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GLOBAL ANALYSIS AND ECONOMICS

Pareto Optimum and a Generalization of Morse Theory

One has considered for centuries the problem of maximizing a function via differential calculus. Morse theory could be regarded as a globalization of this problem. Relatively recently, economists have considered in a special case the problem of 'optimizing' several functions at once, obtaining in this way what is called the Pareto Optimum. Our goal here is to place this problem in the setting of global analysis, or several differentiable real functions on a manifold. We extend the notion of Pareto Optimum to a larger set which we call the critical Pareto set θ. This set θ is the analogue and generalization of the set of critical points of a single differentiable function while the old Pareto optimum is the analogue and generalization of a maximum of a single function. This expansion of the economists, setting allows for the systematic introduction of calculus and global differentiable methods to the process of optimizing several functions on a manifold. For example, we obtain a natural notion of dynamics in this setting to generalize that of a gradient flow. Our approach contrasts with the more usual equilibrium or static approach mathematical economists take toward the study of a pure exchange economy.

Precisely the problem we consider is the following: One is given real differentiable functions $u_i: W \to R$ defined on a manifold W say for $i=1, \ldots, m$. What is the nature of curves $\varphi: R \to W$ with the derivative $\mathrm{d}/\mathrm{d}t(u_i \circ \varphi)(t)$ positive for all i, t? For what $x \in W$ does there exist such a φ with $\varphi(0)=x$? The critical Pareto set θ is defined as the set of $x \in W$, for which there is no such φ. The main problem is the study of θ. Another way of looking at this is: how and when can one gradually improve the values of several functions simultaneously? One could consider this subject as part of game theory.

These questions lead to attractive mathematical problems. But especially one obtains a new way of studying utility, Pareto sets in economics, where traditional assumptions of convexity and monotonicity need not play such a key role. Also I believe that the questions of optimizing several functions at once, transcend economics; in other social problems, opti-

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mization of several functions rather than one permits one to go beyond a 1-dimensional point of view. The question is one of many values in partial conflict vs maximization of a single value.

Our intention is to follow this article by one on price systems in economics treated from a related point of view.

I would like to warn the reader of the slightly tentative nature of some of the later parts of this paper. For example, while Section 5 seems simple and straightforward enough, I haven't carefully verified the axioms of the theory of stratified sets for these cases. Also 'Theorem' 2 of Section 6 will need hard work before it could be regarded as a solid theorem (if true).

Let me end this introduction by thanking Gerard Debreu for getting me into this subject and for many helpful discussions.

SECTION 1

Here we give review of the notion of pure exchange economy and the classical Pareto Optimum. The mathematician reader can skip this section if he is not interested in economics or concrete applications and the econometrist reader will know these things. For more details, one can see Debreu [3].

Commodity space will be an open set in Cartesian space R^l. There will be l different commodities in this pure exchange economy, each measured in quantity by a real number (fixing a unit of measurement) which can be considered a coordinate on R^l. We are concerned with only positive amounts of each commodity and thus define commodity space as the positive orthant P of R^l. So $P=\{x\in R^l \mid x$ has each coordinate positive$\}$. A point of P represents a bundle of commodities, which might be possessed, for example, by a certain consumer in the economy.

It is assumed that there are a finite number of consumers, say m, with the possessions of the ith consumer denoted by $x_i \in P$. An unrestricted state of the economy is a point $x=(x_1, \ldots, x_m)$ in the Cartesian product P^m (a manifold of dimension ml). We suppose, however, that the total resources in this economic model are fixed, say a point $w \in P$. Thus the *attainable states* form a subset W of P^m defined by $W=\{x\in P^m \mid \sum x_i = w\}$. W is an open subset of an affine subspace with compact closure in $(R^l)^m$. W is the basic state space we consider in this paper.

Each consumer is supposed to have his preferences represented by a function $u_i : P \to R$, his utility function which we suppose as differentiable as necessary. Thus consumer i prefers x_i' to x_i if and only if $u_i(x_i') > u_i(x_i)$. Consumer i is indifferent to commodity bundles in the same level surface of u_i. Hence the $u_i^{-1}(c)$ are called indifference surfaces. In fact it is these surfaces that are given primarily in economics and it is these surfaces on which our analysis ultimately rests. However the u_i are convenient for purposes of communication.

Let $\pi_i : P^m \to P$ be the projection $\pi_i(x) = x_i$; then we have induced functions on W, still denoted by u_i, defined as the composition

$$W \xrightarrow{\text{inclusion}} P^m \xrightarrow{\pi_i} P \xrightarrow{u_i} R.$$

One considers exchanges in W, which will increase the utility of each consumer or increase each u_i on W. A state $x \in W$ is called Pareto optimal if it has the property; there is no $x' \in W$ with $u_i(x') \geqslant u_i(x)$, all i, and $u_j(x') > u_j(x)$, some j. The idea is that if $x \in W$ is not Pareto, then it is not economically stable; there will be some trade which will take place and tend to make it Pareto.

We consider this situation more generally in the succeeding sections; we study a manifold W with m real valued smooth functions $u_1, \ldots, u_m$ defined on W and look at this situation from an (extended) Pareto point of view.

We end this section with a proposition communicated to me by Truman Bewley.

PROPOSITION. Using the notation of this section, let utility functions u_i, $i = 1, \ldots, m$ be defined and continuous on $\bar{P}$ and suppose:

(a) 'convexity': $u_i^{-1}[c, \infty)$ is strictly convex for each i, c;

(b) 'mononicity': Define for $x', x \in P$, $x' > x$ if $x' - x \in P$ and similarly for $y', y \in R^m$. Then $x' > x$ implies $u(x') > u(x)$, where $u = (u_1, \ldots, u_m)$.

Under these conditions the set of Pareto Optima is homeomorphic to a closed $(m-1)$-simplex.

SECTION 2

Here we consider W a smooth (C^∞) manifold with smooth functions $u_i : W \to R$, $i = 1, \ldots, m$, where $m \leqslant \dim W$. A prime example is the manifold W of attainable states of a pure exchange economy of the previous section where the $u_1, \ldots, u_m$ are utility functions of m consumers. The main goal of this section is to introduce a differentiable extension and generalization of the Pareto Optimum which we call the critical Pareto set, θ.

Let $u: W \to R^m$ be defined by $u=(u_1, \dots, u_m)$ and Pos $\subset R^m$ be the set of $(y_1, \dots, y_m) \in R^m$ such that $y_i > 0$, each i.

Now let $H(x) = Du(x)^{-1}(\text{Pos})$, where $Du(x): T_x(W) \to R^m$ is the derivative of u at x, considered as a linear map from the tangent space of W at x to R^m. Thus $H(x)$ is an open cone in $T_x(W)$. Then the *critical Pareto set* θ is defined by $\theta = \{x \in W \mid H(x) \text{ is empty}\}$. Clearly θ is a closed subset of W.

We have the following alternate descriptions for $H(x)$. Let $H_i(x) = \{v \in T_x(W) \mid Du_i(x)(v) > 0\}$. Then $H(x) = \bigcap_i H_i(x)$ and, of course, also $H(x) = \{v \in T_x(W) \mid Du(x)(u) \in \text{Pos}\}$. Thus one sees that $\varphi: R \to W$ has increasing (infinitesimally) utility for each i if and only if $\varphi'(t) \in H(\varphi(t))$, all t. We say that φ is *admissible* in this case. It is clear that an admissible curve doesn't admit any type of recurrence; e.g., if φ: $[a, b] \to W$ is admissible then $\varphi(a) \neq \varphi(b)$. An admissible curve could be thought of as a sequence of small trades in the example of Section 1.

Thus $x_1 \in \theta$ is the condition that there is no curve through x increasing infinitesimally all the u_i's. If $m=1$, then θ is precisely the set of critical points of the function u. The field of cones $x \to H(x)$ on W is continuous in the following sense.

PROPOSITION (trivial). Let X be a continuous vector field on W with $X(x_0) \in H(x_0)$, some $x_0 \in W$. Then $X(x) \in H(x)$ for all x in some neighborhood of x_0.

Note that if x is Pareto in the classical sense than $x \in \theta$. So we have not lost anything. But our critical Pareto set θ is bigger in general than the old; we consider now certain natural subsets of θ which are significant economically.

Suppose $\varphi: [a, b) \to W$ is an admissable path and $\lim_{t \to b} \varphi(t) = x$. Then we say that φ ends at x. In this case we also say that φ starts at $\varphi(a) = w$. If $w \notin \theta$, let $\theta(w)$ be the subset of θ of x for which there exist admissable φ starting at w and ending at x. Thus $\theta(w)$ is the set of Pareto critical points accessible from the initial state w by a sequence of (infinitesimally) small trades.

Next we can define naturally in our context the notion of stability for $x \in \theta$. Say $x \in \theta$ is *stable* or $x \in \theta_S$ if given a neighborhood $U(x)$ of x in θ, there exists a neighborhood $V(x)$ in W such that if φ is any admissible path in W starting in $V(x)$ and ending in θ, then φ ends in $U(x)$. Clearly θ_S will be the most significant part of θ from the economic point of view. Note that θ_S is an open subset of θ. Note also that for $m=1$, just as θ is the critical point set, θ_S is the set of local maximums of u, at least in the non-degenerate case.

Let $\theta_S(w)=\theta_S\cap\theta(w)$. We will later show that frequently one can assert that $\theta_S(w)\neq\phi$.

It seems to be the case that if the conditions of the final proposition of Section 1 are satisfied, then $\theta=\theta_S=$classical Pareto Optimum.

Section 3

The goal of this section is to develop the idea of a 'Hessian' in our context. Using this we are able to obtain a criterion for a point $x\in W$ to be a *stable* Pareto point. For the case of one function $u: W\to R$ this amounts to saying that $x\in W$ is a local maximum, and stably so, if the first derivative $Vu(x)$ is zero and the second derivative is negative definite.

The context we are in now is that of a smooth map $u: W\to R^m$ from a manifold to Euclidean space. It is convenient to make an assumption, the rank assumption, as follows:

DEFINITION. Say that $x\in\theta\subset W$ satisfies the *rank assumption* if rank $Du(x)\geqslant m-1$ and that u satisfies the rank assumption if x does for all $x\in\theta$.

Some remarks are in order. First it is clear that if $x\in\theta$, then rank $Du(x)\leqslant m-1$. Thus one could write equality in the rank assumption instead of inequality.

We make a little excursion into the way we use the expressions 'almost all' and 'generic property'. One can put a natural C^r-topology on the space of maps $u: W\to R^m$, $1\leqslant r\leqslant\infty$, which makes this space a Baire space [1]. A Baire space has the property that a countable intersection of open dense sets is dense and is called a Baire set. 'Almost all' refers to being an element of some Baire set. A *generic property* for u is one that is true for all elements of some Baire set.

For almost all u in this Baire sense, almost all points of θ will satisfy the rank assumption. However, one cannot say that it is a generic property for u to satisfy the rank assumption.

On the other hand if $(\dim W+4)/2>m$, then almost all u will satisfy the rank assumption. The above facts follow from Whitney or Thom (cf. Calabi [2] or Levine [4]). In the case of a pure exchange economy, this dimension condition will always be met (at least if there is more than one commodity) as a simple counting argument shows. Thus the rank assumption does not seem too serious a restriction.

We now define the Hessian of $u: W \to R^m$ at $x \in \theta$. Suppose $x \in \theta \subset W$ satisfies the rank assumption. Then $Du(x): T_x(W) \to R^m$ has rank $m-1$. The second derivative can easily be shown to define an invariantly (independent of chart) defined symmetric bilinear map H_x on $T_x(W)$ with values in the 1-dimensional vector space $R^m/ImDu(x)$. This 1-dimensional vector space $R^m/ImDu(x)$ has a canonically defined positive ray (or orthant, etc.) from the fact that image $Du(x)$ does not intersect Pos $\subset R^m$. This last is key for our whole development and allows us to define negative definite, index, nullity etc. for x. These ideas do not extend to the theory of singularities of maps because in this general case there is no natural definition of a positive part of $R^m/ImDu(x)$.

We can now state our theorem.

THEOREM. Suppose $u: W \to R^m$ is C^r, r large enough and x is in the critical Pareto set θ. Suppose also that x satisfies the rank condition and the generalized Hessian H_x is negative definite. Then x is in the stable Pareto set i.e., $x \in \theta_S$.

For example, the proof could go via the normal forms in Levine [5]. In fact no doubt a direct simple argument using Taylor's theorem would work with $r=2$.

Note from similar considerations that if x is as in the theorem and if an admissable curve $\varphi: [0, 1) \to W$ has x in the closure of its image then actually $\varphi(t) \to x$ as $t \to 1$.

Note finally that if $x \in \theta$ satisfies the rank condition, we have defined for x, the *index*, *nullity*, *nondegeneracy* as the index, nullity, nondegeneracy, respectively, of the bilinear form H_x.

SECTION 4

We devote this section to four examples:

EXAMPLE 1. Let W be E^2, the Euclidean plane with norm $\| \ \|$ given by an inner product. Suppose $m=2$, with $u_i:E^2 \to R$ given by $u_1(x) = -\|x\|^2$, $u_2(x) = -\|x - x_0\|^2$ where $x_0 \in E^2$ is some fixed point in E^2, $x_0 \neq 0$. The 0, x_0 are 'satiation points' for number 1 and number 2 respectively, i.e. points of maximum happiness. Then one can check through the definitions to see that $\theta = \theta_S =$ the closed segment between 0 and x_0. Any admissible path which is *complete* in the obvious sense will end at θ. This example is illustrated in Figure 1 where we also mark off $\theta(w)$, and show a typical admissible path.

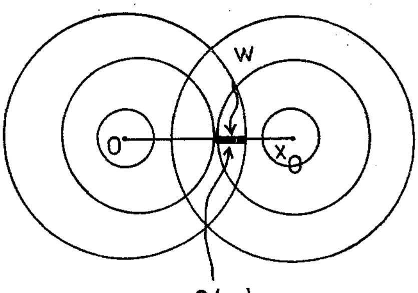

Fig. 1.

EXAMPLE 2. Edgeworth box. This is a standard example from economics and fits into the case of Section 1, where we suppose there are two commodities, two consumers.

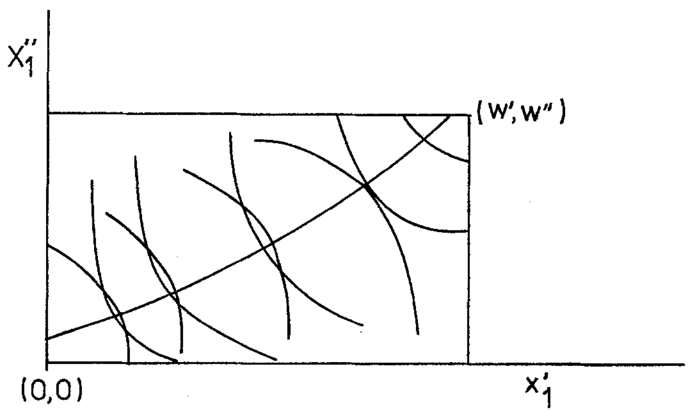

Fig. 2.

Suppose the total resources are denoted by $w \in P$, with $w=(w', w'')$, w' the amount of the first commodity, w'' of the second. Then a state $x \in P^2$ is of form $x=(x_1, x_2)$, $x_1 \in P$, $x_2 \in P$, and we can write $x_i=(x_i', x'')$, where x_i'' is the amount of the first commodity owned by the first consumer, etc. Then W is the set of $(x_1', x_1'', x_2', x_2'')$ which satisfy $0<x_i'$, $0<x_i''$ and $x_1'+x_2'=w'$, $x_1''+x_2''=w''$. Thus we could describe W also with coordinates x_1', x_1'' with $0<x_1'<w'$, $0<x_1''<w''$ as in Figure 2. In Figure 2 are drawn the level curves of u_1, u_2, respectively, where we suppose the standard convexity conditions of traditional economics are met. Here $\theta=\theta_S=$ the arc of common tangents of these level curves. In this case θ is called the Edgeworth contract curve. The situation is locally as in Example 1, away from the end points.

EXAMPLE 3. Here we relax the convexity assumption on u_i in the previous example. Suppose in fact that u_2 is as in Example 2 and u_1 has the

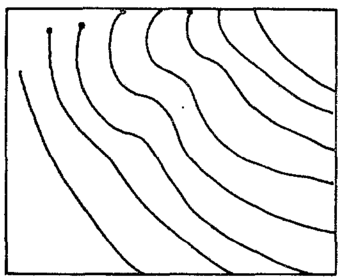

Fig. 3.

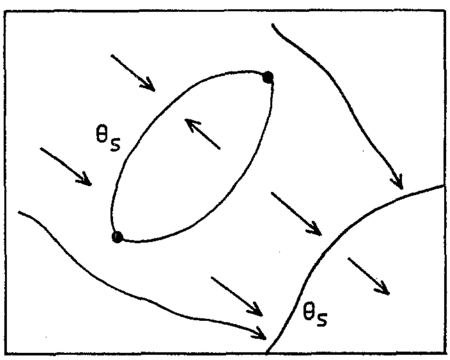

Fig. 4.

qualitative features in Figure 3. Then arguing via common tangents one constructs θ, θ_S as in Figure 4 (θ the circle and segment without any arrows). Some admissible curves are drawn with arrows. These indicate how the circle part of θ gets divided into a stable and unstable part. For some $w \in W$, $\theta(w)$ may have 2 components. Also part of θ_S may not be classical Pareto Optimal and one sees how global problems enter into trading, vs sequences of small trades. Some acquaintance with [8] is helpful in understanding this example.

EXAMPLE 4. Here $W = S^2$, the unit sphere in R^3, u_1, u_2 are two coordinate axes so that $u: W \to R^2$ is projection into the appropriate coordinate plane π. Then θ_S is the intersection of the 1st quadrant of π with S^2 and the only other part of θ is the intersection θ' of the 3rd quadrant of π with S^2. (See Figure 5.)

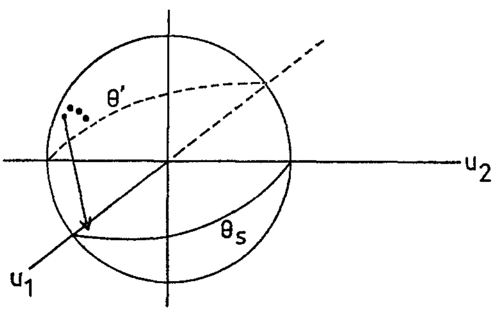

Fig. 5.

Admissible curves will tend to go from θ' to θ_S. This suggests the possibility of a Morse theory for this problem; we pursue this later. Note that a neighborhood of θ_S is the same as Example 1.

SECTION 5

The goal of this section is to give some picture of the structure of θ, θ_S and then to construct a generalisation of a gradient vector field.

For this first part it is useful to assume that $u: W \to R^m$ satisfies the *jet transversality* condition of Thom. (Cf. [4].) This is a generic property and means that the derivatives of u are transversal to the manifolds of singular jets. If in addition, the rank assumption is also true for u, as we shall assume, then the set of points x of W where $Du(x)$ is not surjective, forms a submanifold S_1 of W of dimension $m-1$. Furthermore, this S_1 contains a sequence of submanifolds S_1^q, $q=1, 2, 3, \ldots, m$ with dimension $S_1^q = m-q$. See [5] for details of this case of the theory of singularities of maps. Since $\theta \subset S_1$, these results have a strong bearing on the nature of θ.

We need a further generic property on u as follows:

DEFINITION. Say that $u: W \to R^m$ satisfies *transversality condition* A_1 if it satisfies jet transversality and if the restriction $u/S_1: S_1 \to R^m$ has its 1st derivative transversal to all the coordinate subspaces of R^m.

The point of transversality condition A_1 is that it insures reasonable behavior as $ImDu(x)$ passes into $\mathrm{Pos} \subset R^m$ or gives us a reasonable structure to $\partial\theta$. In fact the following proposition would seem to be valid.

PROPOSITION. Suppose $u: W \to R^m$ satisfies the rank assumption, jet transversality, and transversality condition A_1. Then (if not empty!) the critical Pareto set θ is an $(m-1)$ dimensional manifold with corners in the sense of J. Cerf (thesis) or stratified set in the sense of Thom [7]. Thus θ has the structure of an $(m-1)$-dimensional manifold θ_1 together with its boundary $\partial\theta = \theta - \theta_i$. The boundary is a union of submanifolds of dimension $< m-1$.

Compare this with the examples of Section 4. The reader will be able to construct more interesting examples with $m=3$.

To obtain similar information for θ_S we introduce a generalization of transversality condition A_1, another generic property:

DEFINITION. Say that $u: W \to R^m$ satisfies transversality condition A if it satisfies jet transversality and if the restriction $u/S_1^q: S_1^q \to R^m$ has its 1st derivative transversal to all the coordinate subspaces of R^m, for $q=1, 2, \ldots, m$.

Actually for our immediate purposes $q=1$ and 2 would be good enough. Apparently the following proposition is valid.

PROPOSITION. Suppose $u: W \to R^m$ satisfies the rank assumption, jet transversality and transversality condition A. Then θ_S, the stable Pareto set, if not empty, has the structure of an $(m-1)$-dimensional stratified set. Furthermore θ_S can be characterized as the (open) set of $x \in \theta$ with index and nullity zero.

The simplest cases are in Section 4.

Given $u: W \to R^m$, a *gradient vector field* for u is a smooth tangent vector field X defined over W with the property that $X(x) \in H(x)$ for $x \notin \theta$ and $X(x)=0$ for $x \in \theta$. It is an easy argument to show that one can construct a gradient vector field for any u.

SECTION 6

The goal of this section is to try to gain a global theory of the critical Pareto set and stable Pareto points.

THEOREM 1. Suppose $u: W \to R^m$ satisfies the rank assumption, jet transversality and transversality condition A with W compact. Then for any $w \in W$ with $w \in \theta$, $\theta_S(w) \neq \phi$.

We give an outline of how a theorem generalizing Morse theory to this case might go. This Morse theory perspective requires the introduction of the notion of cycle. Always S^n denotes the n-sphere.

DEFINITION. Suppose $u: W \to R^m$. Then a *cycle* is a continuous injective map $f: S^1 \to W$ such that S^1 can be written as a finite disjoint union of intervals I_α and on each I_α, either $u_i \circ f$ is non-decreasing each i or the image of I_α is in θ.

The thoughtful reader will be able to construct an example of u with a cycle, even a map $u: S^2 \to R^2$ satisfying the rank and transversality conditions.

Define θ_λ to be the set of $x \in \theta$ with index λ and nullity zero, and θ_n the set of $x \in \theta$ with nullity positive.

Next let $\pi_{i_1, \dots, i_k}$ be the coordinate subspace defined by setting $y_1, \dots, y_k$ all equal to zero in R^m.

Let

$$\Sigma_{i_1,\dots,i_k} = \{x \in S_1 \mid Du(x)(T_x(S_1)) \text{ is not transversal to } \pi_{i_1,\dots,i_k}\}.$$

By our transversality assumptions all the $\Sigma_{i_1,\dots,i_k}$ are manifolds and meet transversally with the S_1^q in S_1.

Let G be the set of $x \in \Sigma_i$ some i such that $x \notin \Sigma_{j_k}$ every j, k and $x \notin S_1^q$ each $q > 1$. Then Λ will be the main part of the boundary of θ (i.e. the union of the (m-2)-dimensional strata of $\partial\theta$).

We want to assign a + or − to each point of Λ. This goes as follows. Let $x \in \Lambda$ and suppose $x \in \Sigma_i$. This implies $Du(x)(T_x(S_1)) = \Pi_i$. If the normal to $u(\theta)$ at $u(x)$ is positive (along the y_i axis) then we say that $x \in \Lambda^+$, otherwise $x \in \Lambda^-$. Then let θ' = the closure of $(\theta_n \cup \Lambda^+)$ and $\theta'_\lambda = \theta' \cap \bar{\theta}_\lambda$.

'THEOREM' 2. Suppose $u: W \to R^m$ has no cycles, satisfies the rank assumption, jet transversality, transversality condition A with W compact. Let $M_i = \sum_\lambda \dim H_{i-\lambda}(\theta_\lambda, \theta'_\lambda)$ with coefficients an arbitrary field. Then the M_i satisfy the Morse relations. That is if B_i denotes the ith Betti number of W,

$$\begin{aligned} M_0 &\geqslant B_0 \\ M_1 - M_0 &\geqslant B_1 - B_0 \\ &\cdots \\ \textstyle\sum (-1)^i M_i &= \textstyle\sum (-1)^i B_i. \end{aligned}$$

Note that a Morse function $u: W \to R$ clearly satisfies the hypotheses of 'Theorem' 2. Thus 'Theorem' 2 contains the usual Morse theory.

We check now how this specializes to $m = 2$. First observe that $M_0 = \dim H_0(\theta_0, \theta'_0)$,

$$\begin{aligned} M_1 &= \dim H_1(\theta_0, \theta'_0) + H_0(\theta_1, \theta'_1) \\ M_i &= \dim H_1(\theta_{i-1}, \theta'_{-1}) + \dim H_0(\theta_i, \theta'_i) \\ &\quad \text{for} \quad 0 < i < n = \dim W \end{aligned}$$

and $M_m = \dim H_1(\theta_{n-1}, \theta'_{n-1})$.

This follows from the fact that θ is 1-dimensional and the index is strictly less than n.

Now what are the possible cases for the topology of $(\theta_\lambda, \theta'_\lambda)$? Let us call points of S_1^2 and Λ^+ both generalized cusps by a slight abuse of language. Then we have for the components θ_λ^k of θ_λ:

Case 1. θ_λ^k is a circle with ∂ empty.

Let σ_λ denote the number of these.

Case 2. θ_λ^k is an interval no end point a generalized cusp.

Let α_λ denote the number of these.

Case 3. θ_λ^k is an interval with one end point a generalized cusp.

Let β_λ denote the number of these, and

Case 4. θ_λ^k is a interval with both end points a generalized cusp.

Let γ_λ denote the number of these.

Then

$$\begin{aligned} M_0 &= \sigma_0 + \alpha_0 \\ M_i &= (\sigma_{i-1} + \gamma_{i-1}) + (\sigma_i + \alpha_i), \qquad 0 < i < n \\ M_n &= \sigma_{n-1} + \gamma_{n-1}. \end{aligned}$$

Thus the Morse relations give relations between the σ_λ, α_λ, γ_λ and the Betti numbers of W.

For example the Euler characteristic χ_W satisfies

$$\chi_W = \sum (-1)^i \alpha_i - \sum (-1)^i \gamma_i.$$

By taking $\dim W = 2$, one gets a further simplication, still interesting.

Here is the idea of how the proof of 'Theorem' 2 would go. Choose a gradient system on W for u. Then define for each strata θ_α of θ, $W^u(\theta_\alpha)$ as the set of points tending to θ_α as $t \to -\infty$ for the gradient dynamical system. Thus $M = \bigcup_\alpha W^u(\theta_\alpha)$. At this point one can see Theorem 1. Now one follows the argument in [6] using strongly the hypothesis that there are no cycles and defining the L_k in the same way via the $W^u(\theta_\alpha)$. Finally one evaluates the $M_i = \sum_k \dim H_i(L_k, L_{k-1})$ and applies the algebraic argument of [6].

Perhaps a more complicated version of Morse theory could take into account existence of cycles.

We end this paper with the following remarks.

First, to what extend could one proceed with these things without the rank assumption? Certainly some things could go through, but in general the complexities of the theory of singularities of maps would make for tough going.

Next one could object that the main example of Section 1 does not satisfy the compactness condition on W of this section. While that is true, I feel it is important so see basic ideas at first in their simplest form. Later perhaps these ideas can carry over into more technical situations. For the example of Section 1, one needs to consider carefully the implications of various boundary condition on $u: W \to R^m$.

Finally, one could ask how does the 'core' of theoretical economics relate to what I have done here? I don't think the concept of core fits in very well to this approach. The reason is that here we have change, much change, as a basic element of this approach to Pareto theory. It is an essentially dynamic approach we are considering. On the other hand the core, it seems to me, involves an essentially static or equilibrium approach. For example the initial resources of *each* consumer are needed to define the core. But these change after a single trade, and after several trades may be forgotten. This after some initial but not terminal exchange, the idea of coalition to describe the core loses validity.

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