma_k^i(s,x(s,\eta))-\sigma_k^i(s,x(s,\xi)))(\sigma_k^j(s,x(s,\eta))-\sigma_k^j(s,x(s,\xi)))\right\}ds\\
&=I_1+I_2+I_3+I_4+I_5,\quad\text{say.}
\end{aligned}\tag{2.5}$$

In the following, we will estimate $E[I_k;\Omega_L]$, $k=1,\dots,5$.
By the definition of Ω_L and the function $\phi(u)$, we have for I_1,

$$E[I_1;\Omega_L]\leqq\phi\left(\frac{1}{L}\right)+\phi(L).$$

Since I_2 is a martingale with zero mean, we obtain that

$$E[I_2;\Omega_L]=0.$$

Note that $|\phi'(u)|\leqq 1+\frac{1}{u}$, $0<u$. Then, by the condition (2.1), we get for I_3,

$$\begin{aligned}
E[|I_3|;\Omega_L]&\leqq E\left[\int_0^t nK_{t,L}\sup_{\frac{1}{m}\leqq u\leqq 2L}\left\{\left(1+\frac{1}{u}\right)u\right\}ds;\Omega_L\right]\\
&\leqq nK_{t,L}2t(1+L).
\end{aligned}$$

Also, by the condition (2.1), we have for I_4,

$$\begin{aligned}
E[|I_4|;\Omega_L]&\leqq E\left[\int_0^t n^2rK_{t,L}^2\sup_{\frac{1}{m}\leqq u\leqq 2L}\left\{\left(1+\frac{1}{u}\right)u\right\}ds;\Omega_L\right]\\
&\leqq n^2rK_{t,L}^2\,2t(1+L).
\end{aligned}$$

Since $\phi''(u)=\frac{1}{u^2}$, we get for I_5,

$$E[|I_5|;\,\Omega_L]\leqq E\left[\int_0^t n^2 r K_{t,L}^2\,ds;\,\Omega_L\right]\leqq n^2 r K_{t,L}^2\,t.$$

Thus, we obtain from (2.5) and the above estimates that

$$E[\phi(|x(\tilde{t},\eta)-x(\tilde{t},\xi)|);\,\Omega_L]\leqq K(t,L)<+\infty, \tag{2.6}$$

where $K(t,L)$ is a positive constant which depends only on t and L.

Note that $\phi(u)$ is a non-negative function. Then, on the other hand, the inequality

$$\begin{aligned}&E[\phi(|x(\tilde{t},\eta)-x(\tilde{t},\xi)|);\,\Omega_L]\\&\quad\geqq\phi\left(\frac{1}{m}\right)P(\sigma_{\frac{1}{m}}<t\wedge\tilde{\sigma}_L,\Omega_L)\quad\text{holds.}\end{aligned} \tag{2.7}$$

Since $\phi\left(\frac{1}{m}\right)$ tends to $+\infty$ as m goes to $+\infty$, we get from (2.4), (2.6) and (2.7) that $P(\tau<t\wedge\tilde{\sigma}_L,\Omega_L)=0$.

Thus, letting L tend to $+\infty$, we have from the above that $P(\tau<t\wedge\zeta)=0$.

Since t is an arbitrary positive number, this implies immediately that $P(\tau<\zeta)=0$.

Thus, we have proved

$$P(|x(t,\eta)-x(t,\xi)|>0,\qquad 0\leqq t<\zeta)=1.\quad\text{Q.E.D.}$$

Remark 2.1. In the one-dimensional case, under the local Lipschitz condition (2.1), the relation $\xi(\omega)<\eta(\omega)$, a.s. implies

$$P(x(t,\xi)<x(t,\eta),\qquad 0\leqq t<\zeta)=1.$$

Remark 2.2. The same proof as in the above gives us a small generalization of this Theorem. Instead of (2.1), let

$$\begin{aligned}&|\sigma_j^i(t,x)-\sigma_j^i(t,y)|\leqq\rho_{T,L}(|x-y|),\\&\qquad 1\leqq i\leqq n,\ 1\leqq j\leqq r,\ |x|<L,\ |y|<L,\ t\in[0,T],\\&|b^i(t,x)-b^i(t,y)|\leqq\kappa_{T,L}(|x-y|),\\&\qquad 1\leqq i\leqq n,\ |x|<L,\ |y|<L,\ t\in[0,T],\end{aligned} \tag{2.8}$$

where the functions $\rho_{T,L}(u)$ and $\kappa_{T,L}(u)$ satisfy the same conditions as those for $\rho(u)$ and $\kappa(u)$ in Theorem 1.1, as well as

$$\lim_{u\downarrow 0}\left\{\sup_{u\leqq v\leqq 1}\frac{\rho_{T,L}^2(v)}{v}\int_v^1\frac{ds}{\rho_{T,L}^2(s)}\Big/\int_u^1\int_v^1\frac{ds}{\rho_{T,L}^2(s)}dv\right\}=0 \tag{2.9}$$

for all $T, L \geqq 1$. Then, Theorem 2.1 is valid when we replace (2.1) by the above conditions.

The function $\rho(u)$ in Remark 1.1 (b) also satisfies (2.9).

§3. Solutions of SDE as Homeomorphisms on *R*

Consider the following one dimensional stochastic differential equation;

$$dx(t) = \sigma(t, x(t))\, dB(t) + b(t, x(t))\, dt. \tag{3.1}$$

We assume that two continuous functions $\sigma(t, x)$ and $b(t, x)$ satisfy the following conditions:

$$\begin{aligned} &|\sigma(t,x) - \sigma(t,y)| \leqq K_{T,L}|x-y|, \ (t,x),(t,y) \in [0,T] \times [-L,L], \\ &|b(t,x) - b(t,y)| \leqq K_{T,L}|x-y|, \ (t,x),(t,y) \in [0,T] \times [-L,L], \end{aligned} \tag{3.2}$$

and

$$\sigma^2(t,x) + b^2(t,x) \leqq K_T(1+x^2), \ (t,x) \in [0,T] \times R, \tag{3.3}$$

where $K_{T,L}$ depends only on T and L, and K_T depends only on T.

It is well known that under the above conditions, the existence and the pathwise uniqueness of solutions for (3.1) are assured and that the explosion time of the solutions is equal to $+\infty$.

Let $x(t, x)$ be a family of solutions of (3.1) such that $x(0, x) = x$.

It is also well known that there exists a version of $x(t,x), (t,x) \in [0,\infty) \times R$, which is continuous in $(t,x) \in [0,\infty) \times R$.

Hereafter, we assume that $x(t, x)$ is continuous in (t, x).

Theorem 3.1. *Suppose that two real continuous functions $\sigma(t, x)$ and $b(t, x)$ satisfy the conditions* (3.2) *and* (3.3). *Then the mapping $x \rightsquigarrow x(t, x)$ gives a homeomorphisms on R for all $t \geqq 0$, a.s. ω.*

Proof. We will divide the proof into three steps.

(1°) In the first step, we show that the mapping $x \rightsquigarrow x(t,x)$ is strongly increasing and is the continuous function w.r.t. x for all $t \geqq 0$, a.s. ω.

Let Q be the set of all rational numbers. By Remark 2.1,

$$P(x(t,r) < x(t,q), \quad 0 \leqq t < +\infty, \quad r < q,\ r,q \in Q) = 1.$$

This implies that the mapping $x \rightsquigarrow x(t,x)$ is a strictly increasing function on Q. Then, by the continuity w.r.t. x, the mapping $x \rightsquigarrow x(t,x)$ is strictly increasing and continuous w.r.t. $x \in R$ for all $t \geqq 0$, a.s. ω.

Its inverse mapping is also strictly increasing and continuous w.r.t. x for all $t \geqq 0$, a.s. ω.

(2°) In this step, we show that

$$\lim_{x \to \pm\infty} x(t,x) = \pm\infty \qquad \text{for all } t \geqq 0, \quad \text{a.s. } \omega, \text{ holds.} \tag{3.4}$$

We owe much to H. Kunita (by private communication) for the proof of this step.

Let $0<T<+\infty$ be fixed. Define the function $g(x)$ by $g(x)=\frac{1}{1+x^2}$.

By Ito's formula, we have

$$\begin{aligned} g(x(t,x))=g(x)-\int_0^t \frac{2x(s,x)}{(1+x^2(s,x))^2}\sigma(s,x(s,x))\,dB(s) \\ +\int_0^t g(x(s,x))\left\{\frac{-2x(s,x)}{1+x^2(s,x)}b(s,x(s,x))\right. \\ \left.-\frac{\sigma^2(s,x(s,x))}{1+x^2(s,x)}+\frac{4\sigma^2(s,x(s,x))\,x^2(s,x)}{(1+x^2(s,x))^2}\right\}ds. \end{aligned}$$

By condition (3.3), there exists a positive constant C_T depending only on T, such that

$$\begin{aligned} & g(x(t,x))\leqq g(x) \\ & \quad +\left|\int_0^t \frac{2x(s,x)\,\sigma(s,x(s,x))}{(1+x^2(s,x))^2}\,dB(s)\right|+C_T\int_0^t g(x(s,x))\,ds, \qquad 0\leqq t\leqq T. \end{aligned}$$

Then, we have

$$\begin{aligned} g^2(x(t,x))\leqq 3g^2(x)+3\left(\int_0^t \frac{2x(s,x)\,\sigma(s,x(s,x))}{(1+x^2(s,x))^2}\,dB(s)\right)^2 \\ +3C_T^2\left(\int_0^t g(x(s,x))\,ds\right)^2, \qquad 0\leqq t\leqq T. \end{aligned} \tag{3.5}$$

Now, we define the function $h(t,x)$ by $h(t,x)=E[\sup_{0\leqq s\leqq t} g^2(x(s,x))]$. Then, we get from (3.5)

$$\begin{aligned} h(t,x)\leqq 3g^2(x)+3E\left[\sup_{0\leqq s\leqq t}\left(\int_0^s \frac{2x(u,x)\,\sigma(u,x(u,x))}{(1+x^2(u,x))^2}\,dB(u)\right)^2\right] \\ +3C_T^2E\left[\left(\int_0^t g(x(s,x))\,ds\right)^2\right], \qquad 0\leqq t\leqq T. \end{aligned}$$

By Doob's inequality and that of Schwarz, we obtain from the above,

$$\begin{aligned} h(t,x) & \leqq 3g^2(x)+12E\left[\left(\int_0^t \frac{2x(s,x)\,\sigma(s,x(s,x))}{(1+x^2(s,x))^2}\,dB(s)\right)^2\right] \\ & \quad +3C_T^2T\int_0^t E[g^2(x(s,x))]\,ds \\ & \leqq 3g^2(x)+12E\left[\int_0^t \frac{1}{(1+x^2(s,x))^2}\,\frac{4x^2(s,x)\,\sigma^2(s,x)}{(1+x^2(s,x))^2}\,ds\right] \\ & \quad +3C_T^2T\int_0^t h(s,x)\,ds, \qquad 0\leqq t\leqq T. \end{aligned}$$

Then, using the condition (3.3) again, we get

$$h(t,x)\leqq 3g^2(x)+C_T'\int_0^t E[g^2(x(s,x))]\,ds+3\,C_T^2\,T\int_0^t h(s,x)\,ds$$

$$\leqq 3g^2(x)+C_T''\int_0^t h(s,x)\,ds, \qquad 0\leqq t\leqq T,$$

where C_T' and C_T'' are positive constants which depend only on T.

Then, by Gronwall's inequality, we obtain

$$h(T,x)\leqq\frac{1}{3(1+x^2)^2}e^{C_T''T}.$$

This means

$$\lim_{x\to\pm\infty} E\left[\sup_{0\leqq t\leqq T}\frac{1}{(1+x^2(t,x))^2}\right]=0.$$

Then, by Fatou's lemma we get

$$\underline{\lim}_{x\to\pm\infty}\sup_{0\leqq t\leqq T}\frac{1}{(1+x^2(t,x))^2}=0 \quad \text{a.s.} \tag{3.6}$$

Now, we will show that

$$\lim_{x\to\infty}\inf_{0\leqq t\leqq T} x(t,x)=+\infty.$$

Immediately from (3.6), we have

$$\overline{\lim}_{x\to\infty}\inf_{0\leqq t\leqq T}|x(t,x)|=\infty \quad \text{a.s.} \tag{3.7}$$

On the other hand, using the increasing property of the mapping $x\rightsquigarrow x(t,x)$, we observe that

$$\inf_{0\leqq t\leqq T}|x(t,x)|\leqq\inf_{0\leqq t\leqq T}x(t,x)\vee|x(t,0)|\leqq\inf_{0\leqq t\leqq T}x(t,x)\vee\sup_{0\leqq t\leqq T}|x(t,0)|,$$
$$\text{for } x\geqq 0, \quad \text{a.s.} \tag{3.8}$$

Combine (3.7) with (3.8), and note that $\sup_{0\leqq t\leqq T}|x(t,0)|<\infty$, a.s. holds. Then, we get

$$\overline{\lim}_{x\to\infty}\inf_{0\leqq t\leqq T}x(t,x)=+\infty, \quad \text{a.s.} \tag{3.9}$$

Noticing that the mapping $x\rightsquigarrow\inf_{0\leqq t\leqq T}x(t,x)$ is increasing, we can conclude from (3.9),

$$\lim_{x\to\infty}\inf_{0\leqq t\leqq T}x(t,x)=+\infty, \quad \text{a.s. } \omega.$$

By analogous arguments, we can also prove that

$$\lim_{x\to-\infty}\sup_{0\leqq t\leqq T}x(t,x)=-\infty, \quad \text{a.s. } \omega.$$

Thus, we have proved

$$\lim_{x\to\pm\infty} x(t,x) = \pm\infty \qquad \text{for all } t\geqq 0, \quad \text{a.s. } \omega. \tag{3.4}$$

(3°) The results obtained in (1) and (2) imply the mapping $x\rightsquigarrow x(t,x)$ is a continuous, one to one and onto mapping from R to R for all $t\geqq 0$, a.s. ω. Its inverse mapping is also continuous for all $t\geqq 0$, a.s. ω. This shows that the mapping $x\rightsquigarrow x(t,x)$ gives a homeomorphism for all $t\geqq 0$, a.s. ω. Q.E.D.

Remark 3.1. In multi-dimensional case, H. Kunita and S.R.S. Varadhan show the same result under global Lipschitz condition for coefficients (c.f. [10]).

§ 4. A Strong Comparison Theorem for Solutions of One Dimensional SDE

In this section, we will show that under certain conditions the usual comparison theorem induces a strong comparison theorem for solutions of SDE.

We consider the following two stochastic differential equations;

$$dx_1(t) = \sigma(t, x_1(t))\,dB(t) + b_1(t, x_1(t))\,dt, \tag{4.1}$$

$$dx_2(t) = \sigma(t, x_2(t))\,dB(t) + b_2(t, x_2(t))\,dt. \tag{4.2}$$

Theorem 4.1. *Suppose we are given the following;*

(i) *a real continuous function $\sigma(t,x)$ defined on $[0,\infty)\times R$ such that*

$$|\sigma(t,x)-\sigma(t,y)| \leqq \rho(|x-y|), \qquad x, y\in R,\ t\geqq 0, \tag{4.3}$$

where $\rho(u)$ is an increasing function defined on $[0,\infty)$ such that $\rho(0)=0$ and

$$\int_0^1 e^{\varepsilon B(y)}dy = +\infty \qquad \textit{for any } \varepsilon>0, \tag{4.4}$$

where

$$B(y) = \int_y^1 \frac{du}{\rho^2(u)} = -\int_1^y \frac{du}{\rho^2(u)},$$

(ii) *two real continuous functions $b_1(t,x)$ and $b_2(t,x)$ defined on $[0,\infty)\times R$ such that*

$$b_1(t,x) < b_2(t,x). \tag{4.5}$$

Suppose $x_1(t,\xi)$ and $x_2(t,\eta)$ be solutions of (4.1) *and* (4.2) *respectively, defined on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_t)$ with the same $\mathfrak{F}_t$-Brownian motion $B(t)$ such that $B(0)=0$, and with initial conditions $x_1(0,\xi)=\xi$ $x_2(0,\eta)=\eta$ respectively.*

Then, the property

$$P(x_1(t,\xi)\leqq x_2(t,\eta), 0\leqq t<\zeta) = 1, \tag{4.6}$$

implies

$$P(x_1(t,\xi) < x_2(t,\eta), 0<t<\zeta) = 1, \tag{4.7}$$

where $\zeta = \sup\{t;\ \sup_{s\in[0,t]} |x_1(s,\xi)| < +\infty$ and $\sup_{s\in[0,t]} |x_2(s,\eta)| < +\infty\}$.

Proof. We will divide the proof into several steps.

(1°) Define τ_a by $\tau_a = \inf\{0 < t < \zeta,\ x_2(t,\eta) - x_1(t,\xi) > a\}$ $(\inf \phi = \zeta)$, where $a \geqq 0$. In this step, we will show that

$$P(\lim_{a \downarrow 0} \tau_a = 0) = 1. \tag{4.8}$$

Noticing that τ_a increases with a, we put $\tau^* = \lim_{a \downarrow 0} \tau_a$.
By the definition of τ_a, we observe that $0 \leqq x_2(t,\eta) - x_1(t,\xi) \leqq a$, $0 \leqq t < \tau_a$.
Then, letting a decrease to 0, we have $x_2(t,\eta) - x_1(t,\xi) = 0$, $0 \leqq t < \tau^*$.
Thus we get

$$P(\tau^* \leqq \tau_0) = 1. \tag{4.9}$$

Now, for a fixed $t \geqq 0$, we have on $\{\tau_0 > t\}$

$$\begin{aligned} 0 = x_2(t,\eta) - x_1(t,\xi) &= \eta - \xi \\ &+ \int_0^t \{\sigma(s, x_2(s,\eta)) - \sigma(s, x_1(s,\xi))\}\, dB(s) \\ &+ \int_0^t \{b_2(s, x_2(s,\eta)) - b_1(s, x_1(s,\xi))\}\, ds = N_1 + N_2 + N_3. \end{aligned} \tag{4.10}$$

Since $x_2(s,\eta) = x_1(s,\xi)$, $0 \leqq s \leqq t < \tau_0$, we observe that $N_2 = 0$.
By (4.6), we have $N_1 = \eta - \xi = x_2(0,\eta) - x_1(0,\xi) \geqq 0$.
Thus, we get from (4.10) that

$$E\left[\int_0^t \{b_2(s, x_2(s,\eta)) - b_1(s, x_1(s,\xi))\}\, ds;\ \tau_0 > t\right] \leqq 0. \tag{4.11}$$

On the other hand, we know from (4.5) that $b_2(s, x_2(s,\eta)) - b_1(s, x_1(s,\xi)) > 0$, $s < \tau_0$. So, (4.11) implies $P(\tau_0 > t) = 0$, for any $t > 0$. Thus we have $P(\tau_0 = 0) = 1$. Combining this with (4.9), we get $P(\tau^* = 0) = 1$. Hence

$$\tau_* = \lim_{a \downarrow 0} \tau_a = 0, \quad \text{a.s.} \tag{4.8}$$

(2°) In this step, we will prove that

$$P(x_1(t,\xi) < x_2(t,\eta), \tau_a \leqq t < \zeta) = P(\tau_a < \zeta) \qquad \text{for any } a > 0. \tag{4.12}$$

First, we will fix $1 < T < +\infty$ and $1 < L < +\infty$. Since $b_2(t,x) - b_1(t,x)$ is continuous in (t,x) and strictly positive, there exists a positive number $\varepsilon > 0$ such that

$$b_2(t,x) - b_1(t,x) \geqq \varepsilon, \qquad (t,x) \in [0,T] \times [-L,L]. \tag{4.13}$$

Noticing that $b_i(t,x)$, $i = 1, 2$, is uniformly continuous on $[0,T] \times [-L,L]$, we can choose a positive number $\delta > 0$ such that

$$|b_i(t,x) - b_i(t,y)| < \frac{\varepsilon}{4}, \qquad t \in [0,T],\ x, y \in [-L,L],\ |x - y| < \delta, \quad i = 1, 2.$$

Combining this with (4.13), we get

$$b_2(t,x)-b_1(t,y)\geqq\frac{\varepsilon}{2},\qquad t\in[0,T],\ x,y\in[-L,L],\ |x-y|<\delta. \tag{4.14}$$

Now, we define the function $\phi(x)$ by

$$\phi(x)=\int_x^1 e^{\varepsilon B(y)}dy,$$

where ε is the same constant as in (4.14).

We will note several properties of $\phi(x)$ in the following;

(i) since $\phi'(x)=-e^{\varepsilon B(x)}<0$, $\phi(x)$ is decreasing function on $(0,\infty)$,

(ii) since $\phi''(x)=\dfrac{\varepsilon}{\rho^2(x)}e^{\varepsilon B(x)}>0$, $\phi(x)$ is convex on $(0,\infty)$,

(iii) by (4.4), $\phi(0+)=\int_0^1 e^{\varepsilon B(y)}dy=+\infty$

(iv) also by (4.4)

$$\phi''(0+)=\frac{\varepsilon}{\rho^2(0+)}e^{\varepsilon B(0+)}=+\infty.$$

Now, put $\sigma_{\frac{1}{m}}=\inf\left\{0<t<\zeta;\ x_2(t,\eta)-x_1(t,\xi)=\frac{1}{m}\right\}$ $(\inf\phi=\zeta)$, and $\tau=\inf\{0<t<\zeta;\ x_2(t,\eta)-x_1(t,\xi)=0\}$ $(\inf\phi=\zeta)$.

Then, by the same way as in the first step of the proof of Theorem 1.1, we have

$$\sigma_{\frac{1}{m}}\uparrow\tau\quad \text{a.s. on}\quad \{\tau_a<\tau<\zeta\}. \tag{4.15}$$

Let $T'=T\wedge\sigma_{\frac{1}{m}}\wedge\tilde{\sigma}_L<\zeta$, where $\tilde{\sigma}_L=\sup\{t;\ \sup_{s\in[0,t]}|x_1(s,\xi)|<L$ and $\sup_{s\in[0,s]}|x_2(t,\eta)|<L\}$.

Then, by Ito's formula, we have on $\{T'>\tau_a\}$

$$\begin{aligned}
\phi(x_2(T',\eta)-x_1(T',\xi))&=\phi(a)\\
&+\int_{\tau_a}^{T'}\phi'(x_2(s,\eta)-x_1(s,\xi))\{\sigma(s,x_2(s,\eta))-\sigma(s,x_1(s,\xi))\}\,dB(s)\\
&+\int_{\tau_a}^{T'}\phi'(x_2(s,\eta)-x_1(s,\xi))\{b_2(s,x_2(s,\eta))-b_1(s,x_1(s,\xi))\}\,ds\\
&+\frac{1}{2}\int_{\tau_a}^{T'}\phi''(x_2(s,\eta)-x_1(s,\xi))\{\sigma(s,x_2(s,\eta))-\sigma(s,x_1(s,\xi))\}^2\,ds\\
&=I_1+I_2+I_3+I_4,\quad \text{say.}
\end{aligned}$$

Since $E[I_2; T'>\tau_a]=0$, we get from the above equality

$$E[\phi(x_2(T',\eta)-x_1(T',\xi)); T'>\tau_a] \\ \leqq \phi(a)+E[I_3; T'>\tau_a]+E[I_4; T'>\tau_a]. \tag{4.16}$$

Define $\Lambda_1=\Lambda_1(\omega)$ and $\Lambda_2=\Lambda_2(\omega)$ by

$$\Lambda_1(\omega)=\{\tau_a \leqq s \leqq T'; 0 \leqq x_2(s,\eta)-x_1(s,\xi)<\delta\}$$

and

$$\Lambda_2(\omega)=\{\tau_a \leqq s \leqq T'; x_2(s,\eta)-x_1(s,\xi) \geqq \delta\}.$$

Then, we have for I_3

$$\begin{aligned} &E[I_3; T'>\tau_a] \\ &\quad =E[\int_{\Lambda_1(\omega)} \phi'(x_2(s,\eta)-x_1(s,\xi))\{b_2(s,x_2(s,\eta))-b_1(s,x_1(s,\xi))\}\,ds; T'>\tau_a] \\ &\qquad +E[\int_{\Lambda_2(\omega)} \phi'(x_2(s,\eta)-x_1(s,\xi))\{b_2(s,x_2(s,\eta))-b_1(s,x_1(s,\xi))\}\,ds; T'>\tau_a] \\ &\quad =J_1+J_2, \quad \text{say.} \end{aligned}$$

By (i) and (4.14), we have for J_1

$$\begin{aligned} J_1 &\leqq \frac{\varepsilon}{2} E[\int_{\Lambda_1(\omega)} \phi'(x_2(s,\eta)-x_1(s,\xi))\,ds; T'>\tau_a] \\ &= -\frac{\varepsilon}{2} E[\int_{\Lambda_1(\omega)} e^{\varepsilon B(x_2(s,\eta)-x_1(s,\xi))}\,ds; T'>\tau_a]. \end{aligned} \tag{4.18}$$

Noticing that $|\phi'(x)|=e^{\varepsilon B(x)}$ is decreasing in x, we have for J_2

$$J_2 \leqq Te^{\varepsilon B(\delta)} \sup_{(t,x)\in[0,T]\times[-L,L]} \{|b_2(t,x)|+|b_1(t,x)|\}. \tag{4.19}$$

By (4.3) and (4.4), we have for I_4

$$\begin{aligned} &E[I_4; T'>\tau_a] \\ &\quad =E[\tfrac{1}{2}\int_{\Lambda_1(\omega)} \phi''(x_2(s,\eta)-x_1(s,\xi))\{\sigma(s,x_2(s,\eta))-\sigma(s,x_1(s,\xi))\}^2\,ds; T'>\tau_a] \\ &\qquad +E[\tfrac{1}{2}\int_{\Lambda_2(\omega)} \phi''(x_2(s,\eta)-x_1(s,\xi))\{\sigma(s,x_2(s,\eta))-\sigma(s,x_1(s,\xi))\}^2\,ds; T'>\tau_a] \\ &\quad \leqq \tfrac{1}{2}\varepsilon E\left[\int_{\Lambda_1(\omega)} e^{\varepsilon B(x_2(s,\eta)-x_1(s,\xi))} \frac{\rho^2(x_2(s,\eta)-x_1(s,\xi))}{\rho^2(x_2(s,\eta)-x_1(s,\xi))}\,ds; T'>\tau_a\right] \\ &\qquad +\tfrac{1}{2}\varepsilon E\left[\int_{\Lambda_2(\omega)} e^{\varepsilon B(x_2(s,\eta)-x_1(s,\xi))} \frac{\rho^2(x_2(s,\eta)-x_1(s,\xi))}{\rho^2(x_2(s,\eta)-x_1(s,\xi))}\,ds; T'>\tau_a\right] \\ &\quad \leqq \tfrac{1}{2}\varepsilon E[\int_{\Lambda_1(\omega)} e^{\varepsilon B(x_2(s,\eta)-x_1(s,\xi))}\,ds; T'<\tau_a] \\ &\qquad +\tfrac{1}{2}\varepsilon e^{\varepsilon B(\delta)} T. \end{aligned} \tag{4.20}$$

By (4.16), (4.17), (4.18), (4.19) and (4.20), we get

$$E[\phi(x_2(T',\eta)-x_1(T',\xi));T'>\tau_a]$$
$$\leqq \phi(a)+Te^{\varepsilon B(\delta)}\sup_{(t,x)\in[0,T]\times[-L,L]}\{|b_2(t,x)|+|b_1(t,x)|\}$$
$$+\tfrac{1}{2}\varepsilon e^{\varepsilon B(\delta)}T=K(a,T,\delta,L),\quad \text{say,} \tag{4.21}$$

where $K(a,T,\delta,L)$ does not depend on m.

On the other hand, by (i), we have

$$E[\phi(x_2(T',\eta)-x_1(T',\xi));T'>\tau_a]$$
$$\geqq \phi\left(\frac{1}{m}\right)P(\tau_a<\sigma_{\frac{1}{m}}\leqq T\wedge\tilde{\sigma}_L)+\phi(2L).$$

Combining this with (4.21) and letting m tend to $+\infty$, we get

$$P(\tau_a<\tau\leqq T\wedge\tilde{\sigma}_L)=0. \tag{4.22}$$

Let L and T go to $+\infty$. Then we get from (4.22) that

$$P(\tau_a<\tau<\zeta)=0. \tag{4.23}$$

This implies immediately (4.12).

(3°) Combine

$$\lim_{a\downarrow 0}\tau_a=0 \tag{4.8}$$

with

$$P(x_1(t,\xi)<x_2(t,\eta),\tau_a\leqq t<\zeta). \tag{4.12}$$

Then we can conclude that

$$P(x_2(t,\eta)-x_1(t,\xi)>0,0<t<\zeta)=1.\quad \text{Q.E.D.}$$

Remark 4.1. Suppose that $\rho(u)$ satisfies

$$\int_{0+}\frac{du}{\rho^2(u)}=+\infty. \tag{4.24}$$

Then, the relation $\xi(\omega)\leqq\eta(\omega)$ a.s. implies

$$P(x_1(t,\xi)\leqq x_2(t,\eta),0\leqq t<\zeta)=1. \tag{4.6}$$

The function $\rho(u)=u^\alpha$ ($\frac{1}{2}\leqq\alpha\leqq 1$) satisfies (4.24); that is to say, if σ is Hölder continuous in x uniformly in t with order $\frac{1}{2}\leqq\alpha\leqq 1$, then (4.24) holds. (c.f. [7], [16]).

Remark 4.2. (a) The function $\rho(u)=u^\alpha$ ($\frac{1}{2}<\alpha\leqq 1$) satisfies (4.4). But the function $\rho(u)=u^{\frac{1}{2}}$ does not satisfy (4.4).

(b) The condition (4.4) is best possible for (4.7) in the following sense.

Let $\sigma(t,x)=\rho(x)$ where $\rho(u)$ satisfies the conditions in Remark 1.1(c) but (4.4) fails, i.e. $\int_0^1 e^{\varepsilon_0 B(y)}dy<\infty$ for an $\varepsilon_0>0$.

Let also $b_1(t,x)\equiv 0$, $b_2(t,x)\equiv\frac{\varepsilon_0}{2}$, $\xi\equiv 0$ and $\eta\equiv x_0>0$. Then they satisfy all conditions in Theorem 4.1 except for (4.4) (c.f. [7], [16]). Further, as in Remark 1.1(c), $x_1(t,0)\equiv 0$ and $x_2(t,x_0)$ is a realization of the diffusion corresponding to the generator $\frac{1}{2}\left(\rho^2(x)\frac{d^2}{dx^2}+\varepsilon_0\frac{d}{dx}\right)$ starting at x_0. But the state 0 is regular for this diffusion, since

$$\int_0^1 e^{\varepsilon_0 B(x)}\int_x^1 e^{-\varepsilon_0 B(y)}\,\rho^{-2}(y)\,dy\,dx<\infty.$$

Hence, the natural scale being given by $s(x)=\int_1^x e^{\varepsilon_0 B(y)}\,dy$, it follows that

$$\begin{aligned}P(x_1(t,0)=x_2(t,x_0)\quad &\text{for some } 0<t<\zeta)\\ &=\lim_{L\to\infty}\left(\int_{x_0}^{L} e^{\varepsilon_0 B(y)}\,dy\Big/\int_0^L e^{\varepsilon_0 B(y)}\,dy\right)>0.\end{aligned}$$

This means that (4.7) fails.

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