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PROBABILITY FUNCTIONS AND THEIR ASSUMPTION SETS – THE BINARY CASE

It is argued in Leblanc 1983b that statements are accorded probabilities *in light of assumptions* – or, as mathematicians often put it, *under conditions*.[1] It is further argued that each *singulary* probability function in Kolmogorov 1933 or, equivalently, Popper 1955 comes with (a set of) assumptions, to wit, those statements evaluating to 1 under P. With P a singulary probability function of the Kolmogorov-Popper sort and A a statement from a certain language L, Leblanc thus takes $P(A)$ to be the probability that P accords to A in light of the assumptions in $\{A : P(A) = 1\}$. His rationale for interpreting $P(A)$ in this manner is two-fold. He first contends that any assumption set in light of which a rational agent would accord probabilities *must* be deductively closed and, for convenience's sake, may be presumed consistent as well. He then establishes that a set S of statements of L is consistent and deductively closed if and only if there is a *singulary* probability function P for L such that $S = \{A : P(A) = 1\}$. The second result is called by Leblanc *The Fundamental Theorem on Assumption Sets, Case One*.

A like approach is taken here towards *binary* probability functions. We presume that *pairs of statements*, as well as single statements, are accorded probabilities *in light of assumptions*, and that each binary probability function P in Popper 1959 comes with (a set of) assumptions, to wit, those statements A of L such that – no matter the statement B of L – $P(A/B) = 1$. We thus take $P(A/B)$ to be the probability that P accords to A *in light of the assumptions in* $\{A : (\forall B)(P(A/B) = 1)\}$ *and in the light of* B, B a *stated* assumption which may but need not figure among the members of $\{A : (\forall B)(P(A/B) = 1)\}$, and when it does *not* figure among those *unstated* assumptions is operative only if flagged as in $P(A/B)$. Our rationale for interpreting $P(A/B)$ in this manner is two-fold. We take it as in Leblanc 1983b that *a rational agent would accord probabilities in light of all and only those sets of statements of L that are consistent and deductively closed*. And we establish (in Theorem 5 below) that *a set S of statements of L is consistent*

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and deductively closed if and only if there is a binary probability function P for L such that $S = \{A: (\forall B)(P(A/B) = 1)\}$ a result we call *The Fundamental Theorem on Assumption Sets, Case Two.*

When arguing that assumption sets must be deductively closed, Leblanc remarked that

> to be rational is to be alert to logical implications, those of what one knows, those of what one believes, and – in the present context – those of what one assumes. So, a rational agent – and it is for rational agents that the probability functions in studies such as this are intended – would own as an assumption any statement of L logically implied by his assumptions, and hence accord probabilities in light of *none but* deductively closed sets of statements of L.

Leblanc could see according probabilities in light of "contradictory" assumptions, so long as the probabilities accorded all equal 1. However, the constraints Kolmogorov and Popper place on *singulary* probability functions ruled that out. **C0**, one of the constraints Popper places on *binary* probability functions, rules it out here as well. The constraint, however, could be dropped without prejudice (other than editorial) to our results, a matter we take up on p. 100.[2]

The authors first thought of assumption sets in 1979–80 while outfitting intuitionistic logic with a "probabilistic" semantics, and they obtained (roughly) the present proof of the Fundamental theorem, Case Two, in the summer of 1980. Proof of Case One was found in early 1982 and is reported in Leblanc 1983b.[3]

* * *

The language L we work with is that in Leblanc 1983b. It has as its *primitive signs* $\aleph_0$ *atomic* statements, the two connectives '$\sim$' and '&', and the two parentheses '(' and ')'; as its *formulas* all finite sequences of primitive signs of L; and as its *statements* (i) the atomic statements just mentioned, (ii) all formulas of the sort $\sim A$, where A is a statement of L, and (iii) all those of the sort $(A \& B)$, where each of A and B is a statement of L. We shall presume the statements of L to have been arranged in a fixed order, to be known as their *alphabetic order*; one way of doing that, due to Smullyan, is reported in Leblanc and Wisdom 1972. To abridge matters, we shall write $(A \supset B)$ for $\sim(A \& \sim B)$; with A the alphabetically earliest statement of L, we shall write T for $(A \supset A)$; and, when clarity permits, we shall drop outer parentheses.

L has as its *axioms* all statements of L of the sorts $A \supset (A \& A)$, $(A \& B) \supset A$, and $(A \supset B) \supset (\sim(B \& C) \supset \sim(C \& A))$, and as its one *rule of inference* Modus Ponens – with B called as in Quine *the ponential of A and $A \supset B$*. Where A is a statement and S a set of statements of L, a finite column of statements of L is counted a *proof of A from S* if (i) every entry in the column is a member of S, an axiom, or the ponential of two earlier entries in the column, and (ii) A is the last entry in the column; A is counted *provable from S* ($S \vdash A$, for short) if there exists a proof of A from S; and A is counted *provable* ($\vdash A$, for short) if $\phi \vdash A$. To define formally terms used above and congeners of these, a set S of statements of L is counted *consistent* if there is no statement A of L such that both $S \vdash A$ and $S \vdash \sim A$, *inconsistent* otherwise; *complete* if for each statement A of L either $S \vdash A$ or $S \vdash \sim A$, *incomplete* otherwise; *deductively closed* if S has among its members each statement of L provable from it; and *maximally consistent* if S is consistent, complete, and deductively closed. Further, we say that A is *consistent with S* if $S \cup \{A\}$ is consistent, and that S and S' are *deductively equivalent* if any statement of L provable from either of S and S' is provable from the other.

One more syntactic notion will be defined on p. 95, *the Lindenbaum extension $L(S)$ of a set S of statements of L*; and one more on p. 101 that of *a state-description of L*.

Turning to semantic (hence, probabilistic) matters, we understand by *a (binary) probability function for L* any function P that maps the pairs of statements of L into reals and meets the following seven constraints (issuing from constraints in Popper 1959):[4]

C0. $(\exists A)(\exists B)(P(A/B) \neq 1)$ (Existence)
C1. $0 \leq P(A/B)$ (Nonnegativity)
C2. $P(A/A) = 1$ (Reflexivity)
C3. If $(\exists C)(P(C/B) \neq 1)$, then $P(\sim A/B) = 1 - P(A/B)$ (Complementation)
C4. $P(A \& B/C) = P(A/B \& C) \times P(B/C)$ (Multiplication)
C5. $P(A \& B/C) \leq P(B \& A/C)$ (Commutation to the left)
C6. $P(A/B \& C) \leq P(A/C \& B)$ (Commutation to the right)

With P a (binary) probability function of L, we take a statement A of L to be *P-normal* if $(\exists B)(P(B/A) \neq 1)$,[5] otherwise to be *$P$-abnormal*; and, as indicated earlier, we take a set S of statements of L to constitute *the assumption set of P* if $S = \{A : (\forall B)(P(A/B) = 1)\}$. With S again a set of statements of L, we take S to constitute *a binary assumption set of L* – for short, *an assumption set of L* – if there exists a (binary) probability function for L of which S is the assumption set.[6] And, with S once more a set of statements and A a statement of L, we say that S *logically implies A in the probabilistic sense* if – no matter the probability function P for L – $(\forall B)(P(A/B) = 1)$ if $(\forall B)(P(C/B) = 1)$ for each member C of S, and that *A is logically true* (i.e., is a tautology) *in the probabilistic sense* if ϕ logically implies A in that sense. S thus logically implies A in the probabilistic sense if and only if A belongs to each assumption set of L of which S is a subset, and A is logically true in that sense if and only if A belongs to each assumption set of L, two results which speak for our account of an assumption set.

Proof that

(1) *If $S \vdash A$, then S logically implies A in the probabilistic sense*,

a probabilistic version of the *Strong Soundness Theorem for L*, will be found in Leblanc 1979 and (considerably improved) in Leblanc 1983b, as will be proof that

(2) *If S logically implies A in the probabilistic sense, then $S \vdash A$*,

a probabilistic version of the *Strong Completeness Theorem* for L. (1)–(2) legitimize the account of logical implication and – with ϕ as S – logical truth. We appeal to (1) in the proof of Theorem 1, and show on pp. 99–100 that (2) follows from Theorem 3(a).

* * *

Half of the Fundamental theorem on Assumption sets, Case Two, is readily proved:

THEOREM 1. *If there is a probability function P for L such that $S = \{A : (\forall B)(P(A/B) = 1)\}$, then S is deductively closed and consistent.*

Proof: Let P be an arbitrary probability function for L such that $S = \{A : (\forall B)(P(A/B) = 1)\}$. (1) Let A be an arbitrary statement of L such that $S \vdash A$. Then, by the Strong Soundness Theorem for L, A belongs to each assumption set of L of which S is a subset. But by hypothesis S is an assumption set of L, and S is of course a subset of itself. Hence $A \in S$. Hence S is deductively closed. (2) Suppose for reductio that S is inconsistent. Then, no matter the statement A of L, $S \vdash A$, hence by (1) $A \in S$, and hence $(\forall B)(P(A/B) = 1)$, against

Constraint **C0**. Hence S cannot be inconsistent.[7]

Proof of the converse of Theorem 1 calls for an extra notion and a few preliminary results.

The reader is doubtless familiar with the so-called *Lindenbaum extension* $L(S)$ of a set S of statements of L. We define $L(S)$ by means of (i)–(iii) below, and – using various Lindenbaum extensions of the sort $L(S \cup \{\cdot\})$ – we construct in Theorem 2 a binary function P_S which (a) invariably meets Constraints **C1–C6** and (b) when S is consistent, meets Constraint **C0** as well.

Let S be an arbitrary set of statements of L, and A_1, A_2, A_3, etc., be in alphabetic order the various statements of L.

(i) $L_0(S)$ is to be S itself,

(ii) for each i from 1 on

$$L_i(S) = \begin{cases} L_{i-1}(S) \cup \{A_i\} \text{ if } L_{i-1}(S) \text{ is inconsistent or } L_{i-1}(S) \cup \{A_i\} \\ \text{is consistent} \\ L_{i-1}(S) \cup \{\sim A_i\} \text{ otherwise,}^{8} \end{cases}$$

and

(iii) $L(S)$ is to be the union of $L_0(S)$, $L_1(S)$, $L_2(S)$, etc.

We shall take for granted various familiar facts about Lindenbaum extensions in general: S is a subset of $L(S)$, $L_i(S)$ $(i = 1, 2, 3, \ldots)$ and $L(S)$ are consistent if S is, $L(S)$ is complete, and $L(S)$ is deductively closed. Further facts about these extensions are separately recorded in Lemmas 1 and 2, and one fact about Lindenbaum extensions of the sort $L(S \cup \{\cdot\})$ is recorded in Lemma 3.

LEMMA 1. *Let S be a set of statements of L; let A_1, A_2, $A_3, \ldots$, be in alphabetic order the various statements of L; and for each i from* 1 *on let B_i be A_i or $\sim A_i$.*

(a) *If S is inconsistent, then $L(S) = S \cup \{A_1, A_2, A_3, \ldots\}$;*

(b) *If S is consistent, then*

$$L_i(S) = L_{i-1}(S) \cup \{A_i\}$$

or

$$L_i(S) = L_{i-1}(S) \cup \{\sim A_i\}$$

according as A_i is consistent with L_{i-1} (S) or $\sim A_i$ is;

(c) *Whether or not S is consistent*, (*i*) $L(S) = S \cup \{B_1, B_2, B_3, \ldots\}$ *and* (*ii*) *each member of S is one of* $B_1, B_2, B_3, \ldots$;

(d) *If S is consistent and* $L(S) \vdash A$, *then A is consistent with each subset of* $L(S)$.

Clause (b) of the lemma hinges upon the fact that $L_{i-1}(S)$ is consistent if S is, and either $L_{i-1}(S) \cup \{A_i\}$ or $L_{i-1}(S) \cup \{\sim A_i\}$ is consistent if $L_{i-1}(S)$ is. (ii) in Clause (c) follows from (a) when S is inconsistent. In the contrary case let member C of S be the alphabetically i-th statement of L. Since $L_i(S)$ is consistent, $L_i(S)$ is sure to be $L_{i-1}(S) \cup \{C\}$ rather than $L_{i-1}(S) \cup \{\sim C\}$, and hence C is sure to be B_i. And Clause (d) hinges upon the fact that $A \in L(S)$ if $L(S) \vdash A$, and each subset of $L(S)$ is sure to be consistent if S is. So, under the present circumstances each subset $S' \cup \{A\}$ of $L(S)$ is sure to be consistent, and hence A is sure to be consistent with each subset S' of $L(S)$.

LEMMA 2. *If S and S′ are deductively equivalent, then* $L(S) = L(S')$.

Proof: Let S and S' be deductively equivalent. When S is inconsistent, so of course is S', and hence by Lemma 1(a) $L(S) = L(S')$. So, suppose S is consistent, and let

$$L(S) = S \cup \{B_1, B_2, B_3, \ldots\},$$

where B_1, B_2, B_3, etc., are as in Lemma 1. (1) Since S and S' are deductively equivalent, B_1 is consistent with S' as well as with S, and hence by Lemma 1(b) $L_1(S') = S' \cup \{B_1\}$. (2) Since as a result $L_1(S)$ and $L_1(S')$ are deductively equivalent, B_2 is consistent with $L_1(S')$ as well as with $L_1(S)$, and hence by Lemma 1(b) again $L_2(S') = S' \cup \{B_1, B_1\}$. And so on. Hence for each i from 1 on $L_i(S') = S \cup \{B_1, B_2, \ldots B_i\}$, and hence

$$L(S') = S' \cup \{B_1, B_2, B_3, \ldots\}.$$

But by Lemma 1(c) each member of S is among B_1, B_2, B_3, etc., as is each member of S'. Hence $L(S) = L(S')$. Hence Lemma 2.

LEMMA 3. *If* $L(S \cup \{A\}) \vdash B$, *then* $L(S \cup \{A\}) = L(S \cup \{B \,\&\, A\})$.

Proof: When $S \cup \{A\}$ is inconsistent, so is $S \cup \{B \,\&\, A\}$, and hence by Lemma 1(a) $L(S \cup \{A\}) = L(S \cup \{B \,\&\, A\})$. So, suppose $S \cup \{A\}$ is consistent, and for arbitrary i equal to or larger than 1 let

$$L_i(S \cup \{B \,\&\, A\}) = S \cup \{B \,\&\, A\} \cup \{C_1, C_2, \dots C_i\}.$$

Since $L_i(S \cup \{A\})$ is sure to be consistent if $S \cup \{A\}$ is, C_1 is consistent with $S \cup \{B \,\&\, A\}$, C_2 with $S \cup \{B \,\&\, A\} \cup \{C_1\}$, etc. But, if so, then C_1 is consistent with $S \cup \{A\}$, C_2 with $S \cup \{A\} \cup \{C_1\}$, etc., and hence by Lemma 1(b)

$$L_i(S \cup \{A\}) = S \cup \{A\} \cup \{C_1, C_2, \dots, C_i\}.$$

Suppose then that

$$L_i(S \cup \{A\}) = S \cup \{A\} \cup \{C'_1, C'_2, \dots, C'_i\},$$

and suppose $L(S \cup \{A\} \vdash B$. Then by Lemma 1(d) B is consistent with $S \cup \{A\} \cup \{C'_1\}$, and hence C'_1 is consistent with $S \cup \{B \,\&\, A\}$. Hence by the same reasoning C'_2 is consistent with $S \cup \{B \,\&\, A\} \cup \{C'_1\}$. And so on. Hence

$$L_i(S \cup \{B \,\&\, A\}) = S \cup \{B \,\&\, A\} \cup \{C'_1, C'_2, \dots, C'_i\}.$$

Hence

$$L(S \cup \{A\}) = S \cup \{A\} \cup S'$$

and

$$L(S \cup \{B \,\&\, A\}) = S \cup \{B \,\&\, A\} \cup S'$$

for some common set S' of statements of L. But by Lemma 1(c) each of A and $B \,\&\, A$ is sure to be a member of S'. Hence $L(S \cup \{A\}) = L(S \cup \{B \,\&\, A\})$. Hence Lemma 3.

With Lemmas 2 and 3 at hand we show that any set S of statements of L generates a function for L meeting Constraints **C1–C6**; and, when S is consistent, one meeting Constraint **C0** as well. The result will readily yield the converse of Theorem 1.

THEOREM 2. *Let S be a set of statements of L, and P_S be the binary function such that, for any two statements A and B of L,*

$$P_S(A/B) = \begin{cases} 1 & \text{if } L(S \cup \{B\}) \vdash A \\ 0 & \text{otherwise.} \end{cases}$$

Then:

(a) *P_S meets Constraints* **C1–C6**;

(b) *If S is consistent, then P_S meets Constraint* **C0** *as well.*

Proof: (a) That P_S meets Constraints **C1–C6** can be shown as follows.
Constraint **C1**: $0 \leq P_S(A/B)$ by the very construction of P_S.
Constraint **C2**: $L(S \cup \{A\}) \vdash A$. Hence $P_S(A/A) = 1$.
Constraint **C3**: Suppose $(\exists C)(P_S(C/B) \neq 1)$. Then $(\exists C)(L(S \cup \{B\}) \nvdash C)$, and hence $L(S \cup \{B\})$ is consistent. Suppose next that $P_S(A/B) = 1$, in which case $L(S \cup \{B\}) \vdash A$. Then $L(S \cup \{B\}) \nvdash \sim A$ by the consistency of $L(S \cup \{B\})$, and hence $P_S(\sim A/B) = 0$. Suppose finally that $P_S(A/B) = 0$. Then $L(S \cup \{B\}) \nvdash A$, hence by the completeness of $L(S \cup \{B\})$, $L(S \cup \{B\}) \vdash \sim A$, and hence $P_S(\sim A/B) = 1$. Hence, if $(\exists C)(P_S(C/B) \neq 1)$, then $P_S(\sim A/B) = 1 - P_S(A/B)$.
Constraint **C4**: Suppose *first* that $P_S(A \,\&\, B/C) = 1$. Then $L(S \cup \{C\}) \vdash A \,\&\, B$, and hence both $L(S \cup \{C\}) \vdash A$ and $L(S \cup \{C\}) \vdash B$. But, if $L(S \cup \{C\}) \vdash B$, then by Lemma 3 $L(S \cup \{C\}) = L(S \cup \{B \,\&\, C\})$, and hence $L(S \cup \{B \,\&\, C\}) \vdash A$. Hence both $P_S(A/B \,\&\, C)$ and $P_S(B/C)$ equal 1, and hence $P_S(A \,\&\, B/C) = P_S(A/B \,\&\, C) \times P_S(B/C)$. Suppose *next* that $P_S(A \,\&\, B/C) = 0$. Then $L(S \cup \{C\}) \nvdash A \,\&\, B$, hence either $L(S \cup \{C\}) \nvdash A$ or $L(S \cup \{C\}) \nvdash B$ (or both), and hence either $P_S(A/C) = 0$ or $P_S(B/C) = 0$ (or both). Now suppose $P_S(B/C) \neq 0$, in which case $P_S(A/C) = 0$. Then $P_S(B/C) = 1$, hence $L(S \cup \{C\}) \vdash B$, hence by Lemma 3 $L(S \cup \{C\}) = L(S \cup \{B \,\&\, C\})$, hence $L(S \cup \{C\}) \vdash A$ if and only if $L(S \cup \{B \,\&\, C\}) \vdash A$, hence $P_S(A/C) = P_S(A/B \,\&\, C)$, and hence $P_S(A/B \,\&\, C) = 0$. Hence, if $P_S(B/C) \neq 0$, then $P_S(A/B \,\&\, C) = 0$. Hence $P_S(B/C) = 0$ or $P_S(A/B \,\&\, C) = 0$. Hence again $P_S(A \,\&\, B/C) = P_S(A/B \,\&\, C) \times P_S(B/C)$.
Constraint **C5**: $L(S \cup \{C\}) \vdash A \,\&\, B$ if and only if $L(S \cup \{C\}) \vdash B \,\&\, A$, hence $P_S(A \,\&\, B/C) = P_S(B \,\&\, A/C)$, and hence $P_S(A \,\&\, B/C) \leq P_S(B \,\&\, A/C)$.
Constraint **C6**: Both when consistent and when not, $S \cup \{B \,\&\, C\}$ and $S \cup \{C \,\&\, B\}$ are deductively equivalent. Hence by Lemma 2 $L(S \cup \{B \,\&\, C\}) = L(S \cup \{C \,\&\, B\})$, hence $L(S \cup \{B \,\&\, C\}) \vdash A$ if and only if $L(S \cup \{C \,\&\, B\}) \vdash A$, hence $P_S(A/B \,\&\, C) = P_S(A/C \,\&\, B)$, and hence $P_S(A/B \,\&\, C) \leq P_S(A/C \,\&\, B)$.

(b) Suppose S is consistent. Then $S \cup \{T\}$ is consistent, hence so is $L(S \cup \{T\})$, hence $L(S \cup \{T\}) \nvdash \sim T$, hence $P_S(\sim T/T) = 0$, hence $(\exists A)(\exists B)(P_S(A/B) \neq 1)$, and hence P_S meets Constraint **C0** as well as Constraints **C1-C6**. Hence (b).

THEOREM 3. (a) *If a set S of statements of L is consistent, then there is a probability function P for L such that* $\{A : S \vdash A\} = \{A : (\forall B)(P(A/B) = 1)\}$. (b) *If S is consistent and deductively closed, then* *there is a probability function P for L such that* $S = \{A : (\forall B)(P(A/B) = 1)\}$.

Proof: Let S be consistent.

(a) Let P_S be the function defined in the preamble of Theorem 2, and suppose *first* that $S \vdash A$. Then $(\forall B)(L(S \cup \{B\}) \vdash A)$, and hence $(\forall B)(P_S(A/B) = 1)$ by the definition of P_S in Theorem 2. Suppose *next* that $S \nvdash A$. Then $S \cup \{\sim A\}$ is consistent, and hence so is $L(S \cup \{\sim A\})$. But $L(S \cup \{\sim A\}) \vdash \sim A$. Hence $L(S \cup \{\sim A\}) \nvdash A$, hence $P_S(A/\sim A) = 0$ by the definition of P_S, and hence $(\exists B)(P_S(A/B) \neq 1)$. Hence $S \vdash A$ if and only if $(\forall B)(P_S(A/B) = 1)$. Hence (a).

(b) Suppose S is deductively closed. Then $S = \{A : S \vdash A\}$. Hence (b) by (a).

Hence:

THEOREM 4. *Any set of statements of L that is consistent and deductively closed is the assumption set of a probability function for L.*

Hence Case Two of the Fundamental Theorem on Assumption Sets:

THEOREM 5. *A set of statements of L is the assumption set of a* (binary) *probability function for L if and only if it is consistent and deductively closed.*

As the reader may verify, $S \vdash A$ if and only if A belongs to each consistent and deductively closed set of L of which S is a subset. Theorem 5 thus guarantees that

(3) *If* $S \vdash A$, *then A belongs to each assumption set of L of which S is a subset,*

and

(4) *If A belongs to each assumption set of L of which S is a subset, then* $S \vdash A$,

and hence – given the account of logical implication on p. 94 – the theorem yields each of (1) and (2) on that page.

That Theorem 5 yields (1) signifies little since we called on that version of the Soundness Theorem for L when proving Theorem 1. That Theorem 5 yields (2) is of more interest, as indeed may be the following proof of (2) by means of just Theorem 3(a). Suppose that $S \nvdash A$, in which case $S \cup \{\sim A\}$ is sure to be consistent. Then by Theorem

3(a) there exists a probability function for L, call it $P_{S\cup\{\sim A\}}$, such that (i) $(\forall C)(P_{S\cup\{\sim A\}}(B/C)=1)$ for each statement B of L that belongs to S and hence is provable from S, and (ii) $(\forall C)(P_{S\cup\{\sim A\}}(\sim A/C)=1)$ and hence $P_{S\cup\{\sim A\}}(\sim A/T)=1$. But, as a brief argument using Constraint **C0** would show, T is P-normal, this for each probability function P for L.[9] Hence by Constraint **C3** $P_{S\cup\{\sim A\}}(A/T)=0\neq 1$. Hence there exists a probability function P for L such that $(\forall C)(P(B/C)=1)$ for each member B of S and yet $(\exists C)(P(A/C)\neq 1)$. Hence, by Contraposition, A is sure – if logically implied by S in the probabilistic sense – to be provable from S.

Returning to a point raised on p. 92, suppose that Constraint **C0** were dropped. Theorems 1 and 2(a) would then guarantee that a set of statements of L constitutes an assumption set of L if and only if it is deductively closed. The result is easily accommodated here. We just pointed out that $S\vdash A$ if and only if A belongs to each consistent and deductively closed set of statements of L with S as a subset. However, it can also be shown that $S\vdash A$ if and only if A belongs to each deductively closed set of statements of L – be the set consistent or not – with S as a subset. So, with **C0** dropped from the list of constraints on p. 93. A would still be provable from S if and only if it belongs to each assumption set of L with S as a subset. In view of this result and others, one might willingly jettison **C0**.[10]

* * *

The assumption sets of L can be sorted as follows:

Group One, to consist of $\{A: \vdash A\}$, a set included in all the assumption sets of L,
Group Two, to consist of the remaining assumption sets of L that are *incomplete*, and
Group Three, to consist of all the assumption sets of L that are *complete* – hence, to consist of the "maximally consistent" sets of statements of L.

The set in Group One is the "smallest" assumption set of L: it is the only assumption set of L no proper subset of which constitutes an assumption set of L. The sets in Group Three, on the other hand, are the "largest" assumption sets of L as befits *maximally* consistent sets: they are the only assumption sets of L no proper superset of which constitutes an assumption set of L.

The probability functions for L with $\{A: \vdash A\}$ as their assumption set are called in Leblanc and van Fraassen 1979 *Carnap's probability functions* for L.[11] They are the probability functions for L that meet the following constraint (due to van Fraassen and to Fine, independently):

C7. No matter the state-description C of L, $P({\sim}C/C) = 0$,

a *state-description* of L being a conjunction of the sort $(\ldots(\pm A_1 \,\&\, \pm A_2) \,\&\, \ldots) \,\&\, \pm A_n$, where $n \geq 1$, $A_1, A_2, \ldots, A_n$ are in alphabetic order the first n atomic statements of L, and for each i from 1 through n, $\pm A_i$ is A_i or ${\sim}A_i$.[12]

LEMMA 4. *Let C be an arbitrary state-description of L.*

(a) *If* $P({\sim}C/C) = 0$, *then – no matter the statement A of L –* $(\forall B)(P(A/B) = 1)$ *if and only if* $\vdash A$.

(b) *If – no matter the statement A of L –* $(\forall B)(P(A/B) = 1)$ *if and only if* $\vdash A$, *then* $P({\sim}C/C) = 0$.

Proof: (a) Let $P({\sim}C/C) = 0$, and hence $(\exists B)(P(B/C) \neq 1)$; and suppose first that $\vdash A$. Then $(\forall B)(P(A/B) = 1)$ by the Soundness Theorem for L. Suppose next that $\nvdash A$. Then there is sure to be a state-description C of L such that $\vdash C \supset {\sim}A$, hence by a familiar result such that $P({\sim}A/C) = 1$,[13] and hence by **C3** (and $(\exists B)(P(B/C) \neq 1)$) such that $P(A/C) = 0 \neq 1$. Hence $(\exists B)(P(A/B) \neq 1)$. Hence $(\forall B)(P(A/B) = 1)$ if and only if $\vdash A$. Hence (a).[14]

(b) Let $(\forall B)(P(A/B) = 1)$ if and only if $\vdash A$, this for any statement A of L. Since $\nvdash {\sim}C$, $(\exists B)(P({\sim}C/B) \neq 1)$, and hence by a familiar result $P({\sim}C/C) = 0$.[15] Hence (b).

Hence:

THEOREM 6. $\{A: \vdash A\}$ *is the assumption set of all and only those probability functions for L known as Carnap's probability functions.*[16]

The probability function P_S in Theorem 2 being 2-valued, each assumption set of L – i.e., $\{A: \vdash A\}$, each set in Group Two, and each one in Group Three – is the assumption set of at least one 2-valued probability function for L. $\{A: \vdash A\}$ and the assumption sets in Group Two are also the assumption sets of probability functions boasting more than 2 values. Not so, however, the sets in Group Three. All probability functions with these sets as assumption sets are indeed 2-valued (Theorem 7(b)).

Further, $\{A: \vdash A\}$ is the assumption set of $2^{\aleph_0}$ probability functions, as is each of the $2^{\aleph_0}$ sets in Group Two. Each of the $2^{\aleph_0}$ sets in Group Three, by contrast, is the assumption set of exactly one probability function (Theorem 8(a)). However, since $2^{\aleph_0} \times 2^{\aleph_0}$ equals $2^{\aleph_0}$, the probability functions with a set from Group Two as their assumption set are $2^{\aleph_0}$ in number, as are those with $\{A: \vdash A\}$ or with a set from Group Two as their assumption set.

LEMMA 5. *Let S be a maximally consistent set of statements of L, and P be any probability function for L such that $S = \{A: (\forall B)(P(A/B) = 1)\}$. Then, for any statements A and B of L, $P(A/B)$ equals 1 or 0 according as $B \supset A$ belongs to S or not.*

Proof: Let A and B be arbitrary statements of L, and suppose *first* that B is P-abnormal. Then by definition $P(A/B) = 1$, and hence $P(A/B) = 1$ if $B \supset A \in S$. Suppose *then* that B is P-normal and $B \supset A \in S$. Then $\sim B \in S$ or $A \in S$; hence, by the hypothesis on S and that on P, $(\forall C)(P(\sim B/C) = 1)$ or $(\forall C)(P(A/C) = 1)$; and hence $P(\sim B/B) = 1$ or $P(A/B) = 1$. But $P(\sim B/B) = 0 \neq 1$ by **C2**, **C3**, and the P-normality of B. Hence $P(A/B) = 1$. Suppose *finally* that B is P-normal but $B \supset A \notin S$. Then $\sim A \in S$ by the hypothesis on S, hence $(\forall C)(P(\sim A/C) = 1)$ by the hypothesis on S and that on P, hence $P(\sim A/B) = 1$, and hence $P(A/B) = 0$ by **C3** and the P-normality of B. Hence Lemma 5.

Hence:

THEOREM 7. *Let S be a maximally consistent set of L. Then:*

(a) *S is the assumption set of exactly one probability function for L;*

(b) *The one probability function for L of which S is the assumption set is 2-valued.*

Proof: (a) Suppose $S = \{A: (\forall B)(P_1(A/B) = 1)\}$ and $S = \{A: (\forall B)(P_2(A/B) = 1)\}$. Then by Lemma 5, $P_1(A/B) = P_2(A/B)$ for any statements A and B of L. (b) By (a) and Lemma 5.

* * *

It is because of Theorem 5, to wit:

(5) $(\exists P)(S = \{A : (\forall B)(P(A/B) = 1)\})$ *if and only if S is consistent and deductively closed,*

and our feeling that all and only consistent and deductively closed sets of statements of L qualify as assumption sets, that we appointed $\{A : (\forall B)(P(A/B) = 1)\}$ the assumption set of P. Alternatively, we could have required of the assumption set S of P that

(6) $S = \{A : (\forall B)(\forall C)(P(A \& B/C) = P(B/C))\}$

and

(7) $S = \{A : (\forall B)(\forall C)(P(B/A \& C) = P(B/C))\}$.

Showing that S meets desiderata (6)–(7) *if and only if*

(8) $S = \{A : (\forall B)(P(A/B) = 1)\}$

would have been an easy task, and Theorem 5 would now assure us that a set of statements of L counts as an assumption set if and only if it is consistent and deductively closed.

The route is a longer one, but going about things thusly would effectively rule out

$$\{A : P(A/T) = 1\}$$

from consideration as the assumption set of P. $\{A : P(A/T) = 1)\}$ does have credentials. It can indeed by shown, as Leblanc noted in early 1982, that

(9) $S \vdash A$ *if and only if – no matter and probability function P for L –* $P(A/T) = 1$ *if* $P(B/T) = 1$ *for each member B of S,*[17]

and

(10) $(\exists P)(S = \{A : P(A/T) = 1)\}$ *if and only if S is consistent and and deductively closed.*

Because of (9) one might take *S to logically imply A in the probabilistic sense* if – no matter the probability function P for L – $P(A/T) = 1$ if $P(B/T) = 1$ for each member B of S. Despite (10), however, one should not understand $\{A : P(A/T) = 1\}$ as the assumption set of L: it does not follow form **C1–C6** that

(11) *If* $P(A/T) = 1$, *then* $(\forall B)(\forall C)(P(B/A \& C) = P(B/C))$.

For proof, assign 2 to A, 0 to B, and 1 to C. According to the matrices on p. 338 of Popper 1959, $P(A/T)$ will then equal 1, and yet $P(B/A \& C)$ will equal 1 while $P(C/B)$ equals 0.

Interestingly enough, $\{A:(\forall B)(P(A/B) = 1)\}$ and $\{A: P(A/T) = 1\}$ are the same, Morgan recently noted, when P is what we acknowledge in Morgan and Leblanc 1983a as an intuitionistic probability function for L. But intuitionistic logic is not our present concern.[18,19]

NOTES

[1] All pertinent texts are listed in the References. We refer to them in the body of the paper and in the notes by author's name and date of publication.

[2] Theorem 5 holds with $(\exists P)(S = \{A: P(A/T) = 1\})$ in place of $(\exists P)(S = \{A:(\forall B)(P(A/B) = 1)\})$. But, as we shall see on pp. 13–14, $\{A: P(A/T) = 1\}$ is unsuitable as the assumption set of P.

[3] A paper from which we freely borrow when covering matters already treated there.

[4] See p. 349. That Popper's constraints **A1–A3**, **B1–B2**, and **C** are equivalent to **C0–C6** is shown in Harper, Leblanc, and van Fraassen 1983 and in Leblanc 1981. In earlier writings of Leblanc's **C5**-**C6** appeared as equalities; we use inequalities here to preserve consistency with Popper 1955, Leblanc 1983a, and Leblanc 1983b; in those texts the singulary counterpart of **C5** runs $P(A \& B) \leq P(B \& A)$ (rather than $P(A \& B) = P(B \& A)$).

[5] Given this definition **C3** may of course be made to read: "If B is P-normal, then $P(\sim A/B) = 1 - P(A/B)$."

[6] From now on we shall often drop the (already parenthesized) 'binary'.

[7] The proof is an adaption to the binary case of the proof of Theorem 1 in Leblanc 1983b.

[8] $L(S)$ is commonly defined only for consistent S, in which case $L_i(S)$ is taken to be $L_{i-1}(S) \cup \{A_i\}$ when $L_{i-1}(S) \cup \{A_i\}$ is consistent, otherwise to be $L_{i-1}(S) \cup \{\sim A_i\}$.

[9] See the proof of T5.33(c) in Leblanc 1983a.

[10] Only one set of statements of L is both inconsistent and deductively closed: the set of all the statements of L. The probability function with that set as its assumption set would of course be the function P such that $(\forall A)(\forall B)(P(A/B) = 1)$. See Leblanc 1983b, pp. 381 and 395, for more on requiring assumption sets to be consistent and for ways of lifting that requirement in the singulary case.

[11] Also *Popper's probability functions in the narrow sense*, an unfortunate appellation which we hope will not gain currency.

[12] In the presence of **C7**, **C0** becomes of course redundant.

[13] The result trivially holds true when C is P-abnormal. So suppose C is P-normal, and suppose $\vdash C \supset \sim A$ (i.e., $\vdash \sim(C \& A)$). Then by the Soundness Theorem for L $P(\sim(C \& A)/C) = 1$, hence by **C3** $P(C \& A/C) = 0$, and hence by **C4** $P(C/A \& C) \times P(A/C) = 0$. But, as we establish a few lines hence, $P(C/A \& C) = 1$. Hence $P(A/C) = 0$, and hence by **C3** and the P-normality of C $P(\sim A/C) = 1$. For proof that

$P(C/A \& C) = 1$, note that by **C2** $P(A \& C/A \& C) = 1$ and hence by **C4** $P(A/C \& (A \& C)) \times P(C/A \& C) = 1$. But by **C1** and **C3** any probability lies in the interval [0, 1], and hence each of two probabilities must equal 1 if their product does. Hence $P(C/A \& C)$ has to equal 1.

[14] The first correct proof of (a), an embarrassingly long one, is in Leblanc and van Fraassen 1979.

[15] We prove that if $P(\sim C/C) \neq 0$, then $(\forall B)(P(\sim C/B) = 1)$. The result trivially holds true when B is P-abnormal. So suppose B is P-normal (Hypothesis One) and $P(\sim C/C) \neq 0$ (Hypothesis Two). By Hypothesis One and **C2–C3**, $P(\sim B/B) = 0$, hence by **C4** $P(C \& \sim B/B) = 0$, hence by **C5** $P(\sim B \& C/B) = 0$, and hence by **C4** $P(\sim B/C \& B) \times P(C/B) = 0$. But by Hypothesis Two and **C2–C3**, $(\forall B)(P(B/C) = 1)$, hence $P(\sim B \& B/C) = 1$, hence by **C4** and the result obtained at the close of note 13, $P(\sim B/B \& C) = 1$, and hence by **C6** $P(\sim B/C \& B) = 1$. Hence $P(C/B) = 0$, and hence by **C3** (and the P-normality of B) $P(\sim C/B) = 1$.

[16] For further information on Carnap's probability functions, see Leblanc and van Fraassen 1979, and Harper, Leblanc, and van Fraassen 1983.

[17] The proofs of the Soundness and Completeness Theorems in Leblanc 1983b are easily edited to yield (9). As for (10), establish first that the results of putting $P(\cdot/T)$ for $P(\cdot)$ in Constraints **C1–C6** of Leblanc 1983b follow from Constraints **C0–C6** in this paper. Putting $P(\cdot/T)$ everywhere for $P(\cdot)$ in the proof of Theorem 4 in Leblanc 1983b will then yield (10).

[18] Proving Theorem 5 for a language L with quantifiers as well as connectives is our next order of business. Bas van Fraassen reported in October 1982 that such a proof can be retrieved from his 1982 paper. However, the constraints he places there on binary probability functions are more restrictive than the ones commonly used.

[19] The paper is an elaboration of part of Leblanc's talk at the Conference on Foundations. While working on these matters Leblanc held a research grant from the National Science Foundation (Grant SES 8007179) and was on partial research leave from Temple University. Thanks are due to Tom McGinness, Muffy E. Siegel, and Bas van Fraassen for reading an earlier draft of the paper.

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