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NONCOMPENSATORY PREFERENCES*

1. INTRODUCTION

This paper provides a general characterization of noncompensatory preference structures in multi-attribute preference theory, then examines a number of conditions that might hold for such structures. It will be assumed throughout that $>$ ('is preferred to') is an asymmetric binary relation on a product set $X = X_1 \times X_2 \times \ldots \times X_n$, with each X_i nonempty and $n \geqslant 2$. The indifference relation $\sim$ on X is defined from $>$ by $x \sim y$ iff neither $x > y$ nor $y > x$. We shall refer both to i and to X_i as an attribute.

Loosely speaking, $(X, >)$ will be said to be a noncompensatory preference structure if it satisfies a simple independence condition pertaining to conditional preferences on the X_i and if $>$ between any two n-tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ depends solely on the coordinates i for which x_i is conditionally preferred to y_i and for which y_i is conditionally preferred to x_i. Since this prohibits compensating trade-offs among different attributes, noncompensatory structures are probably much less common than compensatory ones. However, as shown by Chipman (1960), Coombs (1964), Green and Wind (1973), MacCrimmon (1973), Fishburn (1974), and others who are cited in these studies, there are many situations in which a noncompensatory preference or choice model may be a useful guide for decision making or a realistic descriptor of a decision agent's preferences.

The definition of noncompensatory preference structure given below does not presume that the preference relation $>$ is transitive or acyclic. This is in keeping with examples and arguments based on sensory or judgmental thresholds, individual feelings about what constitutes a significant difference between levels of an attribute, and other aspects (e.g., Davidson *et al.*, 1955; Coombs, 1964; Weinstein, 1968; Tversky, 1969; Schwartz, 1972; Fishburn, 1974) that can give rise to cyclic preferences in multiattribute contexts. The effects of ordering assumptions for $>$ on noncompensatory preferences will be considered later in the paper where

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the most restrictive ordering assumption is shown to imply that $(X, \succ)$ has a lexicographic structure.

2. NONCOMPENSATORY PREFERENCE STRUCTURES

For notational convenience let $(x_i, (a_j)_{j\neq i})$ denote the n-tuple $(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$. Then for each i and all $x_i, y_i \in X_i$, we define a binary relation $\succ_i$ on X_i by

$$x_i \succ_i y_i \quad \text{iff } (x_i, (a_j)_{j\neq i}) \succ (y_i, (a_j)_{j\neq i})$$
$$\text{for all } (a_j)_{j\neq i} \in \bigtimes_{j\neq i} X_j.$$

Each $\succ_i$ is asymmetric since $\succ$ is presumed to be asymmetric. For each ordered pair $(x, y) = ((x_1, \ldots, x_n), (y_1, \ldots, y_n))$ of n-tuples in X let $P(x, y)$ be the set of all $i \in \{1, \ldots, n\}$ for which x_i stands in the relation $\succ_i$ to y_i:

$$P(x, y) = \{i: x_i \succ_i y_i\}.$$

The ordered pair $(P(x, y), P(y, x))$ then identifies two disjoint subsets of $\{1, \ldots, n\}$ such that all i in the first have $x_i \succ_i y_i$, and all i in the second have $y_i \succ_i x_i$.

DEFINITION 1. $(X, \succ)$ is an N.P.S. (*noncompensatory preference structure*) if and only if, for all $x, y, z, w \in X$,

(1) $[(P(x, y), P(y, x)) = (P(z, w), P(w, z))] \Rightarrow [x \succ y \text{ iff } z \succ w]$.

Thus, preference between x and y for an N.P.S. depends only on the specific i for which $x_i \succ_i y_i$ and the specific i for which $y_i \succ_i x_i$. This is the characteristic feature of all noncompensatory preference structures. A secondary feature of such structures is the conditional independence property

(2) $(x_i, (x_j)_{j\neq i}) \succ (y_i, (x_j)_{j\neq i}) \text{ iff } (x_i, (y_j)_{j\neq i}) \succ (y_i, (y_j)_{j\neq i})$

for all i and all $x_i, y_i \in X_i$, which is an immediate consequence of (1) and the asymmetry of each $\succ_i$. Conditional independence has wide applicability beyond the context of noncompensatory preferences. For example, (2) arises frequently in compensatory situations, and it is necessary for the standard additive utility model (e.g., Debreu, 1960; Luce and Tukey, 1964; Fishburn, 1970) that has $x > y$ iff $\sum_i u_i(x_i) > \sum_i u_i(y_i)$, where u_i is a real-valued function on X_i.

Attribute X_i will be called *essential* iff $x_i >_i y_i$ for some $x_i, y_i \in X_i$. Since nonessential attributes contribute nothing to our analysis of an N.P.S., we shall assume henceforth that every X_i is essential. It then follows that the set of all possible pairs $(P(x, y), P(y, x))$ for $x, y \in X$ is the set

$$S = \{(A, B): A, B \subseteq \{1, \dots, n\} \text{ and } A \cap B = \phi\}.$$

With $\gg$ and $\approx$ defined on subsets of $\{1, \dots, n\}$ by

$A \gg B$ iff $(A, B) \in S$ and $x > y$ whenever $(P(x, y), P(y, x)) = (A, B)$,

$A \approx B$ iff $(A, B) \in S$ and $x \sim y$ whenever $(P(x, y), P(y, x)) = (A, B)$,

an N.P.S. is described by specifying the one of $A \gg B$, $B \gg A$ and $A \approx B$ that holds for each $A, B \subseteq \{1, \dots, n\}$ for which $A \cap B = \phi$, subject to the limitations in the following lemma. (Asymmetry of $>$ and essentiality are presupposed.)

LEMMA 1. $(X, >)$ is an N.P.S. iff $\phi \approx \phi$, $\{i\} \gg \phi$ for each $i \in \{1, \dots, n\}$, and exactly one of $A \gg B$, $B \gg A$ and $A \approx B$ holds for every $A, B \subseteq \{1, \dots, n\}$ for which $A \cap B = \phi$.

Proof. Suppose $(X, >)$ is an N.P.S. Then exactly one of $A \gg B$, $B \gg A$ and $A \approx B$ holds for each disjoint pair $A, B \subseteq \{1, \dots, n\}$. Moreover, $\phi \approx \phi$ since $x \sim x$ and $(P(x, x), P(x, x)) = (\phi, \phi)$, and $\{i\} \gg \phi$ since $x_i >_i y_i$ requires $(x_i, (a_j)_{j \neq i}) > (y_i, (a_j)_{j \neq i})$ with $(\{i\}, \phi) = (P((x_i, (a_j)_{j \neq i}), (y_i, (a_j)_{j \neq i})), P((y_i, (a_j)_{j \neq i}), (x_i, (a_j)_{j \neq i})))$. Conversely, (1) follows from the conditions in the theorem on $\gg$ and $\approx$.

Let $x_i \sim_i y_i$ mean that neither $x_i >_i y_i$ nor $y_i >_i x_i$. Then, when $(X, >)$ is an N.P.S., it follows immediately from $\phi \approx \phi$ and $\{i\} \gg \phi$ that

(3) $\quad x_i \sim_i y_i \quad \text{for all } i \Rightarrow x \sim y,$

$\quad x_i >_i y_i \quad \text{and} \quad x_j \sim_j y_j \quad \text{for all } j \neq i \Rightarrow x > y.$

However, when $A \cup B$ contains two or more $i \in \{1, \ldots, n\}$ and $A \cap B = \phi$, the lemma allows any one of $A \gg B$, $B \gg A$ and $A \approx B$ to hold. In particular, an N.P.S. can have $y \succ x$ along with $x_i \succ_i y_i$ for all i. For example, with $X = \{x_1, y_1\} \times \{x_2, y_2\}$, suppose that

$$(x_1, x_2) \succ (y_1, x_2)$$

$$(y_1, x_2) \succ (y_1, y_2)$$

$$(y_1, y_2) \succ (x_1, x_2)$$

$$(x_1, x_2) \succ (x_1, y_2)$$

$$(x_1, y_2) \succ (y_1, y_2)$$

$$(x_1, y_2) \sim (y_1, x_2).$$

Then $(X, \succ)$ is an N.P.S. with $x_1 \succ_1 y_1$, $x_2 \succ_2 y_2$ and $(y_1, y_2) \succ (x_1, x_2)$. For this N.P.S.,

$$\{1\} \approx \{2\}, \{1\} \gg \phi, \{2\} \gg \phi \quad \text{and} \quad \phi \gg \{1, 2\}.$$

Although $\phi \gg \{1, \ldots, n\}$ may seem strange, and can occur only if $\succ$ has cycles, there is nothing in Definition 1 that prevents it. The next section examines conditions that forbid such behavior and which, in varying degrees, give coherence to the structure of noncompensatory preferences.

Despite the terminology used here, it should be remarked that an N.P.S. allows interactions among attributes. For example, if $\{1\} \gg \{2\}$ and $\{1\} \gg \{3\}$, it may be true that $\{2, 3\} \gg \{1\}$, so that the combination of X_2 and X_3 outweighs X_1 although X_1 outweighs X_2 or X_3 separately. It could also be true here that $\phi \gg \{2, 3\}$, in which case 'good values' of X_2 and X_3 considered separately become undesirable in combination. Hence an N.P.S. allows compensatory effects and interdependencies within the attribute set even though changes within attributes that preserve the $\succ_i$ or $\sim_i$ relationships cannot alter these effects.

3. SPECIAL CONDITIONS FOR NONCOMPENSATORY PREFERENCES

In the interests of brevity, I shall first define a number of special conditions together and then present a theorem that shows the implications among these conditions. A proof of the theorem will be given in the final section.

The following preliminary definitions are needed. First, we define a binary relation $\sqsupset$ on S by

$$(A, B) \sqsupset (C, D) \quad \text{iff } (A, B), (C, D) \in S, A \supseteq C \text{ and } B \subseteq D \text{ with at least one inclusion a proper inclusion.}$$

Thus, e.g., $(\{1, 2, 3\}, \phi) \sqsupset (\{1, 2, 3\}, \{5\})$, $(\phi, \{1, 2\}) \sqsupset (\phi, \{1, 2, 4\})$ and $(\{1, 2, 3\}, \{6\} \sqsupset (\{3\}, \{4, 6\})$. In going from (A, B) to (C, D) when $(A, B) \sqsupset (C, D)$, A is contracted to yield C and B is expanded to give D.

Second, a binary relation R on a set Y is *acyclic* iff there are no $a_1, a_2, \ldots, a_m \in Y$ for which $a_1Ra_2, a_2Ra_3, \ldots, a_{m-1}Ra_m$ and a_mRa_1; R is a *strict partial order* iff it is asymmetric and transitive; and R is a *weak order* (asymmetric sense) iff it is asymmetric and negatively transitive (for all $a_1, a_2, a_3 \in Y$, if a_1Ra_2 then either a_1Ra_3 or a_3Ra_2).

DEFINITION 2. An N.P.S. $(X, \succ)$ is

C1. *regular* iff $A \gg \phi$ for all nonempty $A \subseteq \{1, \ldots, n\}$;

C2. *monotonic* iff $[(A, B) \sqsupset (C, D)$ and $C \gg D] \Rightarrow A \gg B$;

C3. *strongly monotonic* iff $[(A, B) \sqsupset (C, D)$ and either $C \gg D$ or $C \approx D] \Rightarrow A \gg B$;

C4. *additive* iff $[A \cap C = \phi, (A \cup C) \cap (B \cup D) = \phi, A \gg B$ and either $C \gg D$ or $C \approx D] \Rightarrow A \cup C \gg B \cup D$;

C5. *superadditive* iff $[(A \cup C) \cap (B \cup D) = \phi, A \gg B$ and $C \gg D] \Rightarrow A \cup C \gg B \cup D$;

C6. *decisive* iff $[(A, B) \in S$ and $(A, B) \neq (\phi, \phi)] \Rightarrow A \gg B$ or $B \gg A$;

C7. *attribute acyclic* iff $\gg$ on the subsets of $\{1, \ldots, n\}$ is acyclic;

C8. *acyclic* iff $\succ$ on X is acyclic;

C9. *partially ordered* iff $\succ$ on X is a strict partial order;

C10. *weakly ordered* iff $\succ$ on X is a weak order;

C11. *doubly essential* iff for each $i \in \{1, \ldots, n\}$ there are $a_i, b_i, c_i \in X_i$ such that $a_i \succ_i b_i$ and $b_i \succ_i c_i$;

C12. *lexicographic* iff there is a permutation σ on $\{1, \ldots, n\}$ such that, for all $x, y \in X$, $x \succ y$ iff $x_i \not\sim_i y_i$ for some i and $x_{\sigma(i)} \succ_{\sigma(i)} y_{\sigma(i)}$ for the smallest i for which $x_{\sigma(i)} \not\sim_{\sigma(i)} y_{\sigma(i)}$.

Regularity (C1) is the natural extension of (3) or $\{i\} \gg \phi$: if $x_i \succ_i y_i$ for all $i \in A \neq \phi$ and $x_j \sim_j y_j$ for all $j \notin A$, then $x \succ y$. Monotonicity (C2) says that if $x \succ y$ and if x and y are changed to x' and y' in favor of x' on an attribute-by-attribute basis, then $x' \succ y'$. Strong monotonicity (C3) augments this by asserting that if $x \sim y$ before the changes then $x' \succ y'$ afterwards.

Additivity (C4) is one of many conditions that might be called 'additive'. It says that if the attributes in A are collectively more important than those in B, and the attributes in C are collectively at least as important as those in D, and if $A \cap C = \phi$ and $(A \cup C) \cap (B \cup D) = \phi$, then the attributes in $A \cup C$ are collectively more important than those in $B \cup D$. This can be strengthened or modified in various ways. One modification, which is a very powerful strengthening of the strict part of C4, is superadditivity (C5). The strength of C5 lies in not requiring $A \cap C = \phi$ in its hypotheses. Thus, if $\{1\} \gg \{2, 3\}$ and $\{1\} \gg \{4, 5, 6\}$, then C5 requires $\{1\} \gg \{2, 3, 4, 5, 6\}$. As noted later, superadditivity is a key aspect of lexicographic preferences.

Decisiveness (C6) asserts that $x \sim y$ iff $x_i \sim_i y_i$ for all i, so that a conditional preference on at least one attribute prohibits overall indifference. It too is a key aspect of lexicographic preferences.

Attribute acyclicity (C7) forbids the derived preference relation $\gg$ on subsets of attributes from cycling, and C8, C9 and C10 place increasingly restrictive ordering properties on the basic preference relation. The penultimate condition, double essentiality (C11), will be used to facilitate certain implications between other conditions.

If $(X, \succ)$ is a lexicographic N.P.S. and σ is as given in Definition 2, then $X_{\sigma(1)}$ is the dominant attribute, $X_{\sigma(2)}$ is the next most important attribute, and $X_{\sigma(n)}$ is the least important attribute. An extensive survey of lexicographic structures is given by Fishburn (1974).

The following theorem lists the basic implications among the twelve conditions of Definition 2.

THEOREM 1. Suppose $(X, \succ)$ is an N.P.S. Then

(a) $\quad C10 \Rightarrow C4 \Rightarrow C3 \Rightarrow C2 \Rightarrow C1$;

(b) [C9 & C11]$\Rightarrow$C5$\Rightarrow$C1;
(c) [C5 & C6]$\Rightarrow$C4;
(d) C10$\Rightarrow$C9$\Rightarrow$C8$\Rightarrow$C7;
(e) C9$\Rightarrow$C2;
(f) [C5 & C6 & C7]$\Leftrightarrow$C12;
(g) [C10 & C11]$\Rightarrow$C12.

Part (a) says that a weakly ordered N.P.S. is additive, an additive N.P.S. is strongly monotonic, and so forth; and (b) says that a partially ordered and doubly essential N.P.S. is superadditive, and a superadditive N.P.S. is regular. Of importance here are missing implications that are not a consequence of the conclusions of the theorem in conjunction with the transitivity of implication. For example, although weak order (C10) implies all Ck for $k < 10$ (parts (a) and (d)) except for decisiveness (C6) and superadditivity (C5), simple examples will show that neither C5 nor C6 is generally implied by C10. In conjunction with C5$\Rightarrow$C1 in (b), it can be shown that C5 does not generally imply C2, whereas (c) shows that C4 follows from C5 when $(X, \succ)$ is decisive. Similarly, although a partially ordered N.P.S. is monotonic (e), it is not generally strongly monotonic.

The penultimate part of Theorem 1 shows that $(X, \succ)$ is a lexicographic N.P.S. if and only if it is superadditive, decisive and attribute acyclic. And (g) says that a weakly ordered and doubly essential N.P.S. is lexicographic. Taken together, (a), (d), (f) and (g) imply that a weakly ordered and doubly essential N.P.S. satisfies every other condition in Definition 2. But examples show that a lexicographic N.P.S. need be neither acyclic nor doubly essential.

Other implications can be drawn from the theorem. For example, a superadditive and decisive N.P.S. is strongly monotonic (a, c), and a decisive, partially ordered and doubly essential N.P.S. is lexicographic (b, d, f).

4. PROOF OF THEOREM 1

The proof of each implication assumes the hypotheses including asymmetry of $\succ$ and essentiality.

C2$\Rightarrow$C1. Since $\{i\} \gg \phi$ by Lemma 1, regularity follows immediately from monotonicity.

C3$\Rightarrow$C2. This follows directly from the definitions.

C4$\Rightarrow$C3. Since $\{i\} \gg \phi$ for all i, successive applications of C4 show that C4$\Rightarrow$C1. To show that C4$\Rightarrow$C3, suppose $(A, B) \sqsupset (C, D)$ with either $C \gg D$ or $C \approx D$. We need to show that $A \gg B$. Since $(A, B) \sqsupset (C, D)$ requires either $(A, D) \sqsupset (C, D)$ or $(C, B) \sqsupset (C, D)$, suppose first that $(A, D) \sqsupset (C, D)$. Since $A \backslash C \gg \phi$ by C1, C4 implies that $A \gg D$. If $B = D$, this gives $A \gg B$ as desired. If $B \subset D$, and $A \gg B$ is false, so that $B \gg A$ or $B \approx A$, then $D \backslash B \gg \phi$ by C1, and hence $D \gg A$ by C4, which contradicts $A \gg D$. Hence $A \gg B$ when $B \subset D$. Secondly, suppose $(C, B) \sqsupset (C, D)$. If $C \gg B$ is false, then $B \approx C$ or $B \gg C$ along with $D \backslash B \gg \phi$ by C1 yields $D \gg C$, contrary to prior hypothesis. Hence $C \gg B$. If $C = A$ then $A \gg B$, and if $C \subset A$ then the first case applies and gives $A \gg B$.

C10$\Rightarrow$C4. For the hypotheses of C4 suppose that $A \cap C = \phi$, $(A \cup C) \cap (B \cup D) = \phi$, $A \gg B$ and either $C \gg D$ or $C \approx D$. With $a_i \succ_i b_i$ for each i let

$$x_i = a_i \text{ on } A \cup C;\ x_i = b_i \text{ otherwise};$$

$$y_i = a_i \text{ on } B \cup C;\ y_i = b_i \text{ otherwise};$$

$$z_i = a_i \text{ on } C \cup (B \backslash D);\ z_i = b_i \text{ otherwise};$$

$$w_i = a_i \text{ on } B \cup D;\ w_i = b_i \text{ otherwise},$$

where $B \backslash D$ denotes set difference. Then $x \succ y$ since $A \gg B$, $y = z$ if $B \cap C = \phi$, $y \succ z$ if $B \cap D \neq \phi$ (by changing one a_i at a time to b_i for each $i \in B \cup C \backslash [C \cup (B \backslash D)]$ and using transitivity), $z \sim w$ if $C \approx D$, and $z \succ w$ if $C \gg D$. Then $x \succ w$ by the assumption that $\succ$ is a weak order, and therefore $A \cup C \gg B \cup D$, which is the desired conclusion for C4.

C5$\Rightarrow$C1. See first sentence of C4$\Rightarrow$C3 proof.

[C9 & C11]$\Rightarrow$C5. For the hypotheses of C5 assume that $A \gg B$, $C \gg D$ and $(A \cup C) \cap (B \cup D) = \phi$. We wish to show that $A \cup C \gg B \cup D$. Using C11, take $a_i \succ_i b_i$ and $b_i \succ_i c_i$ for each i (then also $a_i \succ_i c_i$ since $\succ_i$ is strict partial order), and let

$$x_i = a_i \text{ on } A \cup C;\ x_i = b_i \text{ on } (B \cup D) \backslash (B \cap D);\ x_i = c_i \text{ otherwise};$$

$$y_i = a_i \text{ on } (C \backslash A) \cup (B \backslash D);\ y_i = b_i \text{ on } A \cup D;\ y_i = c_i \text{ otherwise};$$

$z_i = a_i$ on $B \cup D$; $z_i = b_i$ on $(A \cup C)\backslash(A \cap C)$; $z_i = c_i$ otherwise.

Then $x \succ y$ by $A \gg B$, $y \succ z$ by $C \gg D$, and hence $x \succ z$ by transitivity (C9). But $x \succ z$ yields $A \cup C \gg B \cup D$.

[C5 & C6]$\Rightarrow$C4. The only case of concern for the hypotheses of C4 is $A \cap C = \phi$, $(A \cup C) \cap (B \cup D) = \phi$, $A \gg B$ and $C \approx D$. By C6, $C \approx D \Rightarrow (C, D) = (\phi, \phi)$, in which case $A \cup C \gg B \cup D$.

C8$\Rightarrow$C7. Suppose $(X, \succ)$ is attribute cyclic with $A_1 \gg A_2 \ldots \gg A_m \gg A_1$. Let $a_i, b_i \in X_i$ be such that $a_i \succ_i b_i$ and form $x^1, \ldots, x^m$ as follows:

$$x_i^j = a_i \text{ for all } i \in A_j;\ x_i^j = b_i \text{ otherwise.}$$

Then $A_1 \gg A_2 \gg \ldots \gg A_m \gg A_1$ implies $x^1 \succ x^2 \succ \ldots \succ x^m \succ x^1$, so that $\succ$ is cyclic. Hence if $\succ$ is acyclic then $\gg$ is acyclic.

C10$\Rightarrow$C9$\Rightarrow$C8. These follow easily from the definitions.

C9$\Rightarrow$C2. For the hypotheses of C2 suppose $(A, B) \supseteq (C, D)$ and $C \gg D$. With $a_i \succ_i b_i$ for each i let

$$x_i = a_i \text{ for } i \in A;\ x_i = b_i \text{ otherwise;}$$

$$y_i = a_i \text{ for } i \in C;\ y_i = b_i \text{ otherwise;}$$

$$z_i = a_i \text{ for } i \in D;\ z_i = b_i \text{ otherwise;}$$

$$w_i = a_i \text{ for } i \in B;\ w_i = b_i \text{ otherwise.}$$

Since $C \gg D$, $y \succ z$. If $A = C$ then $x = y$. If $A \supset C$ then, by changing one a_i for i in A but not C at a time to b_i, transitivity of $\succ$ implies that $x \succ y$. Likewise, $z = w$ if $B = D$ and $z \succ w$ if $B \subset D$. Consequently, transitivity of $\succ$ gives $x \succ w$, which implies $A \gg B$.

[C5 & C6 & C7]$\Leftrightarrow$C12. Suppose that C12 holds. Then $(X, \succ)$ is easily seen to be decisive (C6) and superadditive (C5). Contrary to C7, suppose that $A_1 \gg A_2 \gg \ldots \gg A_m \gg A_1$. Then no A_i is empty and, with σ as in Definition 2 for C12, $\min\{i: \sigma(i) \in A_1\} < \min\{i: \sigma(i) \in A_2\} < \ldots < \min\{i: \sigma(i) \in A_m\} < \min\{i: \sigma(i) \in A_1\}$, or $\min\{i: \sigma(i) \in A_1\} < \min\{i: \sigma(i) \in A_1\}$, which is false. Hence $(X, \succ)$ satifies C7. Suppose next C5, C6 and C7 hold. Then C6 and C7 require $\gg$ to be a linear order on $\{\phi, \{1\}, \{2\}, \ldots, \{n\}\}$, say $\{\sigma(1)\} \gg \{\sigma(2)\} \gg \ldots \gg \{\sigma(n)\} \gg \phi$. C5 yields $A \gg \phi$ for all nonempty $A \in \{1, \ldots, n\}$ since C5$\Rightarrow$C1, and C5 implies that $A \gg B$ when $A \neq \phi$, $B \neq \phi$, $\min\{i: \sigma(i) \in A\} < \min\{i: \sigma(i) \in B\}$ and $(A, B) \in S$. Hence $(X, \succ)$ satisfies C12.

[C10 & C11]$\Rightarrow$C12. In view of parts (b), (d) and (f) of Theorem 1, we need only prove that [C10 & C11]$\Rightarrow$C6. Contrary to C6 and C1 (as implied by C10), suppose there are nonempty disjoint $A, B \subseteq \{1, \ldots, n\}$ for which $A \approx B$. Take $a_i \succ_i b_i \succ_i c_i$ for each i and let

$$x_i = b_i \text{ on } A;\ x_i = a_i \text{ on } B;\ x_i = c_i \text{ otherwise};$$

$$y_i = a_i \text{ on } A;\ y_i = c_i \text{ on } B;\ y_i = c_i \text{ otherwise};$$

$$z_i = c_i \text{ on } A;\ z_i = b_i \text{ on } B;\ z_i = c_i \text{ otherwise}.$$

Then $A \approx B$ implies $x \sim y$ and $y \sim z$. However, $x \succ z$ by C1, which along with $x \sim y$ and $y \sim z$ contradicts C10. An alternate proof that a weakly ordered, doubly essential N.P.S. is lexicographic appears in Fishburn (1975).

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