Prelude to Dimension Theory: The Geometrical Investigations of Bernard Bolzano

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Abstract

This paper treats BERNARD BOLZANO's (1781–1848) investigations into a fundamental problem of geometry: the problem of adequately defining the concepts of line (or curve), surface, solid, and continuum.

BOLZANO's interest in this problem spanned most of his creative lifetime. In this paper a full discussion is given of the philosophical and mathematical motivation of BOLZANO's problem as well as his two solutions to the problem. BOLZANO's work on this part of geometry is relevant to the history of modern mathematics, because it forms a prelude to the more recent development of topological dimension theory.

1. Introduction

Dimension theory forms a sturdy branch of the great oak of modern topology. The history of its growth to mathematical maturity belongs largely to the latter half of the nineteenth and the first decades of the twentieth centuries. BERNARD RIEMANN (1826–1866) and GEORG CANTOR (1845–1918) were among the first mathematicians to investigate dimension-theoretic problems. Above all, CANTOR was the prime mover in the early development of dimension theory. He discovered in 1877 that the points of a square, 'clearly 2-dimensional', can be put into one-one correspondence with the points of a line segment, 'obviously 1-dimensional', thereby rendering the simple idea of dimension problematic and so posing the invariance problem of dimension. Among the most important twentieth-century contributors to the growth of dimension theory we must include HENRI POINCARÉ (1854–1912), L.E.J. BROUWER (1881–1966), PAVEL S. URYSOHN (1898–1924), and KARL MENGER (1902–). The combined efforts of these and other mathematicians brought dimension theory to a state of mathematical maturity by around 1930.

It is quite surprising to discover that many years before the main historical development of dimension theory, briefly sketched above, BERNARD BOLZANO (1781–1848) examined a cluster of elementary geometrical problems which led him to formulate a theory of geometry which, in the light of recent topology, looks very much like dimension theory. It is well known to historians of mathematics that the great Bohemian priest, philosopher, logician, and mathematician from Prague prefigured much that is central to modern pure mathematics, especially in analysis and set theory. But in topology also we can reckon BOLZANO as a precursor of more recent developments. Through a critical examination of the basic principles of EUCLIDEAN geometry, BOLZANO was led to propose some new definitions of the elementary geometrical concepts. These definitions have several features in common with the definitions of the dimension concept given later by POINCARÉ, BROUWER, URYSOHN, and especially MENGER.

Unfortunately BOLZANO's geometrical research suffered the same tragedy which befell much of his other mathematical and philosophical work: it did not receive due recognition either from his contemporaries (apart from a few friends) or the immediately subsequent generation of mathematicians. Yet in spite of this lack of influence on the main stream of progress in topology and dimension theory, it is still worth while examining BOLZANO's highly original way of dealing with geometrical problems. His investigations constitute a fascinating prelude, regrettably unheard by later mathematicians, to dimension theory. In this paper I intend to discuss the full details of BOLZANO's contribution to 'topological' geometry.[1]

Note on References: References are given in the text to the bibliography at the end of the paper in the form, Author *(date: section or page reference)*.

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2. Bolzano's Geometrical Problem and the Background to it

What was the problem which led BOLZANO to see geometry in a topological light and to formulate a rudimentary dimension theory? It was quite simply a question of definitions. In brief, BOLZANO's problem was to define in a precise way the geometrical concepts of line (or curve), surface, and solid and more generally to define geometrical extension or continuum. Surprisingly this dull essentialist problem of definitions, conceived within the limits of EUCLIDEAN geo-

[1] This paper is intended to be the first in a series covering the entire history of dimension theory from BOLZANO to BROUWER, URYSOHN, and MENGER, roughly from 1800 to 1930.

metry, led BOLZANO to break from the bonds of traditional geometry and to explore, or shall we say 'invent', the unknown domain of topology. Of course, this special problem of definitions was not the only geometrical problem which captured BOLZANO's interest, but it provided an undercurrent to his geometrical thought throughout his creative life.

BOLZANO's interest in his definitional problem began very early—by his own account at the age of 16!—and continued intermittently throughout his life. His research on the problem neatly divides into two periods. His very first publication, which appeared when he was 23, the pamphlet *Betrachtungen über einige Gegenstände der Elementargeometrie (1804)*, contains hints about the problem, while in his pamphlet of 1817 *Die drey Probleme der Rectification, der Complanation und der Cubirung* there is a tentative solution. In the latter he set down his first quasi-topological definitions of lines, surfaces, and solids, given according to set-theoretical and dimensional properties. The work of 1817 marks the end of the first period of BOLZANO's investigations on this subject.

Later, in the 1830's and 40's, BOLZANO renewed his interest in his geometrical problem. His work here forms a second period of topological investigation in which he revised and improved his earlier theory. The two main papers of this period are 'Uiber Haltung, Richtung, Krümmung und Schnörkelung bei Linien sowohl als Flächen sammt einigen verwandten Begriffen' and 'Geometrische Begriffe, die Jeder kennt und nicht kennt. Versuch einer Erhebung derselben ins deutliche Bewusstseyn'. Both of these were written around 1843–44, but not published in his lifetime. In fact, the first was not published until 1948 *(1948d)* and the second awaits publication in the *Bolzano-Gesamtausgabe*. However, BOLZANO included a short summary of his second 'dimension theory' in the *Paradoxien des Unendlichen,* which appeared posthumously in 1851.

To see why BOLZANO took up his special geometrical problem, we must examine the origins of his interest in mathematics and the problem situation in traditional geometry around 1800 from which his work emerged.

1796 to 1804 were the crucial formative years of BOLZANO's intellectual development (*cf.* WINTER *(1969:21–32)*). As a young student he became intrigued with mathematics and philosophy. From the start BOLZANO looked upon mathematics with the eyes of a philosopher. In this respect he resembles DESCARTES, who saw a firm methodological link between mathematics and philosophy. In his autobiography BOLZANO says *(1836:19)*:

> Mein besonderes Wohlgefallen an der Mathematik beruhte also eigentlich nur auf ihrem rein speculativen Theile, oder ich schätzte an ihr nur dasjenige, was zugleich Philosophie ist.
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> (My special pleasure in mathematics therefore rests only in its purely speculative part, or, in other words, I value in it only that which is at the same time philosophical.)

BOLZANO was always more concerned with examining the logical foundations of mathematics than with creating new results or finding new applications. He was more a philosophical mathematician, a foundationalist, than a 'creative' mathematician. This is not to say that his insights into mathematical foundations were not extremely original. They were. But to appraise BOLZANO's contribution to mathematical thought, one must keep in mind the philosophical overtones of his work.[2]

In an autobiographical statement (written during the latter 1830's) BOLZANO mentions what must have been practically the earliest influence on his philosophical thought, ALEXANDER GOTTLIEB BAUMGARTEN's (1714–1762) little handbook *Metaphysica (1779)*. BAUMGARTEN's textbook is largely a summary of the LEIBNIZ-WOLFF philosophy. LEIBNIZIAN ideas proved to have an enormous effect on BOLZANO's philosophy of logic and his philosophy generally. He came to LEIBNIZ first through BAUMGARTEN. BOLZANO, writing in the third person, says (see WINTER *(1969:23)*[3]):

> Eines der ersten philosophischen Bücher, die Bolzano etwa in seinem 16ten Lebensjahre las, war Baumgartens Metaphysik, wo er mit aller Deutlichkeit einzusehen glaubte, dass viele Erklärungen verfehlt, manche Beweise, z. B. der des Satzes vom Grunde, das zu Beweisende schon voraussetzten, und mehrere ganz unrichtige Behauptungen (z. B. von der Zusammensetzung der Linie, Fläche und Körper aus einer endlichen Menge von Punkten) vorkommen.
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> (One of the first philosophical books which Bolzano read at the age of about 16 was Baumgarten's *Metaphysica,* in which he believed he saw with full clarity that many definitions were wrong, several proofs, e. g., that of the principle of sufficient reason, assumed what they set out to prove, and several entirely incorrect assertions occurred (e. g., that of the construction of the line, surface, and solid from a finite set of points).)

BOLZANO's last criticism is most important for the origins of his geometrical ideas. Apparently it was through this book of philosophy that BOLZANO came to his geometrical problem. In *Metaphysica* BAUMGARTEN devotes several pages *(1779:84–89)* to definitions of geometrical concepts like point, line, and continuum. Points are the simple elements of geometry. A series of points with interposed points making up a continuum is, according to BAUMGARTEN, a line. Similarly, a surface is a continuous series of lines and a solid is a series of surfaces. BOLZANO objected to these definitions, perhaps because they really are very uninformative. Indeed according to BOLZANO BAUMGARTEN operates with only a *finite* set of points. Nevertheless, there is a primitive set-theoretic approach to the basic geometrical figures which BOLZANO also used later in his own definitions.

BOLZANO's main teachers in mathematics at Prague were STANISLAUS VYDRA and F. J. GERSTNER. The latter took a particular interest in BOLZANO's early career. However, it was a book which had most effect on the formation of BOLZANO's early mathematical attitudes: ABRAHAM GOTTHELF KÄSTNER's (1719–1800)

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[2] In a recent paper KITCHER *(1975)* emphasises the philosophical background and motivation for one of BOLZANO's most famous proofs, *Rein analytischer Beweis... (1817a)*. While KITCHER does shed new light on BOLZANO's work, he tends to neglect the fact that BOLZANO was well aware of the state of analysis of his time. Thus he neglects the mathematical motivation for BOLZANO's work. Moreover, KITCHER does not discuss BOLZANO's philosophical investigations prior to 1817, such as *(1810)*.

[3] Professor Dr. BERG has kindly provided me with a better transcription of BOLZANO's 'Zur Lebensbeschreibung'. The important difference between WINTER's transcription and BERG's is that 'unendlichen' becomes 'endlichen'. BERG's transcription will appear in *Bernard Bolzano-Gesamtausgabe* II. A, 12/1.

big compendium of mathematics, *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspectiv*. This work, which several generations of students read as a basic text, went through six editions from 1758 to 1800 *(cf.* FOLTA *(1966)*, WINTER *(1969:21–22))*. BOLZANO learned much of his EUCLIDEAN geometry from the geometrical sections of KÄSTNER's textbook *(1800:176–614)*. In these pages KÄSTNER does not merely restate EUCLID's results, but he restructures the logic of EUCLID's proofs, so much so that there are passages which strike one as prefiguring ideas of PASCH and HILBERT *(cf.* GOE *(1964))*. KÄSTNER's improved EUCLIDEAN logic is what impressed the young BOLZANO *(1836:19)*:

> Kästner bewies dort nämlich, was man sonst insgemein, weil es doch Jeder schon weiss, ganz übergeht; das heisst, er suchte dem Leser den Grund, auf welchem eines seiner Urtheile beruht, zum deutlichen Bewusstseyn zu bringen: und das war mir eben das Liebste.
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> (Kästner proved there what one usually passes over entirely, because everyone knows it already; i.e., he sought to bring home clearly to the reader the basis on which each of his judgments rests: And that was what I liked best.)

Like EUCLID KÄSTNER begins his treatment of geometry with definitions. The first few of these together with the definitions of BAUMGARTEN must have been what originally inspired BOLZANO to take up the problem of defining precisely the concepts of line, surface, solid, and continuum. Interestingly KÄSTNER himself starts off by attempting to define continuum or extension, a concept not defined in EUCLID as such. KÄSTNER's definition runs as follows *(1800:176)*:

> Eine *stetige* Grösse (continuum) heisst, deren Theile alle so zusammenhängen, dass, so einer aufhört, gleich der andere anfängt, und zwischen des einen Ende und des andern Anfange nichts ist, das nicht zu dieser Grösse gehörte.
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> (A continuous quantity (continuum) is something whose parts are so connected that when one stops immediately another begins and between one end and another beginning there is nothing which does not belong to this quantity.)

Of course, this early attempt to define continuity or connectedness does not come up to the later standards of, say, DEDEKIND's definition of cut (1872), but it merits some attention. The KÄSTNER definition was partially motivated by the old criticism of EUCLID's proof of the very first proposition of Book I of the *Elements*. EUCLID, when constructing an equilateral triangle on a given line segment, assumes without justification that the two circles used in the construction will intersect in some point. (Before KÄSTNER, CHRISTIAN WOLFF (1679–1754), for example, knew about this criticism and tried to correct it in his textbook *(1742: 147)*.) To fill up this gap in EUCLID's reasoning, KÄSTNER laid down his definition and added a special axiom *(1800:189, Grundsatz 7)* to his axiom system. Later this whole problem of continuity, implicit in EUCLID's very first proof, impelled BOLZANO to look more closely at the topology of the situation and to come up with a surprising result (see Section 6).

For KÄSTNER a geometric extension (geometrische Ausdehnung) is a space which a continuous quantity fills up. On this basis we can define solid, surface, and line *(1800:177)*:

> *Die körperliche Ausdehnung, ein geometrischer Körper* (solidum; corpus) heisst eine solche Ausdehnung, die das, was sich innerhalb ihrer Gränzen befindet, überall, nach allen Seiten zu umgiebt. Die Ausdehnung der Körper an ihren Gränzen heisst eine *Fläche* (superficies) und die Ausdehnung der Fläche an ihren Gränzen, *eine Linie* (linea).
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> (The solid extension, a geometrical solid (solidum; corpus) is an extension which within its boundaries is surrounded everywhere on all sides. The extension of a solid at its boundaries is called a surface (superficies) and the extension of a surface at its boundaries, a line (linea).)

To complete this sequence, KÄSTNER adds that the point is the boundary of the line and, hence, of all extension. Thus, starting with solids (3-dimensional extensions), we have that the boundaries of these are surfaces, the boundaries of surfaces are lines, and the boundaries of lines are points. A few years later BOLZANO came to criticise this progression in KÄSTNER's definitions (see the next section).

Contrary to BAUMGARTEN, KÄSTNER argues that a line is *not* a set of juxtaposed points ('eine Menge von aneinandergesetzten Puncten macht keine Linie aus' *(1800:178)*) and, similarly, a surface is not a set of lines nor a solid, a set of surfaces. Now even with the boundary definition it is possible to imagine points placed anywhere on a line, *i.e.*, the boundaries need not be conceived as occurring just at the ends. In fact, points are everywhere on a line, so that one can even think of a line as generated by the motion of a single point. But for KÄSTNER these conceptions do not imply that a line is a mere set of points. Indeed, in the first paper in which he set out his boundary definitions, KÄSTNER *(1749)* tried to show that it is contradictory to regard a line as a set of points. If a line were made up of a set of points, then any point would have an immediate neighbouring point. But these neighbouring points, although distinct, would have no distance separating themselves, so that distances measured from any other point of the line to these two points would be the same. That is to say, the two points are really the same. A contradiction! Such metaphysical arguments come close to ZENO's paradoxes.

KÄSTNER *(1782)* thought that he could trace his definitions of surface, line, and point as boundary elements back to a passage in the writings of WILLIAM OF OCKHAM. Going back even further, there is a suggestion of the boundary definitions in EUCLID's *Elements* (Book I, Defs. 3, 6; Book XI, Def. 2). However, the progression of EUCLID's definitions is not uniformly from solids down to points as in KÄSTNER's.[4]

Undoubtedly the most original part of KÄSTNER's treatment of elementary geometry is his discussion of the problem of parallels (see FOLTA *(1966)* and GOE *(1973)*). While he did not solve the logical problem of the role of EUCLID's

[4] In general there are several passages in ancient Greek writings which have a bearing on modern dimension-theoretic problems and theories of dimension. In fact, several modern thinkers, notably POINCARÉ and MENGER, have linked recent dimension theory with some ancient geometrical ideas. I do not wish to ascribe too much importance to this link, for dimension *theory* is primarily a modern subject and its problems are essentially of recent date. Nevertheless, in certain ancient writings,

especially texts of ARISTOTLE and EUCLID, we can find some discussion of ideas concerning dimension. Hence in this note I wish to draw attention to some of these ancient roots of dimension theory, but without pretending to examine them in full historical detail.

Among EUCLID's definitions in the *Elements* we find the following (EUCLID/HEATH *(1926: I,153; III,260)*):

Book I

1. A *point* is that which has no part.
2. A *line* is breadthless length.
3. The extremities of a line are points.
5. A *surface* is that which has length and breadth only.
6. The extremities of a surface are lines.

Book XI

1. A *solid* is that which has length, breadth, and depth.
2. An extremity of a solid is a surface.

In this group of definitions there are suggestions of two 'theories' of dimension: a direct theory given by definitions I.1, I.2, I.5, and XI.1 and an indirect theory hinted at by definitions I.3, I.6, and XI.2. It would be interesting to know what problems motivated these and other ancient 'dimension theories', but undoubtedly a rational reconstruction of the problems faced by the ancients would be a very difficult task. In any case, we may examine the theories as EUCLID has given them.

The first 'theory' offers a direct link between the basic objects of geometry and dimension. Points have no dimension ('no part'); while lines have one ('length'); surfaces, two ('length and breadth'); and solids, three ('length, breadth, and depth'). However, EUCLID only touches on the concept of dimension implicitly. There is no explanation of the concept of dimension and, in fact, EUCLID does not use any general term for dimension. So EUCLID's definitions I.1, I.2, I.5, and XI.1 hardly add up to a genuine *theory of dimension.*

In the *Topics* (VI. 5, 142b22–29; VI. 6, 143b11–144a4; *cf.* HEATH *(1949:86–91)*) ARISTOTLE by implication criticises the main definitions and direct theory of EUCLID's *Elements*. He dislikes the negative character of the EUCLIDEAN definitions. Elsewhere ARISTOTLE offers what he considers to be a better direct theory, which attempts to explain dimension on the basis of divisibility (*On the Heavens,* I.1, 268a4–13, 20–b5, and *Metaphysics,* V.6, 1016b23–31; *cf.* HEATH *(1949:159–160, 206–207)*). It is perhaps significant that ARISTOTLE uses a special word, *διάστασις* for dimension.

From the vantage point of modern dimension theory, EUCLID's 'subsidiary' definitions I.3, I.6, and XI.2, which form a second 'indirect' theory joining the basic geometrical concepts together, are much more interesting than his other definitions. Some modern mathematicians have even read into these definitions a hint at a recursive definition of dimension, since there is a progression from points to lines to surfaces to solids. However, we should not make too much of this reading of EUCLID's 'hints'. The ancients can hardly be said to have had a recursive definition of anything.

According to HEATH (EUCLID/HEATH *(1926:I,155–156)*) the subsidiary definitions are older than EUCLID's main definitions. ARISTOTLE in the *Topics* (VI.4, 141a24–142a9; *cf.* HEATH *(1949: 85–86)*) speaks of the subsidiary definitions as *the* definitions. Moreover, ARISTOTLE gives us a supplement to EUCLID's subsidiary definitions (*Metaphysics,* XI.2, 1060b12–17; *cf.* HEATH *(1949:224)*):

> If we suppose lines or what immediately follows them (I mean the primary surfaces) to be principles, these are at all events not separable substances but are sections and divisions, the one of surfaces, the other of bodies (as points are of lines); they are also extremities or limits of the same things; but all of them subsist in other things, and no one of then is separable.

Thus the various geometrical objects can be sections and divisions as well as extremities of the next higher ones in the hierarchy.

However, ARISTOTLE is critical of these definitions (HEATH *(1949:85–86)*):

> All these definitions explain the prior by means of the posterior, for they say that a point is an extremity of a line, a line of a plane, and a plane of a solid.

They put the cart before the horse, for the progression is from the solid down to the point, rather than from the prior concept of point to the posterior concepts of line, surface, and solid. (See ARISTOTLE's theory of definition at the cited passage.) So the older definitions, which possibly suggest a recursion, are, to ARISTOTLE's mind, *unscientific.* BOLZANO *(1804:46–47)* also criticised this progression from solid to point (Section 3), so in essence BOLZANO is in agreement with ARISTOTLE on the ordering of the concepts (see note 6).

parallel postulate, *i.e.*, he did not genuinely understand its independence, he did realise there was a difficulty surrounding the postulate. KÄSTNER's textbook was read by the great early nineteenth century founders of non-EUCLIDEAN geometry and as a result it apparently had some influence. KÄSTNER's treatment of parallels particularly inspired BOLZANO to examine the problem in his very first publication.

From what we know of BOLZANO's early studies in mathematics, we can see that he must have derived his problem of geometrical definitions mainly from BAUMGARTEN and KÄSTNER. (BOLZANO's annotated copy of KÄSTNER's book still exists.) But at that time there were others interested in the basic definitions of geometrical objects. For example, KARL CHRISTIAN LANGSDORF (1757–1834) in his own *Anfangsgründe (1802)*, a rival to KÄSTNER's text, defines lines *etc.* both according to the boundary definitions and as sets of points lying next to one another. Thinking of the old theory of indivisibles of CAVALIERI, he believes the latter definitions can provide better foundations for analysis *(1802: Vorrede)*. JOHANN SCHULTZ (1739–1805), KANT's friend and Professor of Mathematics at Königsberg, in his defence of the philosopher's *Critique of Pure Reason* takes the boundary definitions as the correct ones and sees in them a justification for KANT's doctrine that our idea of space is a pure intuition *(1789:55–62)*. Thus, while BOLZANO's definitional problem may seem to us trivial or misconceived, around 1800 it appeared to be fundamental to several mathematicians. It seemed to go right to the heart of the foundations of geometry. It is no wonder BOLZANO devoted his energies to it.

3. The 'Betrachtungen' of 1804

Betrachtungen über einige Gegenstände der Elementargeometrie (1804) was BOLZANO's first published work. He was but 23 when it appeared. He dedicated it to his teacher Professor STANISLAUS VYDRA, who had just retired because of increasing blindness and who died later in that year. Of the *Betrachtungen* BOLZANO said it is no 'œuvre achevé', but only a trial (Probe), a mere test balloon for his youthful investigations into geometry *(1804: Einleitende Vorrede, viii)*. Nevertheless, this little pamphlet contains the seeds of many of BOLZANO's mature mathematical ideas. Moreover, he retained a lifelong interest in the subjects dealt with in it, for just a few years before his death he wanted to bring out a new edition.[5]

BOLZANO's *Betrachtungen* is heavily impregnated with philosophy. Indeed from the point of view of his intellectual development, his philosophical ideas here are more interesting than his mathematical theories. In the 'Vorrede' BOLZANO stresses the theoretical role of mathematics and its 'usefulness' in exercising and sharpening the mind. Rigour in pure mathematics is uppermost in his thoughts. He opens with two methodological rules concerning proof, rules which always remained a part of his mathematical methodology. In the first rule BOLZANO says

[5] For example, in a letter to PŘIHONSKÝ of 23 February 1844 BOLZANO suggests a new edition of the *Betrachtungen (1956:245)*.

that no sort of evidence for a statement will relieve him of the obligation to prove it, unless he can clearly see exactly why no proof is necessary. One must strive to derive all the truths of mathematics from their ultimate grounds and to obtain the greatest possible clarity and order among all the concepts of science. Thus, BOLZANO's EUCLIDEAN methodology is complete. He even sees this strict logical ordering of concepts and truths as a way to discovering new truths in mathematics!

In a second methodological rule BOLZANO says that he will not be satisfied with a proof if it cannot be derived using just those concepts contained in the thesis to be proved. If a proof uses other accidental, foreign (fremdartig) concepts as well, then it cannot be satisfactory. For example, he regards it as a mistake in geometry to use the plane when considering the theorems about angles, straight lines, and triangles. The plane is foreign to these ideas. This application of his methodological rule, in fact, accounts for the peculiar treatment of geometry given in the *Betrachtungen*.

Another concept which BOLZANO considers foreign to the whole of geometry is the concept of motion. He gives two special reasons why this concept should be banned from geometrical investigations. First, he reckons that the idea of motion involves the notion of a moving object, which is different from the containing space. But the concept of object is, even according to KANT (a philosopher with whom BOLZANO usually disagreed), an empirical concept and foreign to 'pure' geometry. Second, BOLZANO claims the theory of motion actually assumes the theory of space, *i.e.*, geometry, because in order to show the possibility of a certain motion used to prove some geometrical theorem, one must use the geometrical theorem itself for the proof. Hence, there is a vicious circle. We must deal with pure geometry first, before we can consider any motion. Starting from these rather tenuous philosophical arguments, BOLZANO proceeds to investigating *pure* geometry. This purge of motion from geometry is relevant to BOLZANO's dimension-theoretic definitions of line, surface, and solid. For instead of taking a line as the path of a moving point, as for example KÄSTNER was willing to do, BOLZANO attempts to define the concept of line independently of any idea of motion whatsoever.

BOLZANO's central aim in the *Betrachtungen* is to prove the elementary theorems about triangles and parallel lines solely on the basis of an assumed theory of the straight line. His objective, following his second methodological principle, is *not* to use the concept of plane in these derivations. Ultimately he tries to justify the EUCLIDEAN parallel postulate, to prove that it and no other is true. Thus, BOLZANO stays within the confines of EUCLIDEAN geometry, and subsequently he never ventured into any non-EUCLIDEAN geometrical investigations. So the central part of his book is contained in the first part entitled, 'Versuch die ersten Lehrsätze von Dreyecken und Parallellinien mit Voraussetzung der Lehre von der geraden Linie zu beweisen'.

In the second part, 'Gedanken in Betreff einer künftig aufzustellenden Theorie der geraden Linie', BOLZANO outlines a theory of the straight line, upon which the first part of his pamphlet is based. BOLZANO starts with the concepts of direction and distance. By doing this, he seems to be making a primitive notion of metric space or vector or inner product space. BOLZANO *(1804:48–49)* gives the following definitions:

> I. Dasjenige, was dem Puncte *b* in Beziehung auf *a* so zukömmt, dass es *unabhängig* ist von dem *bestimmten Puncte a* (qua praecise hoc est et non aliud); was folglich auch in der Beziehung auf einen *andern* Punct, z.B. α *gleich* vorhanden seyn kann: *genannt die Entfernung des Punctes b von a.*
> II. Dasjenige, was dem Puncte *b* in Beziehung auf *a* so zukömmt, dass es *abhängig ist bloss von dem bestimmten Puncte a;* wovon nun getrennt werde, was schon in dem Begriffe der Entfernung liegt, d.h. was dem Puncte *b* auch in Rücksicht auf noch einen andern Punct zukommen kann: genannt *die Richtung, in welche b zu a liegt.*
>
> (I. That which is associated with the point *b* in relation to [a point] *a* such that it is independent of the determined point *a* (which is precisely this and not another); hence, which could equally be in relation to another point, e.g., α: is called the 'distance of the point *b* from *a*'.
> II. That which is associated with the point *b* in relation to [a point] *a* such that it is dependent just on the determined point *a*; from which is separated what belongs already to the concept of distance, i.e., what we can associate with the point *b* also with respect to some other point: is called the 'direction in which *b* lies to *a*'.)

BOLZANO's 'philosophical' definitions are hardly models of clarity. Yet his intentions are plain. Moreover, the concepts of distance and direction always remained basic to BOLZANO's geometrical research. On this basis BOLZANO offers a definition of straight line segment *(1804:57)*:

> Ein Ding, welches alle jene, und nur jene Puncte enthält, die zwischen den zwey Puncten *a* und *b* liegen, heisst eine *gerade Linie zwischen a und b.*
>
> (An object which contains all and only those points which lie between the two points *a* and *b*, is called a 'straight line between *a* and *b*'.)

In this definition 'zwischen' ('between') is a technical term, previously defined. Of course, we would judge BOLZANO's theory of the straight line to be all but worthless. It is the result of investigating a pseudoproblem. Nevertheless, it is a significant part of the background to BOLZANO's later development of a theory of points, lines, surfaces, and solids.

In two sections of the second part of the *Betrachtungen (1804:46–48)* BOLZANO remarks on the appropriateness of certain definitions in geometry. These remarks have a bearing on his later theory of geometrical objects. He criticises the boundary definitions of solid, surface, line, and point. Although BOLZANO blames WILLIAM OF OCKHAM for these 'wrong' definitions, in effect, he is criticising KÄSTNER, from whom BOLZANO had learned the definitions and who had ascribed them to OCKHAM. BOLZANO's criticism amounts to this. Obviously, we can think of a surface or a line or a point without also referring to some solid which they bound. So the OCKHAM-KÄSTNER definitions are backwards. Conversely, BOLZANO thinks it would be better if we constructed definitions so that the concept of line is based on the concept of point, the concept of surface on that of the line, *etc.* BOLZANO cites LANGSDORF *(1802)* in support of his criticism, but the criticism is of an older date. We can find it in ARISTOTLE.[6]

[6] For the criticism of ARISTOTLE see note 4. BOLZANO repeats his criticism in *(1837:§79, Anmerk.; 369) (1972:113)*. Apparently BOLZANO discovered only later that ARISTOTLE agreed with his criticism *(0000a:32v)*.

Overall we can see in BOLZANO's *Betrachtungen* the seeds of many of his later geometrical ideas and some of the principles of his mathematical philosophy. The book was well received in Prague, in particular by his teacher GERSTNER, and was a good recommendation for BOLZANO's obtaining the professorship in mathematics which VYDRA left on his retirement in 1804. However, BOLZANO's friend JANDERA got the post, mainly because JANDERA had been VYDRA's assistant. Since he was passed over, BOLZANO applied for a new professorship of religious instruction (Religionslehre) at Prague, which he was given at first provisionally in 1805 and then, after some difficulties, permanently on 23 September 1806 (WINTER *(1969:34–40)*).

4. Bolzano's Philosophy of Definitions and Concepts

Throughout his career BOLZANO's mathematical research was strongly biased by his philosophical outlook. After publishing what we might call a piece of 'philosophical mathematics' as his first work, it is not surprising that BOLZANO chose to state his mathematical philosophy in his second publication: *Beyträge zu einer begründeteren Darstellung der Mathematik. Erste Lieferung (1810).* (There exist in manuscript two versions of a planned sequel to the *Beyträge.*) The *Beyträge* contains a number of suggestive ideas which BOLZANO later developed in his *magnum opus* of philosophy, *Wissenschaftslehre (1837).* In this section I shall examine the philosophical ideas most relevant to BOLZANO's geometrical problem, as contained in the *Beyträge, Wissenschaftslehre,* and a manuscript, 'Versuch einer Erklärung der Begriffe von Linie, Fläche und Körper' *(0000a),* which summarises his philosophy in relation to his problem.

BOLZANO opens his *Beyträge* by claiming that, while mathematics among all the sciences stands closest to perfection, it nevertheless embraces several gaps and imperfections—even among its most elementary theories. One of the gaps is in geometry *(1926:8):*

> Hier mangelt es zur Stunde noch an einer bestimmten Erklärung der wichtigen Begriffe: Linie, Fläche, Körper.
>
> (At the present time there is still lacking a precise definition of the most important concepts: line, surface, solid.)

This marks the first time BOLZANO gave a clear statement of his geometrical problem in print. In this philosophical essay he does not attempt to solve the problem, but the philosophy of definitions put forward is directly relevant to it.

The philosophy of mathematics of the *Beyträge* is characteristically 'EUCLIDEAN', similar to that of PASCAL's *De l'Esprit géométrique (cf.* LAKATOS *(1962)).* BOLZANO even tries to outdo EUCLID by insisting on much greater rigour. The long second part of the *Beyträge (1926:30–76)* comprises a philosophical analysis of mathematical method—method as carried out in a EUCLIDEAN deductive system. BOLZANO is critical of previous analyses of the mathematical method and thus he tries to come to some new conclusions in line with his own philosophical outlook. A hallmark of BOLZANO's entire philosophy is his view, already contained in the early *Beyträge,* that mathematical and scientific knowledge is objective. In mathematics and science we are concerned with objective connections among propositions and not with subjective convictions or beliefs. Hence, proof in mathematics is not aimed at persuading the reader or increasing his conviction; rather it is directed towards revealing objective connections between propositions, in particular, between objective truths. Thus we find BOLZANO writing the following in 1810 *(1926:31)*:

> In dem Reiche der Wahrheit, d.h. in dem Inbegriffe aller wahren Urteile herrscht ein gewisser objektiver, von unserer zufälligen subjektiven Anerkennung derselben unabhängiger Zusammenhang, zufolgedessen einige aus diesen Urteilen die Gründe anderer, und diese die Folge jener sind. Diesen objektiven Zusammenhang der Urteile darzustellen, d.h. eine Menge von Urteilen so auszuwählen und aneinander zu reihen, dass jedes, das ein gefolgertes ist, auch als ein solches aufgeführt werde und umgekehrt, scheint mir der eigentliche Zweck zu sein, den wir bei einem wissenschaftlichen Vortrage verfolgen.

> (In the domain of truth, i.e., in the totality of all true judgments, a certain objective connection prevails, which is independent of our accidental subjective recognition of it, and consequently, some of our judgments are the grounds for others which follow from them. To present this objective connection of judgments, i.e., to choose a set of judgments and to order them so that each one which is a consequence is so represented and conversely, seems to me to be the proper aim to pursue in a science.)

This strict deductivist view of scientific knowledge always remained essential to BOLZANO's mathematical research. It is no wonder that he came to a theory of logical consequence in the *Wissenschaftslehre* similar to TARSKI's of one hundred years later.

BOLZANO's geometrical problem is fundamentally a search for appropriate definitions. But what precisely are definitions according to BOLZANO? In the *Betrachtungen (1804:46)* we find the following principle about the nature of definitions:

> ... eine ächte Definition nur solche Merkmale des zu erklärenden Begriffs enthalten muss, die sein *Wesen* ausmachen, und ohne welchen er gar nicht gedacht werden kann ...

> (A genuine definition must contain only such marks of the concept to be defined that reveal its essence and without which it cannot be conceived at all.)

Clearly, BOLZANO took an essentialist position over definitions. However, his conception of a definition is different in the *Beyträge* of 1810 *(1926:32; cf. 32–40)*:

> Die Logiker verstehen unter einer Erklärung (Definition) in dieses Wortes eigentlichstem Sinne die Angabe der nächsten (zwei oder mehreren) Bestandteile, aus welchen ein gegebener Begriff zusammengesetzt ist.

> (Logicians understand by a 'definition' in the most proper sense of this word the statement of the proximate (two or more) parts of which a given concept is composed.)

BOLZANO's notion of a definition is pretty much the same in the later *Wissenschaftslehre (1837: IV, 330–331) (cf. (1837: §§ 350–351, 554–559; III, 397–405; IV, 330–350))*:

> Die erste Art der Betrachtungen über blosse Vorstellungen und Sätze ... sind die *Erklärungen;* worunter ich ... hier eben nichts Anderes verstehe, als Sätze, welche bestimmen, ob eine gewisse Vorstellung oder ein Satz einfach oder aus Theilen zusammengesetzt sey, und in dem letzteren Falle, aus was für Theilen, und in welcher Verbindung derselben er bestehe.
>
> (The first kind of considerations concerning mere ideas and propositions are definitions; by which I understand here nothing other than statements which declare whether a certain idea or proposition is simple or composed of parts, and in the latter case, of what parts it consists and how they are connected.)

So definitions are reports of analysis of concepts. A definition tells us either that a certain concept or idea is simple or that it is complex and is composed of certain other ideas joined together in a particular way. Thus a definition is either true or false, depending on whether or not it gives a correct report of an analysis of a concept. BOLZANO's theory of definitions and the analysis of concepts is one of the less original parts of his philosophy. It is deeply rooted in Continental rationalist philosophy of the seventeenth and eighteenth centuries and is most closely akin to LEIBNIZ' logical theory of concepts and definitions.[7]

BOLZANO's theory of definitions takes us into his world of propositions and ideas in themselves, probably the most characteristic part of his philosophy. There are indications of the theory of objective propositions and ideas already in the early *Beyträge,* but these are only fully explored in the *Wissenschaftslehre (1837: Erster Theil, Zweiter Theil)*. According to the latter work, propositions and ideas in themselves are both mind-independent *(1837: §§ 19, 48; I, 76–80, 215–218) (1972: 20–23, 61–62)*. They are to be strictly differentiated from written, spoken, or mental propositions and ideas. An objective idea or proposition is independent of whether anyone has ever thought of it or expressed it. They are more or less the objective contents, or meanings, of expressed or conceived ideas and propositions. Clearly, it is difficult to pinpoint the exact ontological status of objective propositions or ideas. For example, BOLZANO emphasises that, strictly speaking, they do not exist and they do not have reality. Only subjective ideas and propositions really exist. The objective counterparts to subjective propositions and ideas only 'subsist' in what we might call a world of meanings. Apparently, BOLZANO's objective entities are close to being in POPPER's world three. Now a proposition in itself asserts that something is or is not the case, so it can be true or false, whereas an idea in itself is only a part of a proposition and, consequently, does not have a truth value. Propositions and ideas in themselves provide the central core of BOLZANO's philosophical system. Subjective ideas and propositions, spoken, written, or only thought, are secondary in importance. They are, in fact, the manifestitations in the mind of the objective entities. BOLZANO thinks that

[7] Compare particularly LEIBNIZ, 'Dialogue on the Connection between Things and Words' *(1969: 182–185)* and 'On Universal Synthesis and Analysis, or the Art of Discovery and Judgment' *(1969: 229–234)*.

definitions are one way to get at the objective world via the subjective mental world.

The basic principle of BOLZANO's theory of definition is the tenet that we can break down or analyse (at least some) complex ideas into simpler ideas until we reach absolutely simple ideas. (Propositions, too, can be analysed and thus defined, but we can neglect this aspect of BOLZANO's theory.) Thus all ideas are composed of simple ideas, the atomic building blocks of BOLZANO's conceptual universe. In attempting to discover a definition, we try to find these simple components of an idea. The simple ideas, while not 'existing', subsist independent of our minds. They are not our mere inventions. Consequently, in a search for definitions we are trying to discover 'real' definitions, which point beyond our subjective world. Accordingly, definitions are not arbitrary. If correct, they are true reports of concept analyses. Moreover, a true definition for some particular idea is unique, since it gives the complete breakdown of the parts of the idea.

Do simple ideas in themselves really occur? In the *Beyträge, Wissenschaftslehre,* and 'Versuch einer Erklärung....' BOLZANO tries to prove that they do. The argument of the *Beyträge (1926:33)* is straightforward, if inconclusive. If there were no simple concepts (=ideas in *Wissenschaftslehre*), then we could continue dividing our concepts into infinity. But, according to BOLZANO, this is inconceivable. Apparently, he thinks such an infinite regress is absurd. BOLZANO's argument for simple ideas in the *Wissenschaftslehre (1837: §61; I,263–265)* is less direct and rather obscure. In this later work BOLZANO is willing to countenance complex objects with infinitely-many parts, so that a finite number of subdivisions will not reduce them to simples. BOLZANO uses an analogy from geometry. Any line, surface, or solid is made up of infinitely-many parts, *i.e.* points, which cannot be arrived at by subdivisions. BOLZANO stakes his argument for simple ideas on the bald assertion that any complexity must be explained by parts that are simple. Complex parts of complex wholes can never give an adequate explanation for the wholes. Now this is hardly an argument, more a metaphysical dogma. However, the example from geometry is interesting. Points are the simple constituents of other geometrical objects. Perhaps the concept of point will be a simple part of the concepts of line, surface, and solid. Here and elsewhere BOLZANO suggests that 'point' is a simple concept, but he does not state this explicitly.

In the *Beyträge* there is not much indication of whether it is difficult to find a definition, but in the *Wissenschaftslehre (1837: §§350–351; III,397–405)* BOLZANO emphasises that the task is normally quite difficult. Only when we construct concepts through 'synthetic' definitions is it easy to provide a definition. 'Analytic' definitions, *i.e.* definitions which yield the correct analysis of an idea, are virtually always hard to find. As a result BOLZANO gives several heuristic rules to aid in the search for true definitions. For example, in the *Beyträge* he sets down two rules to help distinguish simple from complex ideas. In the *Wissenschaftslehre (1837: §§350–351; III,397–405)* as well as 'Versuch einer Erklärung....' he offers fuller hints for discovering definitions. In particular, he says we must make sure that a definition is not too wide or too narrow, but that it covers exactly the objects which fall under the concept. Additionally, after a definition has been found, we must justify it, *i.e.* prove it to be correct. After reading BOLZANO's cautious text, one feels that he found it difficult to put down any really helpful rules for discovering definitions. Perhaps he vaguely realised it is nearly impossible to find a definition in his sense. So he offered a way out. Instead of trying to give definitions in all cases, it is easier to provide approximate 'descriptions' ('Verständigungen') in order to acquaint someone with the meaning of a term *(1837: § 668; IV,542–551)*.

BOLZANO's old-fashioned essentialist theory of definitions with its reliance on a postulated world of absolutely simple ideas is fraught with so many difficulties that virtually no contemporary logician or philosopher is likely to agree with it. BOLZANO's assumption of hard and fast concepts with clearly distinguishable simple parts is too sweeping, too metaphysical to accept. Yet BOLZANO's theory of definitions and the analysis of concepts directly motivated his research in geometry. He viewed his task as a search for true definitions of lines, surfaces, and solids and species of these. BOLZANO himself realised his task was difficult, for he came up with two theories. As we shall see (Section 7), the first proved inadequate. For example, his definition of line turned out to be too narrow. Hence, he constructed new definitions in a second theory, which again he did not find entirely satisfactory. The surprising feature of BOLZANO's work is that he was able to obtain topologically interesting insights and results *in spite of his restrictive theory of definitions.*

5. First Solution: The Geometrical Definitions of 1817

During[8] the same year in which he brought out his well-known *Rein analytischer Beweis,* 1817, when he was at the height of his teaching career, BOLZANO published another significant work: *Die drey Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen, ohne die Annahmen des Archimedes und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst; zugleich als Probe einer gänzlichen Umstaltung der Raumwissenschaft, allen Mathematikern zur Prüfung vorgelegt (1817).* In this pamphlet BOLZANO seeks rigorous proofs—proofs which achieve the standards laid down in the *Beyträge*—for the usual integral formulae for the length of a curve, the area of a surface, and the volume of a solid. All previous demonstrations are unacceptable to BOLZANO and it is evident from his citations that he has surveyed a vast number of these. In particular, he censures those who use infinitesimals. Such criticism was an integral part of BOLZANO's early arithmetisation programme. Also he finds fault with those who base their arguments on the so-called axioms of ACHIMEDES, *viz.*, the assumption that of two curved lines with the same chord the one enclosed within the chord and the outer curve must be the shorter and a similar assumption for surfaces. BOLZANO's extensive critical remarks in the 'Vorrede' demonstrate his desire to stay within the bounds of the rigorous 'philosophical' programme of his *Beyträge.* However, judged by later standards of rigour (STOLZ *(1881: 267–268)*), his analytic results are not altogether successful.[9]

[8] In the text I have passed over BOLZANO's attempt to prove the 3-dimensional character of physical space, *(1845) = (1948b)*. In this paper he does not use the dimension-theoretic basis of neighbour, *etc.* which he employed for his attempts to define line, surface, solid, and continuum. In fact, he does not give a definition of dimension as such in *(1845)*. Rather he uses ideas of analytic geometry somewhat akin to the theory of vector spaces.

[9] One can compare early reviews of BOLZANO *(1817): Allgemeine Literatur-Zeitung (1819: columns 180–184); Leipziger Literatur-Zeitung (1822: 1393–1403).*

Embedded in this work of analysis, we find BOLZANO's first published solution to his fundamental geometrical problem, a group of modern-sounding 'topological' definitions of the basic objects of geometry *(1817:20–30,51–54,66)*. BOLZANO's definitions show the high standard of rigour which he sought in the foundations of geometry and, inasmuch as they appear in a book on mathematical analysis, they suggest that he was groping for some kind of 'geometrico-topological' basis for the entire edifice of analysis. BOLZANO's whole approach to his geometrical problem strikes one as very similar to KARL MENGER's intuitive approach to dimension during the early 1920's.

Contrary to his own early criticism of BAUMGARTEN (see Section 2), BOLZANO begins his analysis of geometrical concepts by assuming that any geometrical figure is a set of points *(1817:20)*:

> *Raumding* heisst überhaupt jedes *System* (jeder Inbegriff) *von Puncten* (sie mögen in endlicher oder unendlicher Menge vorhanden seyn).
>
> ('Spatial object' is in general the name for any system (any totality) of points (it may consist of a finite or an infinite set).)

Thus BOLZANO immediately puts his geometry on a set-theoretic basis. But the main objects of geometry, lines, surfaces, and solids, are not *mere* sets of points, for they are connected in a certain way. BOLZANO sees the objective of his definitions as a way to explain how points are composed together to make lines, surfaces, and solids. For example, the definition of line reads as follows *(1817:20)*:

> Ein Raumding, zu dessen jedem Puncte es [,] anzufangen von einer gewissen Entfernung für alle kleineren abwärts, wenigstens *einen,* und höchstens nur eine *endliche* Menge von Puncten als Nachbarn gibt, heisst eine *Linie überhaupt* (Fig. 1-7.).
>
> (A spatial object with the property that every point of it has at least one and at most only a finite set of points as neighbours, corresponding to each distance smaller than a certain given distance, is called a 'line' in general.)

By a 'neighbour' of some point of a spatial object with respect to a given distance, BOLZANO really means a point of the intersection of the spatial object with the surface of a sphere with radius equal to the given distance. Thus BOLZANO uses spherical neighbourhoods to determine his neighbours, although he doesn't say this explicitly. This interpretation of his neighbour concept is only revealed by the way he applies it and through the diagrams he gives. In modern terms he defines a line or curve as a point set with the property that the boundaries of arbitrarily small spherical neighbourhoods have finite intersections with the point set. Continuing with BOLZANO's main series of definitions, we have *(1817:51,66)*:

> Ein Raumding, zu dessen jedem Puncte es, anzufangen von einer gewissen Entfernung für alle kleineren abwärts, wenigstens *eine*, und höchstens nur eine *endliche* Menge *getrennter Linien* voll Puncte gibt, heisst eine *Fläche überhaupt* (Fig. 13-19.).
>
> (A spatial object with the property that every point of it has at least one and at most only a finite set of separate lines full of points, corresponding to each distance smaller than a certain given distance, is called a 'surface' in general.)

Ein Raumding, zu dessen jedem Puncte es, anzufangen von einer gewissen Entfernung für alle kleineren abwärts, wenigstens *eine* durchaus zusammenhängende Fläche voll Puncte gibt, heisst *Körper überhaupt* (Fig. 20 und 21.).

(A spatial object with the property that every point of it has at least one absolutely connected surface full of points, corresponding to each distance smaller than a certain given distance, is called a 'solid' in general.)

These definitions of the typical geometrical objects of 1, 2, and 3 dimensions are acute, especially if we consider the time at which they were written.

It is interesting to compare BOLZANO's definition of line (1817) with MENGER's definition of a regular curve as a continuum (1925), each of whose points has the property that every neighbourhood of the point contains a neighbourhood (also of the point) whose boundary has a finite intersection with the continuum (MENGER *(1925)*, *(1932:96)*). To be sure, there are subtleties in MENGER's definition which are missing in BOLZANO's. MENGER's definition requires that for each neighbourhood of a point on the curve we can find a neighbourhood within the given one such that its boundary has a finite intersection with the curve. BOLZANO's definition says that *every spherical neighbourhood* with sufficiently small radius has a boundary which has a finite intersection with the curve. To see the distinction between the two definitions, let us look at BOLZANO's figure 7. As he indicates *(1817:22)* point c has a whole line full of neighbours at the distance cr. Nevertheless, this figure still qualifies as a line, because for distances smaller than cr, e. g., co, the point c has only two neighbours. Now we can modify the example to disqualify it as a line and to see how BOLZANO's definition differs from MENGER's. The modified figure accords with BOLZANO's thinking of a few years later, although he never gave the figure as such. In BOLZANO's figure 7 just insert circular arcs around c between cr and cs at distances $(\frac{1}{2})^n \overline{cr}$. This new figure is neither a line nor a surface in BOLZANO's realm of concepts. Yet on intuitive grounds it seems to be 'linelike'.

BOLZANO sometimes looks upon a group of unconnected curves as a single line. His figure 1, a hyperbola, is such a case. One may conceive of the hyperbola as 'connected' by a single equation, although it is separated into two branches. Nevertheless, BOLZANO regards connectedness, in the sense of being in one piece, as an important property for lines and defines it as follows *(1817:20–21)*:

Ein Raumding, dessen jeder Theil, der sich nach eben gegebener Erklärung als Linie ansehen lässt, mit dem noch übrigen Theile, der sich dann gleichfalls als Linie muss ansehen lassen, wenigstens *einen Punct* gemein hat, heisst *eine durchaus zusammenhängende Linie* (Fig. 2-7.).

(A spatial object is called an 'absolutely connected line', if every part which can be considered as a line according to the given definition has at least one point in common with the remaining parts, which must themselves also be considered as lines.)

In order to make sense of BOLZANO's definition of connectedness for lines, we must assume that he thinks a line segment or an arc always includes its endpoints. If we do not make this assumption, then his definition of connectedness falls to pieces. Even the segment [0,1] would be disconnected. In fact, BOLZANO does define lines with endpoints *(1817:21)*:

> Eine Linie..., in der es Puncte gibt, die anzufangen von einer gewissen Entfernung für alle kleineren abwärts, nur *einen* Nachbarn haben, heisst eine begrenzte Linie (Fig. 2, 5, 6.).
> Jene Puncte in ihr (z. B. *e, e*) heissen *Grenz-* (oder *End-*) *puncte*; die übrigen (z. B. *m, m*) *innere* Puncte.
>
> (A line in which there are points which have only single neighbours, corresponding to each distance smaller than a certain distance, is called a 'bounded line' [or more accurately, a 'line with endpoints'].
> Those points are called 'boundary-' or 'endpoints'; the rest, 'interior points'.)

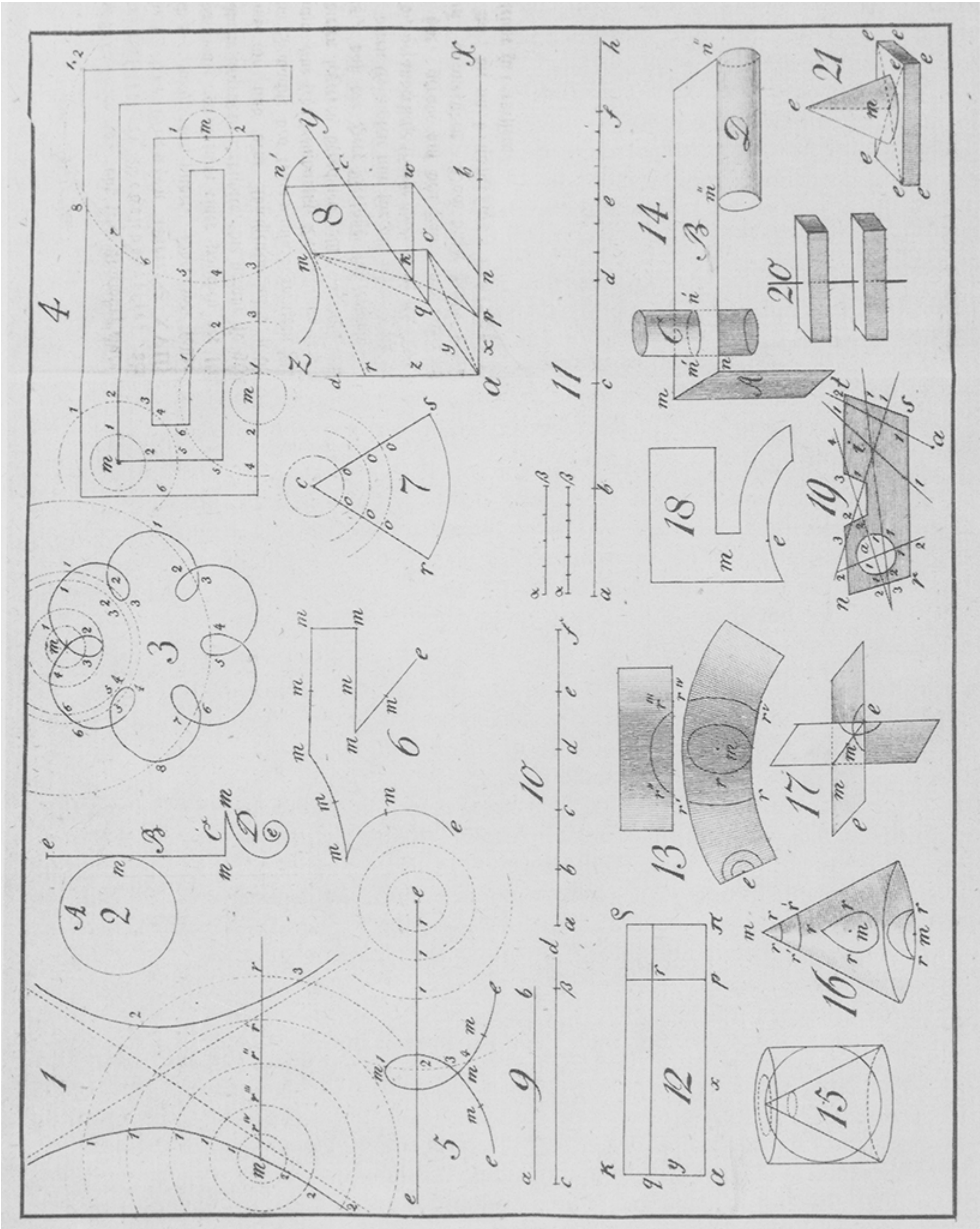
Yet in the opening definition of line (given above) BOLZANO does not explicitly require segments to contain their endpoints; only his figures suggest that this is a requirement. During the 1840's when he made a second attempt to define the concept of line, he definitely did not wish to require that segments or arcs necessarily include their endpoints.

BOLZANO also defines connected surfaces and solids, relying on the tacit assumption that nonclosed surfaces include their perimeters and solids, their boundary surfaces *(1817:51–52,66)*:

> Ein Raumding, dessen jeder Theil, der sich nach eben gegebener Erklärung als Fläche ansehen lässt, mit dem noch übrigen Theile, der sich dann gleichfalls als Fläche muss ansehen lassen, wenigstens *eine Linie* gemein hat, heisst eine einzige durchaus *zusammenhängende Fläche* (Fig. 14-19.).
>
> (A spatial object is called a 'single absolutely connected surface', if every part which can be considered as a surface according to the given definition has at least one line in common with the remaining parts, which must themselves also be considered as surfaces.)
>
> Ein Raumding, dessen jeder Theil, der sich nach eben gegebener Erklärung als Körper ansehen lässt, mit dem noch übrigen Theile, der sich nun gleichfalls als Körper muss ansehen lassen, wenigstens *eine Fläche* gemein hat, heisst ein *einziger durchaus zusammenhängender Körper* (Fig. 21.).
>
> (A spatial object is called a 'single absolutely connected solid', if every part which can be considered as a solid according to the given definition has at least one surface in common with the remaining parts, which must themselves also be considered as solids.)

A further implication of BOLZANO's definitions of lines, surfaces, and solids and their connected varieties is that the lines, surfaces, and solids must be homogeneously 1-, 2-, and 3-dimensional. For example, a square surface with a line segment sticking out from it or two disjoint circular areas joined together by a line cannot be connected surfaces according to BOLZANO's definitions. Whether BOLZANO fully understood this implication is hard to tell, but he probably was vaguely aware of it.

Among his several further definitions BOLZANO lays down defining conditions for simple, closed, and simple closed lines as well as surfaces. His quasi-topological method is eminently suitable for this task. For example, simple closed lines are lines made up of points having exactly two neighbours for sufficiently small



The sheet of figures from BOLZANO's *Die drey Probleme (1817)*.

distances and such that the distances between pairs of points on the line never exceed some specified maximum (Fig. 4,7) *(1817:21)*. This definition is all the more interesting, when one finds that some years later BOLZANO thought of stating the JORDAN curve theorem for simple closed curves and asserting that the 'theorem' needed a proof!

BOLZANO's 'topological' theory of geometrical figures as unfolded in the twenty definitions of his pamphlet of 1817 is certainly general, in fact, so general that it is of no use to him in the analytic part of the work. He admits this himself *(1817:21)*:

> Diese Erklärungen... muss man nicht nothwendig inne haben, um die hier vorkommende Theorie verstehen und beurtheilen zu können. Gleichwohl schien es mir zweckmässig, ihnen hier einen Platz zu vergönnen; wäre es auch nur, um gelegenheitlich einen kleinen Vorbegriff von der Beschaffenheit jener gänzlichen *Umstaltung der Geometrie* zu geben, an der ich schon seit Jahren gearbeitet, bisher aber erst sehr wenige Proben dem Publico mitgetheilt habe.
>
> (One does not have to keep these definitions in mind in order to be able to understand and judge the present theory [of analysis]. Nevertheless, it seemed appropriate to put them here, even if only to give a glimpse of the structure of the entire domain of geometry, upon which I have worked for several years, but about which I have communicated very few examples to the public before now.)

In his rectification theory BOLZANO confines his attention to 'determined' or 'determinable spatial objects', roughly speaking, those figures which can be specified by sets of equations. So in *Die drey Probleme* the general topological theory quickly drops out of the picture.

BOLZANO undoubtedly thought he achieved more through his twenty definitions than we might think he did. After all, he felt that he was searching for the *true* definitions of geometry. However, in the pamphlet of 1817 there are *no* theorems or proofs which rely on his definitions and a mere set of definitions does not constitute a complete mathematical theory. Perhaps we shall find more in his second period of research on his geometrical problem.

6. Bolzano's Later Programme in Geometry

Two years after BOLZANO published his highly significant mathematical tracts of 1817, he lost his post as Professor of Religious Instruction at Prague. Emperor FRANZ I signed the final decree on 24 December 1819. It was almost inevitable that BOLZANO's liberal views on moral and political matters should come into conflict with those of the civil and church authorities during this time of the 'Austrian-Catholic Restoration' (see WINTER *(1969:52–71)*) With the enforced end of his teaching career, BOLZANO could devote his full energies to a study of logic —logic as the foundation of all science. The result of his researches into logic and the philosophy of science, which took up most of his time during the decade 1820 to 1830, was the *Wissenschaftslehre,* first published in 1837.

Around 1830, after the decade devoted to studies in logic, BOLZANO turned his attention to mathematics, among other things.[10] During the 1830's and '40's, the last two decades of his life, BOLZANO investigated a number of mathematical topics, wrote several short papers, and contemplated the execution of a large-scale mathematical treatise, the *Grössenlehre*. This is the time of his second period of interest in his geometrical problem. There are five relevant documents which belong to this second period. Among these we have two programmatic works: 'Versuch einer Erklärung der Begriffe von Linie, Fläche und Körper' *(0000a)* and 'Anti-Euklid' *(1967)*. The former paper was probably written around the time of composing the *Wissenschaftslehre*. It contains a single reference to his book on logic *(0000a:20r)* and is partly a summary of passages of the *Wissenschaftslehre* which are relevant to the geometrical problem. Thus it seems most likely that the 'Versuch' was composed around 1830 or a few years after this date. The other programmatic paper, 'Anti-Euklid', (only recently discovered) is almost certainly of a later date, perhaps stemming from the late 1830's or early 1840's. It gives the broad outlines of a mathematical programme which BOLZANO worked on during the 1840's, but was not able to complete. He hoped that his mathematical friends would be able to finish the task.

Falling under this final mathematical programme are three works: 'Uiber Haltung, Richtung, Krümmung und Schnörkelung bei Linie sowohl als Flächen sammt einigen verwandten Begriffen' *(1948d)*, 'Geometrische Begriffe, die Jeder kennt und nicht kennt. Versuch einer Erhebung derselben ins deutliche Bewusstseyn' *(0000b)*, and *Paradoxien des Unendlichen (1851)*. The first two papers, left incomplete by BOLZANO, were written during 1843–44, and are the result of a burst of mathematical activity which he had during the 1840's. They are two of his final attempts to get his mathematical ideas down on paper, in order that others might carry out his programme. The *Paradoxien des Unendlichen* contains, *inter alia*, a summary of his last geometrical ideas, in particular, the ideas relating to dimension.

In this section I shall examine BOLZANO's programmatic papers, while in the next, his partial working out of the programme.

More than half of the incomplete 'Versuch einer Erklärung der Begriffe von Linie, Fläche und Körper' is taken up with a summary of BOLZANO's theory of definitions and concepts, a theory which finds fuller expression in the *Wissenschaftslehre* (see Section 4). Then in the second part of the 'Versuch' BOLZANO makes fresh attack on his geometrical definition problem *(0000a:18r–33v)*. Since lines, surfaces, and solids all are kinds of spatial extension, he first proceeds to analyse the concept of extension (Ausdehnung), never returning in this unfinished paper to a specific analysis of the concepts of line, surface, and solid.

What is an extension? In the first place, BOLZANO rejects the idea that an extension must be continuous (stetig) in the sense of connected, because he does not want to exclude curves which fall into two or more parts. For example, he wishes to include the curve defined by the equation $y=\sqrt{(1-x)(2-x)(3-x)}$, comprising two separate pieces, as a single extension. This view is the same as

[10] Compare various letters in the BOLZANO-PŘIHONSKÝ correspondence on the subject of BOLZANO's renewed interest in mathematics during the 1830's and 40's: *(1956:121,154,220,232,240, 241,245,252)*.

the view in the pamphlet of 1817. But in the second place, he does not want to regard finite sets of points as proper extensions. Extensions are certainly types of 'wholes' or 'totalities' (Ganze, Inbegriff), but they are not merely sets (Summe, Menge) with no form or order. They are somehow united (zusammengesetzt) into a whole. The problem is how to explain the connection of the parts in a geometrical extension. Thus why can we not regard a set of, say, 100 points as an extension? BOLZANO's answer is that the parts, *i.e.* the points, are not united (verbunden), but are isolated; they stand alone (vereinzelt oder isolirt stehen). Corresponding to each point in a finite system of points there is a smallest distance for which the point has a neighbour, while for all smaller distances the point has no neighbours. Consequently, in order to have points joined together into an extension the opposite must be the case, *viz.* for none of the points can there be a smallest distance such that for smaller distances the point has no neighbours. But is this condition sufficient to guarantee that a collection of points form an extension? BOLZANO answers 'no'. Consider the following example. Take two points a and b and insert a point c half way between them; then insert points half way between a and c, and c and b; and so on. Each point in this infinite system has neighbours for certain small distances, but not for *all* arbitrarily small distances. For example, a does not have neighbours at the distances $\frac{1}{3}\overline{ab}$, $\frac{1}{5}\overline{ab}$, *etc.* Thus BOLZANO ends up with the following definition of extension *(0000a:22v)*:

> Ein jeder Inbegriff von Punkten würde hiernächst ein *ausgedehntes Raumding* heissen, falls ein jeder Punkt ... mit andern verbunden ist, d.h. Nachbarn in diesem Raumdinge findet, die ihm so nahe, als man nur immer will, treten; oder noch anders, sobald sich für eine jede auch noch so kleine Entfernung Punkte im Raumdinge fänden, die diese Entfernung von ihm haben.
>
> (Every totality of points will be hereafter called an 'extended spatial object', in case each point is united with the others, i.e. it has neighbours in this spatial object which are as close as you please; or, in other words, for every small distance there are points in the spatial object at this distance from it.)

This definition of an extended spatial object is the one which BOLZANO finally settled on for his last attack on his geometrical problem. Although the terms may be different in the papers of the 1840's ('ausgedehntes, continuirliches Raumding', 'Kontinuum'), the definition remains the same. In the rest of the incomplete 'Versuch' BOLZANO tries to justify his definition of extension or continuum. He attempts to show that it is not too narrow and not too wide, but that it is the *right* definition *(0000a:22v–32v)*. Here BOLZANO's essentialism comes to the fore. His deliberations show the close link between his philosophical view and his mathematics.

At this stage in his research we see that BOLZANO was heading for a new conception of geometrical objects based on the concept of isolated point. This concept is implicit in the 'Versuch', and in his paper of 1843–44, 'Uiber Haltung', there is an explicit definition *(1948d:143)*:

> Wenn der Punct i in einem Raumdinge so liegt, dass keine auch noch so kleine Entfernung angeblich ist, von der behauptet werden könnte, *für diese und für alle kleineren Entfernungen,* die es nur überhaupt gibt, besitze i einen

oder etliche Nachbarn: so sage ich, *i* stehe *isolirt* oder *vereinzelt* in diesem Raumdinge.

(If the point *i* so lies in a spatial object that there is no distance, however small, for which it can be asserted that for this distance and for all smaller distances *i* possesses one or several neighbours, then I say that *i* stands isolated or alone in the spatial object.)

Using this definition, we may describe a BOLZANO-extension or continuum as a set with no isolated points.

BOLZANO's concept of isolated point is quite different from that in present-day topology, which has come down to us from CANTOR.[11] Rather BOLZANO's concept of isolated point resembles the notion of a point at which a set is 0-dimensional in the sense of URYSOHN-MENGER dimension theory (*cf.* HUREWICZ & WALLMAN *(1948:10)*):

A separable metric space (in particular, a Euclidean space) has *dimension 0 at a point p,* if p has arbitrarily small neighbourhoods with empty boundaries, i.e. if for each neighbourhood U of p there exists a neighbourhood V of p such that $V \subset U$ and $\mathrm{bdry}\ V = \phi$.

It is not difficult to see that a BOLZANO-isolated point in a set must also be a point at which the set is 0-dimensional in URYSOHN-MENGER's sense. However, the converse is not true, since BOLZANO restricts his attention to spherical neighbourhoods.[12]

In his works of the '40's BOLZANO gives some interesting examples of isolated points. For example, in the 'Uiber Haltung' *(1948d:144)* BOLZANO exhibits an infinite set, all of whose points are isolated: the set of points x of the form

$$x = \frac{m}{2^n},$$

where m, n are integers. In the *Paradoxien des Unendlichen (1921: § 41;82)* we find the following example. Consider line segment az. Let point b be the midpoint between a and z; c, the midpoint between b and z; d, the midpoint between c and z; and so forth. If we take the set consisting of the segment az minus the midpoints but including z, then z is an isolated point and so the set cannot be a BOLZANO-continuum. However, if we remove z from the set, we are left with a set with no isolated points and, hence, a BOLZANO-continuum. Clearly, BOLZANO is well aware of the subtle distinction between his continua and noncontinua.

Some decades after BOLZANO's death GEORG CANTOR felt obliged to criticise the former's notion of continuum *(1883a) = (1932:194)*. CANTOR, of course, thought his own definition of continuum was better.

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[11] The modern concept of isolated point stems from a paper of CANTOR *(1883:51) = (1932:158)*. Briefly a point is isolated in a set if there is a neighbourhood of the point containing no other point of the set (apart from the isolated point under consideration). It is easy to see that an isolated point in the modern sense must also be BOLZANO-isolated.

[12] Consider the following example of the set of rational numbers in the interval $[-1, 0]$ union the set of irrational numbers in the interval $(0, 1)$. This set is 0-dimensional at the point 0, but 0 is not a BOLZANO-isolated point.

> Die Bolzanosche Definition des Kontinuums *(Paradoxien § 38)* ist gewiss nicht richtig; sie drückt einseitig bloss *eine* Eigenschaft des Kontinuums aus, die aber auch erfüllt ist bei Mengen, welche aus G_n dadurch hervorgehen, dass man sich von G_n irgendeine 'isolierte' Punktmenge (man vgl. *Math. Ann.* Bd. 21, S. 51) entfernt denkt; desgleichen ist sie erfüllt bei Mengen, welche aus mehreren getrennten Kontinuis bestehen; offenbar liegt in solchen Fällen kein Kontinuum vor, obgleich nach Bolzano dies der Fall wäre.
>
> (The Bolzano definition of continuum is certainly not correct; it exclusively expresses just *one* property of a continuum, which, however, is also fulfilled by sets which result from G_n by imagining any 'isolated' point set removed. Likewise it is fulfilled by sets consisting of several separated continua. Clearly, in such cases we have no continuum, although according to Bolzano this would be the case.)

CANTOR's criticism is misplaced, at least in part. On the one hand, CANTOR seems to confuse his own notion of isolated point (the modern one) with BOLZANO's. It is by no means true that any set of EUCLIDEAN n-space (G_n) with some set of 'CANTOR'-isolated points removed is a BOLZANO continuum. The straight line minus all the points $(\frac{1}{2})^n$ and—$(\frac{1}{2})^n$ is a set such as CANTOR describes, yet, because 0 is BOLZANO-isolated in it, it is *not* a BOLZANO-continuum. On the other hand, CANTOR is right when he says that a BOLZANO-continuum may be composed of several pieces. BOLZANO fully recognised this possibility; he just did not always require his continua to be connected. Clearly, BOLZANO's and CANTOR's ideas diverge. In fact, CANTOR's ideas are the antecedents of the modern notion of a connected continuum, whereas BOLZANO's are the forerunners of the concept of a set of positive dimension.

We now turn to look briefly at BOLZANO's other late programmatic paper, 'Anti-Euklid' *(1967)*. This short sketch, covering just a few sheets, offers a vigorous critique of EUCLID's presentation of geometry, thereby suggesting an improved version of traditional geometry. BOLZANO criticises the whole basis for the proofs of EUCLIDEAN geometry. Constructively, he seeks a 'learned' or rigorous presentation (eine gelehrte Darstellung) of EUCLIDEAN geometry, given according to the principles of science laid down in the *Wissenschaftslehre*. In a rigorous development of geometry BOLZANO requires that we proceed from simplest concepts to more complex ones through scientific definitions and from simplest truths to complex truths through objective proofs. In essence, this is his old programme for geometry as outlined in his earliest works, the *Betrachtungen* of 1804 and the *Beyträge* of 1810.

BOLZANO cites many examples of concepts which, according to his view, lack rigorous definitions in traditional presentations of geometry. Even the concept of space itself has no definition. Not surprisingly, BOLZANO demands definitions for the well-known simpler concepts of geometry: extension, line, surface, solid, as well as distance and direction. This is just a restatement of his old problem. However, he also goes on to demand definitions for a whole host of other concepts, thereby vastly extending the scope of his geometrical problem. For example, he wants definitions for the two sides of each point on a line, of each line on a surface, of each surface in a solid as well as definitions for closed lines and surfaces and when a point lies on the interior or the exterior of a closed line drawn on a surface. It is certainly most interesting that BOLZANO should want to have a definition for this last 'topological' situation.

In a similar way BOLZANO criticises previous expositions of geometry for not giving adequate proofs or no proofs at all for several interesting propositions. Undoubtedly the most significant theorem which BOLZANO cites as lacking a proof in the geometry textbooks comes at the end of his manuscript (not in *(1967)*):

> *Lehrsatz:* Wenn eine in sich zurückkehrende Linie in einer Ebene liegt, und man verbindet einen Punct derselben Ebene der von ihr eingeschlossen wird mit einem anderen Puncte dieser Ebene, der aber von ihr nicht eingeschlossen wird, durch eine zusammenhängende Linie, so muss diese die zurückkehrende Linie schneiden.
>
> (Theorem: If a closed line lies in a plane and if by means of a connected line one joins a point of the plane which is enclosed within the closed line with a point of the plane which is not enclosed within it, then the connected line must cut the closed line.)

Here we have BOLZANO's prophetic version of the celebrated JORDAN curve theorem. It may seem almost incredible to find it as part of BOLZANO's critical programme for reorganising elementary geometry. Yet the motivation for his statement of the theorem is easy to detect. It is the old criticism of EUCLID's first proof concerning the construction of an equilateral triangle on a given line segment, which WOLFF and KÄSTNER knew about before BOLZANO. But in contrast, BOLZANO's insight into the problem is deeper than that of his predecessors, for he sought to base this and other elementary geometrical theorems on a rigorous foundation of topology. Thus in the mid-1840's, unfortunately when his strength was fading, BOLZANO started to carry out the ambitious programme adumbrated in 'Anti-Euklid'.

7. Second Solution: Geometrical Investigations of the 1840's

We now come to BOLZANO's final working out of his fundamental geometrical problem in works of the 1840's. The two papers written during 1843–44, 'Uiber Haltung' and 'Geometrische Begriffe', are the most significant; *Paradoxien des Unendlichen (1921: §§ 38, 40–41; 71–75, 78–83)* only has a very brief summary of his last results. Unfortunately, these final investigations are incomplete. BOLZANO left neither of the two main papers in anything like a finished state. In the 'Vorwort' to 'Uiber Haltung' he freely admits that this work must be considered as an 'incomplete attempt' ('unvollständiger Versuch'). He even says that he has not carried out the calculations required for a complete solution of the problem under examination. He says he lacks the necessary strength, but hopes that someone else will finish the task. In his last years BOLZANO was certainly not well, but his interest in his research remained unabated.

In the paper 'Uiber Haltung' *(1948d)* BOLZANO is especially concerned with the problem of curvature, particularly the curvature of skew (nonplanar) curves in space. He is not satisfied with previous efforts to characterise the 'double curvature' of skew curves. No arc of such curves, however small, is planar; hence, the usual circle of curvature fails to delineate the full idea of curvature for these curves, just as the straight line tangent to a planar curve fails to embrace the curvature in this instance. BOLZANO also rejects an old suggestion of NEWTON to use a parabola touching the skew curve, for it is no better than the circle. Consequently, what BOLZANO is trying to find is some measure of torsion for spatial curves. His partial 'solution' is to use osculating cylindrical helices as a way of measuring the spatial curvature of skew curves, but he realises that this is not a complete solution *(1948d: 177–181)*.[13]

Curvature is not the only problem which BOLZANO intended to deal with in the 'Uiber Haltung'. He planned to examine a whole class of neighbourhood properties of both curves and surfaces. In fact, he coined the term 'Haltung' as a general name for these neighbourhood properties. In particular, 'Haltung' is supposed to cover the concepts of direction, curvature, and rate of change of curvature ('Schnörkelung', another invented term). BOLZANO only wrote down a partial analysis of the 'Haltung' of lines, mainly investigating direction and curvature. A treatment of the 'Haltung' of surfaces is lacking. On the whole, the specific results on curvature and 'Haltung' which BOLZANO committed to paper are meagre.

Undoubtedly the most significant feature of the 'Uiber Haltung' paper is the very general quasi-topological viewpoint which BOLZANO adopts throughout. It is clear that he intended to weave his topological basis much more solidly into the fabric of his geometrical work. The concepts of distance and neighbour form the foundation of his ideas. Using this basis, BOLZANO defines an isolated point of a set as hinted at in his earlier 'Versuch einer Erklärung' *(0000a)*, as a point for which there is no distance, however small, such that the point has one or more neighbours in the set for this distance and all smaller distances. A continuum or extension is a set with no isolated points, whereas a discontinuum ('discontinuirliches Raumding') is a set all of whose points stand isolated *(1948d: 143–144)*.

Now BOLZANO is able to define the basic objects of elementary geometry in a new way, bringing ideas of dimension to the fore *(1948d: 144–145)*:

> Ein Ausgedehntes, dessen jeder Punct für jede hinlänglich kleine Entfernung der Nachbarn nur so viele hat, dass ihr Inbegriff, für eine jede dieser Entfernungen für sich allein betrachtet, noch kein Ausgedehntes darstellt, nenne ich ein *Raumding von einer einzigen oder einfachen Ausdehnung,* auch eine *Linie*. Ein Raumding, dessen jeder Punct für jede hinlänglich kleine Entfernung der Nachbarn so viele hat, dass ihr Inbegriff für eine jede dieser Entfernungen für sich allein betrachtet selbst noch ein Raumding von einfacher Ausdehnung darstellt, nenne ich ein *Raumding von zweifacher oder doppelter Ausdehnung,* auch eine *Fläche*. Ein Raumding endlich, dessen jeder Punct für jede hinlänglich kleine Entfernung der Nachbarn so viele hat, dass ihr Inbegriff für eine jede dieser Entfernungen für sich allein betrachtet schon ein Raumding

[13] BOLZANO mentions LAGRANGE and CAUCHY in the text of *(1948d)*. Hence compare LAGRANGE *(1797)* and CAUCHY *(1826)* for the background to BOLZANO's main problem. Also compare OLIVIER *(1835)*, SAINT-VENANT *(1845)*. In general BOLZANO *(1948d)* seems to be a bit out of touch with the current literature and problems concerning curvature.

von doppelter Ausdehnung darstellt, nenne ich ein *Raumding von dreifacher Ausdehnung* oder einen *Körper*.

(I call an extension whose every point has only so many neighbours for each sufficiently small distance that the set of these, considered in itself for each of the distances, still does not represent an extension, a 'spatial object of a single or simple extension' or a 'line'. I call a spatial object whose every point has so many neighbours for each sufficiently small distance that the set of these, considered in itself for each of the distances, still represents a spatial object of simple extension, a 'spatial object of twofold or double extension' or a 'surface'. Finally, I call a spatial object whose every point has so many neighbours for each sufficiently small distance that the set of these, considered in itself for each of the distances, already represents a spatial object of double extension, a 'spatial object of threefold extension' or a 'solid'.)

These revised definitions broaden the ones BOLZANO put forward in 1817. The '0-dimensional' basis of isolated point sets is far more interesting than his older basis of finite sets used to define lines. BOLZANO proudly claims *(1948d: 145)* that the new definitions are essentially better than his former ones. The important question then arises: Why the change?

There are probably a number of reasons which made BOLZANO modify his earlier definitions. His discovery of the idea of isolated point and more generally his willingness to deal with infinite sets in his later work almost certainly influenced the change in his thinking. However, there is a specific example in the 'Uiber Haltung' *(1948d: 158–159; fig. 9)* which suggest to me why he altered his theory. The example consists of an *infinite* set of circles MA, MA',..., with increasing diameters, but all tangent to a straight line *(RMS)* at a single point M. If we consider this figure as a single line,[14] then the point M has infinitely many neighbours for each small distance, but these do not form a BOLZANO-continuum. Consequently, we have a line in BOLZANO's new sense, but not in the sense of the pamphlet of 1817. Perhaps an example like this motivated BOLZANO to rework his older theory. Certainly in his later research BOLZANO developed a facility to deal with infinite sets and to see their important role in geometry and analysis. Hence, he was able to handle a wider variety of geometrical objects, considered as mere sets of points.

In the 'Uiber Haltung' *(1948d: 146)* BOLZANO defines the special class of 'determinable' spatial objects in a way similar to that in *Die drey Probleme*. An object of geometry is determined with respect to a right-angled coordinate system and a measure of distance. In particular, lines are determined through one or more pairs of equations of the form

$$f(x,y,z)=0, \qquad \phi(x,y,z)=0,$$

where f and ϕ are functions which are continuous and have derivatives except possibly at an isolated set (finite or infinite) of values x,y,z. The cases of surfaces and solids are similar.

Unlike his treatment in *Die drey Probleme,* BOLZANO does not restrict his attention to determinable geometrical objects in the 'Uiber Haltung'. Instead he

[14] BOLZANO does not quite do this, but it is well within the sphere of his ideas.

wishes to consider lines, surfaces, and solids in full generality (where possible), so that his concept of topological neighbour becomes a more genuine foundation for his geometrical investigations. Thus when it comes to indicating the scope of 'Haltung', he offers a very general description *(1948d:150)*:

> Wohl dürfen wir also einem *jeden Ausgedehnten, das keine körperliche Ausdehnung hat,* in jedem seiner Puncte eine *Beschaffenheit* beilegen, welche wir uns als den *nächsten Grund* davon denken, warum diess Raumding in jedem seiner Puncte für eine jede hinlänglich kleine Entfernung aus dem ganzen Vorrathe von Puncten, die es für diese Entfernung überhaupt gibt, *nur eben jene und sonst keine anderen sich als Nachbarn aneignet.* Es werde mir verstattet, diese Beschaffenheit eines Raumdinges in jedem seiner Puncte *seine Haltung in diesem Puncte* zu nennen.
>
> (We are therefore permitted to attribute to every extension which has no solid part, a property at each of its points, which we think of as the proximate cause why this spatial object has as neighbours, for each of its points for every sufficiently small distance, only those neighbours and, otherwise, no others from the entire stock of points which have this distance from the point. Allow me to call this property of a spatial object at each of its points its 'Haltung' at this point.)

As mentioned above, BOLZANO unfortunately did not carry out a very complete analysis of the general 'Haltung' of lines, nor any analysis of surfaces. Primarily he treats the 'Haltung' of the straight line, the circle, and the helix. So at the end of the 'Uiber Haltung' BOLZANO's topological programme still needs much filling out in details.

'Geometrische Begriffe, die Jeder kennt und nicht kennt' *(0000b)* continues the topological investigations of 'Uiber Haltung', filling in more details. It aims at clarifying some of the ordinary concepts of geometry which we, mathematicians and non-mathematicians alike, know and yet do not really understand in 'clear consciousness'. It treats lines and surfaces in full generality, according to the definitions of 'Uiber Haltung', and special types of these. In effect, 'Geometrische Begriffe' is a long sequence of definitions about lines and surfaces, frequently accompanied by diagrams which nicely illustrate BOLZANO's motives. BOLZANO defines in rapid succession special kinds of lines and surfaces, simple, bounded, closed, simple closed, *etc.*, and then takes up the problem of betweenness for various types of geometrical figures. On the latter problem he considers, for example, the cases when one point is between two others on a line and when a point is said to be enclosed within a closed curve. The lengthy series of definitions in the 'Geometrische Begriffe' emphasises BOLZANO's strong conceptualist views. An important job of the mathematician is to seek good definitions for basic mathematical concepts in order to bring them into 'clear consciousness'. Correct definitions are essential to the proper development of a mathematical science. In 'Geometrische Begriffe' BOLZANO lays down definitions for many of the geometrical concepts which he suggested needed defining in 'Anti-Euklid'.

Unfortunately BOLZANO left 'Geometrische Begriffe' in a far less finished state than 'Uiber Haltung'. It reads like a series of notes, not a complete paper.

There are several concepts which Bolzano tries to define in two or three distinct ways. Apparently a later definition is intended to improve a previous one. Hence, we cannot expect too much from the few folios which Bolzano left us.

As an example of the investigations of 'Geometrische Begriffe' let us consider the definitions for closed and simple closed lines. Recall that in *Die drey Probleme* Bolzano tacitly assumed that lines which are not closed, such as segments and arcs, always include their endpoints, just as surfaces which are not closed always include their boundary curves. In 'Geometrische Begriffe' he comes to realise this tacit assumption. At the beginning of 'Geometrische Begriffe' he gives the following definition for a simple closed curve *(0000b:2r):*

> Eine *einfache in sich zurückkehrende Linie* ist eine solche, deren jeder Punct 2 Nachbarn hat, aber kein Punct Nachbarn besitzt, deren Entfernung grösser als eine gegebene ist.
>
> (A simple closed line is one for which every point has two neighbours, but no point possesses neighbours at a distance greater than a given distance.)

Under this definition an open line segment such as (0,1) or an open arc qualifies as a closed line, which is not Bolzano's intention. Hence, the definition is unsatisfactory. Apparently in order to improve matters, Bolzano offers a definition of closed line in an addendum at the very end of the manuscript *(0000b:9r):*

> Eine Linie *heisst in sich zurückkehrend,* wenn 1, kein Punct derselben eine Entfernung von einem anderen hat grösser als eine gegebene E, wenn 2, jeder Punct für jede hinlänglich kleine Entfernung zwei Nachbarn hat; und wenn endlich 3, kein einzelner Punct, auch kein Inbegriff mehrerer, die nicht für sich selbst wieder eine Linie bilden, zu ihr hinzugefügt werden kann, ohne dass das neue Ganze, das so zum Vorschein kommt, aufhört eine Linie zu seyn und in jedem Punct für jede hinlänglich kleine Entfernung 2 Nachbarn zu haben.
>
> (A line is called 'closed', if (1) no point of it has a distance from any other greater than a given E, (2) every point has two neighbours for every sufficiently small distance, and finally (3) no single point or set of several points, which does not in itself form a line, can be added to it without the new totality ceasing to be a line and having two neighbours for each sufficiently small distance.)

This definition overcomes the difficulty about open segments and arcs. Nevertheless, clause (2) effectively requires the curve to be simple. At another place *(0000b:8r)* Bolzano seems to offer his final attempt at defining closed and simple closed curves. For an arbitrary closed line (with possible multiple points) we need only require an even number of neighbours in clauses (2) and (3); exactly two neighbours are required for *simple* closed lines. These 'final' definitions are quite good and are typical of Bolzano's late topological thinking.

A pair of disjoint figure-eights can be taken as one closed curve under Bolzano's 'final' definition, but, of course, they are not connected. To sharpen his definition Bolzano gives a condition to describe when a closed curve is all in one piece *(0000b:8r):*

> Eine *Linie* heisst eine *einzige* in sich zurückkehrende Linie, wenn kein Theil derselben, der für sich allein schon eine in sich zurückkehrende Linie darstellt, weggelassen werden kann, ohne dass der Uiberrest aufhörte, eine in sich zurückkehrende Linie zu sein. Eben so eine *einzige in sich zurückkehrende Fläche.*
>
> (A line is called a 'single closed line', if no part of it, which by itself already represents a closed line, can be omitted without the remainder ceasing to be a closed line in itself. In a similar way a 'single closed surface'.)

Here we have some improvement on the definitions of connectedness of 1817. Nonetheless, these definitions are very restricted in scope, since they only apply to closed curves and surfaces.

In the mass of definitions of 'Geometrische Begriffe' BOLZANO states virtually only one 'theorem' *(0000b:8r):*

> Jede einfache in sich zurückkehrende Linie, welche in einer Fläche liegt, theilt diese in 2 Theile, welche sich dadurch unterscheiden, dass alle Puncte der Fläche, die nicht auf dieser Linie liegen, entweder auf der Einen oder auf der ihr entgegengesetzten Seite der Linie liegen.
>
> (Every simple closed line which lies on a surface separates the latter into two parts, which are distinguished by the fact that all points of the surface which do not lie on the line lie either on one or the opposite side of the line.)

Unfortunately if we take 'surface' to mean any figure which falls under BOLZANO's very general definition, then this new version of the closed curve theorem is simply false. For example, it fails for certain closed curves on the torus. BOLZANO's diagram accompanying his assertion shows that he was really thinking of quite simple types of surfaces. Yet his surface definition takes in much more. Consequently, his version in 'Anti-Euklid' of the closed curve theorem for the plane—in effect the celebrated JORDAN curve theorem—is preferable to the one just quoted.

BOLZANO does not include in 'Geometrische Begriffe' any proof of either version of the closed curve theorem. However, before asserting the version above, he carefully defines the main concepts of the theorem: the concepts of simple closed line (as given above) and of points lying on the same or opposite sides of a line on a surface *(0000b:8r):*

> *Zwei Puncte in einer gegebenen Fläche heissen* 1) *auf derselben Seite oder* 2) *auf entgegengesetzten Seiten einer in dieser Fläche liegenden Linie,* je nachdem jede durch sie gelegte Ebene (höchstens mit Ausnahme eines oder mehrerer vereinzelt stehender Fälle) entweder 1) keine oder 2, 4, ... überhaupt *eine gerade Anzahl* von Durchschnittspuncten, oder 2) Ein oder 3 ... oder überhaupt eine *ungerade* Anzahl von Durchschnittspuncten mit jener Linie gemein hat, liegend in dem Stücke der Durchschnittslinie der Ebene mit der Fläche, das jene Puncte zu seinen Grenzpuncten hat.
>
> (Two points on a given surface are said to be (1) on the same side or (2) on opposite sides of a line lying on the surface, according to whether every plane passing through them (at most with the exception of one or several isolated

cases) has either (1) none or 2, 4, ... or in general an even number of intersection points or (2) one or 3, ... or in general an odd number of intersection points with the line which lie in the piece of the line of intersection of the plane with the surface that has those points as endpoints.)

Perhaps BOLZANO thought that his definitions cleared the way for an easy proof of the 'theorem' or that it was a direct consequence of the definitions. These thoughts would be in line with his philosophy. But in any case he did not write down a proof and so we cannot be sure of his attitude towards the nature of a proof.

BOLZANO's bare statement, with no supporting proof, of the closed curve theorem is the highpoint of 'Geometrische Begriffe'. Beyond this paper he produced no new results. Hence, we are left with the interesting beginnings of what might have been a fruitful programme in topology, but which BOLZANO did not develop any further. It is clear that BOLZANO had many exciting topological ideas, which could have been the framework for a detailed topological theory. However, in his last years he could not summon up the energy to work out the details.

8. Conclusion: The Significance of Bolzano's Geometrical Investigations

BOLZANO's interest in his fundamental geometrical problem of defining the concepts of line, surface, solid, and continuum spanned his entire creative lifetime, from his earliest thoughts on mathematics and first published works to his last research papers. Having examined BOLZANO's attempted solutions to his problem in detail in the previous sections, I shall now try to put his work into perspective.

A basic feature of BOLZANO's outlook on research in mathematics was his view that mathematics stands in close relation to philosophy. His autobiographical statements reveal that he preferred 'philosophical mathematics' and his published work shows his deep concern with logical and foundational issues in mathematics. One is tempted to call BOLZANO a 'philosopher's mathematician'. In this respect he reminds one of DESCARTES, whose mathematical work was closely linked to his philosophy and special method of discovery.

In the particular case of his basic geometrical problem BOLZANO's philosophy of definitions and concepts strongly influenced, indeed guided, his quest for a solution. BOLZANO's essentialist position over definitions required that he seek the 'true' definitions for the objects of geometry. Burdened with such a view of the desired solution, BOLZANO clearly did not set himself an easy task. However, having adopted this view, he stuck to it and so examined a number of possible definitions, testing them on a wide variety of examples. For example, his first definitions of 1817 eventually did not satisfy him, so he devised new ones in the 1830's and 40's. But throughout his research he kept looking for the definitions which, to his eyes, would cover the geometrical concepts precisely.

Undoubtedly, BOLZANO's essentialist philosophy is to blame for the main shortcomings of his geometrical investigations. From one standpoint the endproduct of his research, a seemingly endless string of definitions with hardly a theorem, must be regarded as disappointingly meagre. Yet if one asks 'what is' questions —What is a line?, What is a continuum?—then one must expect essentialist answers. Moreover, while definitions have a certain value in mathematics, no genuinely fruitful mathematical theory can consist entirely of definitions. Theorems and their proofs are much more important, and BOLZANO's theory is unquestionably lacking in these, *except* for his inspired statement of the closed curve theorem.

In spite of the seeming poverty of BOLZANO's results in the foundations of geometry, I think we must still give him a good measure of credit for his definite achievements. Above all, it is his viewpoint over the whole domain of geometry which is worthy of attention and this viewpoint is topological. The topological and set-theoretical methods which BOLZANO used as a way of dealing with the basic elements of geometry were without question far ahead of his time. His topological basis, derived from his concept of neighbour and then later also from his concept of isolated point, is very deep. The concept of neighbour, which in effect uses the modern notion of the boundary of a spherical neighbourhood coupled with a set-theoretic approach to geometry, allowed BOLZANO to put forward some very clever definitions of line, surface, and solid in 1817. Then later, when he discovered his notion of isolated point, he was able to arrive at an even deeper understanding of the basic figures of geometry. Thus we must take BOLZANO's perspicacious topological viewpoint as his most important achievement. It permitted him to gain insights into geometry which were far more penetrating than those of his contemporaries. It even led him to state the JORDAN curve theorem. Perhaps if BOLZANO had had more time to complete his later programme or if others had been able to take up his theory from where he had left it, then we might be able to reckon him as the initiator of a fully fledged topological theory.

In the final analysis BOLZANO is best seen as an inexperienced explorer in a new territory of mathematics. Although he had several forward-looking ideas, he did not have time or energy to take in the whole of the new domain suggested by his ideas. His viewpoint was expansive, but his conquests were few. Like LEIBNIZ when he was trying to open up his new domain of 'analysis situs', BOLZANO could not provide us with a detailed map of the new land, but only a few impressions of its landscape.

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