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SYMMETRY IN INTERTHEORY RELATIONS*

1. INTRODUCTION

The object of the present paper is to consider the relationship between symmetries of different theories in physics. We shall be interested in discussing such symmetry relationships from two points of view. Firstly there is the formal question of how the symmetries of two theories can be related in virtue of correspondence relations that may exist between the theories. Secondly there is the heuristic aspect, how certain symmetries which project forward from an old theory to a new theory which replaces it serve as a heuristic guideline in the discovery of the new theory.

To pursue the purely formal aspect we shall need to develop appropriate techniques for discussing arbitrary theories, their symmetries and their various correspondence relations. To elucidate the heuristic aspect we shall be concerned with the consideration of historical examples illustrating the rôle of symmetry principles in physics and our interest here will be to distinguish various different types of symmetry, and in the final section of the paper to attempt a broad classification of those symmetries which have heuristic potential.

Firstly however we shall say something about the metaphysical[1] concept of symmetry in the widest context, so that the motivation for the rather precise definition of the symmetry of a physical theory we shall employ later (see Section 6) may become apparent. We begin by explaining that symmetry is basically a two-pronged concept. Formally we may write Symmetry$=\langle\{I\}, \{T\}\rangle$ expressing the symmetry of a physical system or situation as an ordered pair, $\{I\}$ denoting the set of invariants, the features of the situation which remain unchanged, while $\{T\}$ is the set of transformations which express those changes for which the invariants remain fixed[2]. The idea is fundamental because if we want to discuss change in a system we must be able to identify what it is that *is* changing, and this identification of a system during change is only possible by specifying the set of invariants which remain unchanged during the indicated trans-

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formation – if there were no invariants we could not define 'identity'[3].

In applying these ideas to understand what is meant by the symmetry of a physical law we remark informally that the invariant feature is the mathematical form of the law and the symmetry transformations are transformations affecting the variables in terms of which the law is formulated, which leave the form of the law unchanged.

In order to discuss the fundamental significance of symmetry in physics the concept is often linked (see for example [42] and [7]) to two other notions.

(1) Conservation Laws: In classical physics theories which can be derived from a variational principle possess the remarkable property that if the laws are invariant with respect to some continuous symmetry group, then quantities corresponding to the infinitesimal generators of the symmetry are conserved in time, that is themselves display a special sort of symmetry, namely invariance under time-displacement. The connection between the symmetry of laws and conservation principles is even closer in quantum mechanics where discrete as well as continuous symmetries lead to associated conservation laws for the unitary operators which represent the symmetry transformations.

(2) Non-observability of some feature of the physical situation: Clearly if the set of invariants $\{I\}$ express all the structural features of a situation that can be observed, then we cannot distinguish observationally any change as having occurred if the system is subjected to a transformation which is an element of $\{T\}$, i.e. the features which do change under the transformations are non-observable in such a case. For example symmetry under displacement is related to the non-observability of absolute position, and so forth.

We prefer to regard these two ideas of conservation[4] and non-observability as essentially derivative. Our own approach, following Wigner [68], will be to consider a set of possible correlations, or 'solutions' as we shall term them, permitted by a physical theory as the invariant object, and to regard symmetry transformations as inducing a rearrangement of these solutions, which leaves their totality unchanged. This idea will provide an adequate framework for our formal discussion of symmetry and will enable us to analyse precisely the relationships between symmetries of different theories.

2. The changing rôle of symmetry in the history of physics

We turn next to a consideration of the way in which the concept of symmetry has been used in the history of physics, so that our subsequent remarks may be seen in a proper historical context. The first application of a symmetry argument in physics appears to be the proof given by Anaximander (see [3], p. 235) for the stability of the earth. Another example is Archimedes' proof of the laws of equilibrium for the lever [2]. In both cases there is an implicit argument in the account, the structure of which is to show that if equilibrium is not achieved then we can derive a logical contradiction, hence equilibrium follows by *reductio ad absurdam.* Thus Anaximander's reasoning can be very freely paraphrased as follows. If the earth moves up we conclude by symmetry it must equally move down, which is a contradiction. Hence it does not move up. Similarly if the earth moves down we prove that it must move up. Hence it does not move down. But if the earth moves neither up nor down it cannot be moving at all[5]. Q.E.D. An interesting variation on this method of reasoning is provided by Stevin ([59], p. 177f.) in his discussion of the equilibrium of a chain of spheres on a double inclined plane. The conclusion of the argument is not here a logical contradiction, but the fact that the chain once set in motion would continue to revolve perpetually. Stevin comments "which is absurd", but the absurdity is not a logical one. It is perhaps counter-intuitive, but against this, perpetual motion, at that time, was not regarded as rationally impossible, so it seems clear that by absurd Stevin means simply contrary to empirical observation. In this example empirical observation is not directed at the symmetry itself (the invariance of the form of the chain as it slides round the wedge) but is used as an adjunct in the application of a symmetry argument. The idea that one could derive symmetry principles from observed laws, rather than laws from *a priori* symmetry principles can be traced to Galileo. In his *Dialogue Concerning the Two Chief World Systems* (see [24], pp. 186–7) Galileo derives the principle of invariance of physical laws with respect to the uniform motion of the reference frame by considering a variety of phenomena observed on board a moving ship. Further, in his *Dialogues Concerning Two New Sciences* [25] Galileo effectively considers the possibility of scale invariance for mechanical phenomena, and rejects this as conflicting with his laws of resistance to fracture, pointing for example to limitations on the size of land animals, but it is interesting to note that Galileo clearly implies that scale invariance would hold if the elastic properties of the bones of the animals were also appropriately scaled (see [25], p. 130). In this respect Galileo's view agrees with that of Fourier ([23], p. 66) who is usually credited with deriving a principle of scale invariance from a physical law, namely the phenomenon of heat conduction, but does this only on the assumption that the thermometric conductivity is appropriately scaled. Scale invariance is important in the history of symmetry because it is a clear example of a symmetry which is empirically based and is also not universal in the sense that it applies to all phenomena – thus it does not apply to electromagnetic phenomena, since the velocity of light, being, unlike conductivity, a universal constant cannot be scaled.

We have then two views on the rôle of symmetry in physics: (1) the *a priori* approach which seeks to derive laws of nature from symmetry principles, which are in some sense self-evident, or at any rate more plausible than the phenomena which they seek to explain, (2) the empirical approach which derives symmetry principles from known laws of nature and expresses interesting mathematical properties of such laws. The *a priori* approach is exemplified in the application of symmetry principles to derive the laws of probability, as in Bernoulli's Principle of Indifference [5] (see also Laplace [41]). Here the symmetry involved is not about the world but about our knowledge of the world. As our attention turns from one face of the die to another our knowledge relating to the conditions under which the die is thrown remains unchanged, hence our belief that this face rather the first will emerge from the throw is also invariant, since for Bernoulli and Laplace knowledge rationally determines belief. This example should be contrasted with Leibniz's Principle of Sufficient Reason [43] which, in its application to scientific phenomena, effectively contained a hidden assumption about the symmetry of the laws of nature, that is to say in Leibniz's case the symmetry applies to nature, not merely to our knowledge about nature. The opposition between the *a priori* and the empirical approaches to symmetry principles is seen most clearly in the development of the special theory of relativity. To Lorentz [44] and Poincaré [50] the Lorentz transformations were mathematical properties of Maxwell's equations (symmetries derived from laws) but to Einstein [18] the true rôle of the Lorentz transformation was the reverse one of deriving new laws of mechanics, and indeed every other branch of physics, from a symmetry with respect to uniformly moving frames of reference, the transformation laws being derived from a proper analysis of the rôle of electromagnetic waves in specifying simultaneity, and the invariance of the velocity of light under the proposed transformation. From a modern point of view it is clear that both approaches to symmetry are in order. We can distinguish two classes of symmetries which we shall refer to as *universal* and *dynamical*[6]. The universal symmetries are supposed to hold for all physical phenomena and hence can be used as constraints when devising theories relating to any new branch of physics. According to Wigner [69] the only universal symmetries of this kind are included among the so-called geometrical[7] symmetries, and are elements of the proper Poincaré group comprising translations and proper 'rotations' in space-time. The dynamical symmetries hold only for special sorts of interaction, for example isospin invariance in nuclear physics is clearly broken by electromagnetic interactions which distinguish protons from neutrons by the presence of electric charge. However we can again subdivide dynamical symmetries into those having heuristic potential, such as the case of isospin just referred to, in the sense that they can be used as constraints on the construction of theories for a wide class of phenomena, albeit not for all phenomena, and those we may term *accidental* in the sense that while expressing interesting features of some specialized phenomenon, they are in a sense dynamical accidents having no fundamental physical significance. Examples of accidental dynamical symmetries might include the conformal invariance of Maxwell's equations in free space discovered by Bateman [4] and Cunningham [13], the Fock [21] symmetry for the motion of an electron in a Coulomb potential, which explains the degeneracy of the hydrogen spectrum with respect to angular momentum, or the SU(3) symmetry [34] of the three-dimensional harmonic oscillator.

It is clear that the distinction between heuristic and accidental symmetries is not a categorical one. What may appear initially as an accidental symmetry may later transpire to have heuristic potential. The concept of a gauge symmetry (derived from a Lie symmetry group by replacing the infinitesimal parameters by arbitrary space-time functions) would be an example of such a change in status[8]. Alternatively putative heuristic symmetries may be downgraded to accidental status as has been suggested by

Dyson [17] in the case of unitary symmetries in hadron physics. Nevertheless the distinction we have drawn is useful in assessing the rôle to be ascribed at any particular instant in the development of physical theories to a particular symmetry.

3. Correspondence and Symmetry

The subject of intertheory relations has been discussed in several recent papers, for example Tisza [63], Strauss [61], Bunge [9] and Post [54]. There is the formal aspect of detailing the various sorts of correspondence which are possible between theories. But the practical results of such studies may be said to contribute to a methodology of heuristics in the sense of distinguishing the sorts of correspondence which do, or should, obtain between successive theories. The use of the phrase methodology of heuristics may stand in need of some clarification since modern philosophy of science has tended to regard heuristics as belonging properly to some other discipline, psychology or sociology[9]. Thus Popper in a famous passage in *The Logic of Scientific Discovery* denies the possibility of a logic of scientific discovery! As Popper puts it [10] "The question how it happens that a new idea occurs to a man – whether it is a musical theme, a dramatic conflict, or a scientific theory – may be of great interest to empirical psychology; but it is irrelevant to the logical analysis of scientific knowledge". Or again Lakatos [40] "... Modern methodologies or 'logics of discovery' consist merely of a set of ... rules for the appraisal of ready, articulated theories... Outside the legislative domain of these normative rules there is, of course, an empirical psychology and sociology of discovery". Nevertheless, while granting that the question of the mental processes involved in arriving at a 'discovery' in science belongs properly to psychology, we may argue that certain constraints may be placed on the untrammelled working of the creative imagination, which may provide what Post [54] refers to as heuristic guidelines. The important point to notice here is that, as detailed historical analysis shows, theories do not spring fully-armed out of nowhere, but evolve from existing theories with which they are linked by some relation of 'Correspondence'[11]. One of the most fruitful methods for discovering a heuristic guideline is to examine the nature of this correspondence relation between successive theories, and this is clearly part of the general field of intertheory relations which deals not only with the relations between old theories and new theories which replace them but also with the problem of comparing contemporaneous theories relating to different, albeit overlapping, areas of scientific interest.

We can distinguish then two ways in which we can usefully understand the phrase methodology of heuristics: (1) The empirical-historical approach which is concerned with investigating how successive theories in the history of science have in fact been related. (2) A stronger normative or prescriptive sense of the term methodology which attempts to lay down heuristic guidelines, which may themselves be derived from the empirical-historical analysis referred to under (1). For a more complete discussion of the methodology of heuristics along the lines indicated reference may be made to the work of Koertge [37] who has considered a number of examples from the history of physics and chemistry, and of Redhead [55], who has examined the historical development of theories in modern elementary particle physics.

One of the most interesting features of intertheory relations is the question of comparing the symmetries of two theories which stand in different sorts of correspondence. The formal aspect of this problem was discussed by Post [54] who sought to apply Curie's Principle in this particular context. Curie's Principle [14] was originally formulated as a rule relating the symmetries of cause and effect in a physical phenomenon. In terms which are more succint than illuminating it states that a cause cannot be more symmetric than its effect, i.e., symmetries are always transmitted from cause to effect, although asymmetries need not necessarily be so transmitted. An excellent discussion of the significance of Curie's Principle has been given by Chalmers [11], to whose work reference may be made for an account of physical applications of the principle.

Consider now the case of two theories L and S (we follow the terminology of Post [54]) such that S is followed chronologically by L which is regarded by 'any neutral observer' as the 'successor' of S. According to the General Correspondence Principle [54] L and S will stand in some non-trivial relation of correspondence[12]. In a loose way we might regard L as the cause of S in the sense that L serves to explain the successful part of S (this well-confirmed part of S we denote by S^* and assume the absence of so-called Kuhn losses[13] shrinking S^* to S^{**}) and enables us to understand why it was successful. Thus at first sight we might apply Curie's

Principle at once in the form

P 1: the L-theory cannot be more symmetric than S^*.

Now one can easily think of examples where S^* is equal in symmetry to L or is more symmetric than L, both of which possibilities are permitted by this formulation. Thus taking the transition classical mechanics (CM) $\rightarrow$ relativistic mechanics (RM), both theories exhibit symmetry under spatial rotations or in the transition electrostatics (E) $\rightarrow$ electromagnetism (EM) the S-theory E possesses scale invariance which is lacking in the L-theory EM described by the full Maxwell equations. But one can also easily produce examples which contradict our simple preliminary formulation. Referring once again to the transition CM $\rightarrow$ RM, the L-theory, RM, possesses a symmetry under pure Lorentz transformations which is lacking for the S-theory CM. Post [54] attempts to deal with this difficulty by imposing appropriate conditions on the sort of correspondence we are dealing with. He formulates therefore a modified principle which I shall refer to as the Curie-Post Principle, in the form

P 2: the L-theory cannot be more symmetric than S^* in the case where L and S stand in a relation of *consistent* correspondence.

The precise explanation of the distinction between consistent and inconsistent correspondence will be taken up in Section 7, but broadly the case of inconsistent correspondence in which L only 'approximately' explains S^* would apply to the transition CM $\rightarrow$ RM (CM is derived from RM only in the limit of vanishingly small velocities). So this type of counterexample is eliminated in the formulation of the Curie-Post Principle. Nevertheless, as we shall find in Section 7, the Curie-Post Principle is not generally valid even in the case of consistent correspondence unless we further restrict the type of correspondence, or alternatively restrict the type of symmetry to be allowed in the formulation of the principle. But if we allow ourselves to restrict allowable symmetries we can now obtain a new version of the Curie-Post Principle which applies both in cases of consistent and inconsistent correspondence. We shall find in Section 7 that we can arrive at a suitable criterion for delimiting the allowable symmetries in terms of a precise analysis of the so-called Q conditions which play an important rôle in the formulation of the General Correspondence Principle. These Q conditions are restrictions on the L-

theory formulated in the language of L which must be applied before we establish the correspondence between L and S^* via rules of a translation $\mathscr{T}$ between the languages in which the L and S theories are formulated, so that Post summarizes the General Correspondence Principle in the form $S^* = \mathscr{T}(L \mid Q)$.

Before proceeding to a more detailed analysis we want to consider a quite different sort of objection to the Curie-Post Principle raised by Strauss (private communication based on the point of view expounded in [61]). The difficulty of comparing symmetries of theories according to Strauss is that the same theory may have different mathematical formulations, which may admit differing covariance groups, so that one could suggest an extreme counterexample to the Curie-Post Principle in the form of a theory which is more symmetric than itself! A particular example considered by Strauss is that of the Lagrangian and Hamiltonian formulations of classical mechanics. Hamilton's equations are invariant under the group of contact transformations, whereas the covariance group for Lagrange's equations is the more restricted group of point transformations, which do not 'mix' coordinates and momenta. However, if we want to discuss symmetry in intertheory relations we want to distinguish symmetries which are characteristic of a theory independent of any particular mathematical formulation from those symmetries which may depend on the particular formulation. The former class we shall call *physical symmetries* which will comprise in general a proper subset of the more general class of *mathematical symmetries*. The Curie-Post Principle refers then to physical symmetries and not to mathematical symmetries which are not also physical symmetries. In particular the contact transformations of Hamiltonian mechanics are not physical symmetries in a sense which we shall make precise in Section 6, and in this manner we shall avoid the Strauss objection.

The relationship between mathematical symmetries (MS), physical symmetries (PS), heuristic physical symmetries (HPS) and universal physical symmetries (UPS) is clarified in Figure 1. The class of accidental symmetries is the relative complement of HPS with respect to PS and the class of dynamical symmetries is the relative complement of UPS with respect to PS. The purely mathematical symmetries (i.e. the relative complement of PS with respect to MS) do not have heuristic potential in themselves, but it is important to notice that a reformulation of an S-theory may possess mathematical symmetries which project into the L-theory if the latter is obtained by 'stretching' the reformulated S-theory in the technical sense elaborated in [55]. Thus for example the transition Poissonian dynamics → quantum mechanics à la Dirac preserves contact transformations as a symmetry which now reappear as (in general time-dependent) unitary transformations which preserve commutation relations

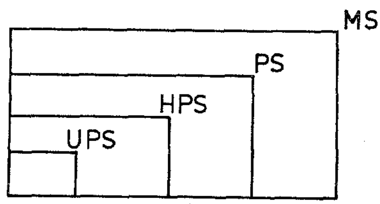

Fig. 1.

in the same way that contact transformations preserve Poisson brackets. Strauss refers to the Poissonian formulation of classical mechanics as a *partial formal anticipation* of quantum mechanics, but the mathematical symmetry is not here used heuristically as one might, for example, use Lorentz invariance as a constraint on any future theoretical development.

In passing we may also notice the interesting possibility of a non-trivial mathematical symmetry which is trivially a physical symmetry (i.e. corresponds to the identity automorphism in the physical structure). This is exemplified by the gauge symmetries which will be referred to in Section 8 below.

4. The Relation of Mathematics to Physics

In order to explain precisely what is meant by a mathematical symmetry of a theory we must first address ourselves to a brief elucidation of the rôle of mathematics in theoretical physics. We begin by outlining a scheme for relating a mathematical structure to a physical theory. For the purposes of this discussion we shall have in mind a 'realist' approach and regard a physical theory as a collection of statements which refer to actual states of affairs in the real world. We may think of a theory as consisting of axioms and deductive chains flowing from these axioms to produce theorems and finally empirical generalizations to be confronted with experiment as expressed in the form of singular observation statements, but we do not wish to enter into an explanation of exactly how we might distinguish theoretical from observation terms in a theory. We shall simply assume that all the terms employed have some ontological reference. We shall also gloss over any discussion of how we might formalize a theory, i.e. express it in a formalized language, and the usual difficulties arising from the lack of expressive power of formalized languages which leads to well-known problems such as incompleteness and non-categoricity. We shall indeed suppose our theory to be presented and discussed as an informal axiomatic system in the terminology of Stoll [60]. This is not to say we decry formal methods, but their use is not necessary to highlight the problems in which we are presently interested. We regard it as an empirical-historical fact that theories in physics can be represented as mathematical structures. More generally we shall envisage the possibility of embedding a theory T in a mathematical structure M' in the sense that there exists an isomorphism[14] (a one-to-one structure-preserving correspondence) between T and a sub-structure M of M'. M' is thus a non-simple conservative extension of M. The situation is represented schematically in Figure 2.

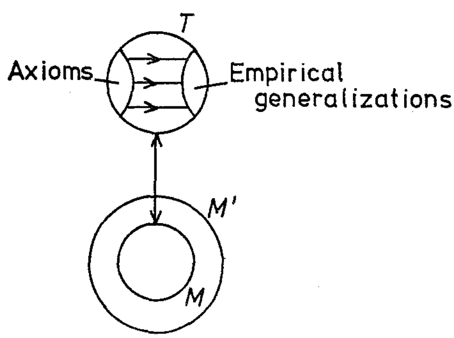

Fig. 2.

If we like we can introduce an uninterpreted calculus C of which T and M are regarded as isomorphic models, or we can introduce a calculus C' for M' and introduce a new theory T' which is partially interpreted via the structure T. These ideas are illustrated schematically in Figure 3.

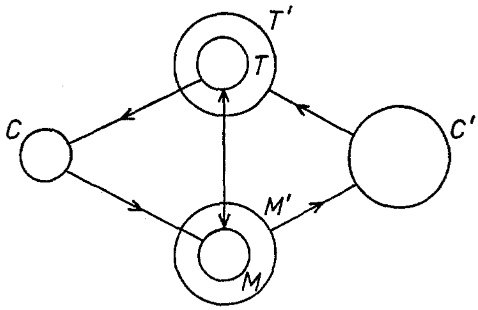

Fig. 3.

The relative complement of T in T' we refer to as the *surplus structure* in the mathematical representation of the theory T. Now we can reverse our line of argument and starting with a given T' ask what is the corresponding T? To a positivist T would involve only observation terms but to a realist theoretical terms may also appear in T. Even to a realist it is not clear in some cases whether the surplus structure should or should not be accorded ontological reference – the concept of the Schrödinger wavefunction in quantum mechanics might be such a borderline example. In other cases terms in T' which start their life effectively as uninterpreted symbols may acquire a reference in reality as that particular branch of science develops – the kinetic theory of matter, as viewed by positivists like Ostwald or Mach, might be cited as an example, the molecules only acquiring a semblance of reality after the discovery of Brownian motion and its quantitative interpretation by Einstein and Smoluchowski. But in other cases the status of the surplus structure is quite unequivocal. The example of analytic S-matrix theory in modern elementary particle physics is an excellent example of the essential rôle that surplus structure, with no possibility of ontological reference, may play in the development of a physical theory. (A detailed study of this example is given in [55].) At all events we shall suppose in what follows that initial agreement has been reached as to what constitutes T and T'. There is a view emphasized by Hilbert and von Neumann [32] that theoretical physics should ideally proceed by a direct axiomatization of physical concepts, with the elimination of all surplus structure. But other physicists, notably Jeans [35] and Chew [12] have stressed the importance of purely mathematical consideration in theoretical physics. Einstein's development of general relativity is another example in this category. Opponents of the view that God is a mathematician are faced with the task of explaining the success of a great deal of theoretical physics, which has not generally developed in accordance with the programme of Hilbert and von Neumann. We do not need in our discussion to take sides in this argument but simply point to the empirical-historical fact that physical theories are always related to mathematical structures in the way we have indicated.

In what follows we shall conveniently identify T with M and T' with M' so that we can effectively reduce the problem of intertheory relations to one of relations between mathematical structures. Our next task will be to develop a method for representing an arbitrary theory in physics, i.e. to formulate a prototype for a natural law, so that we can use the resulting theoretical apparatus to discuss relations between symmetries of successive theories in full generality. We will be operating here on a meta-level and what we will be developing in the next section is a *theory of theories.*

5. A THEORY OF THEORIES

In order to exhibit the canonical theory in physics we can follow a number of different approaches. First of all we could follow the ideas of Birkhoff and von Neumann [6] and exhibit a general theory as a lattice of propositions. Symmetries of a theory appear then as automorphisms of the lattice structure. Quantum mechanics is distinguished from classical mechanics by requiring a non-distributive (i.e. non-Boolean) lattice of propositions. But this method of representing a canonical theory is not well adapted to our particular problem because there is no simple way of characterizing an arbitrary lattice, and it would be very difficult to investigate relationships between symmetries of different theories within the framework of this formulation. Another approach would be to follow the method of Houtappel, van Dam and Wigner [33] who formulate theories in terms of a generalized conditional probability function which they call the Π function which measures essentially the probability that results β_1, β_2,... for observations B_1, B_2,... on a physical system will result if we know already results α_1, α_2,... for observations A_1, A_2,... Symmetry transformations are simply transformations among the observables which leave the Π function invariant. Clearly the motivation here is quantum-mechanical. Classical mechanics is handled somewhat awkwardly by introducing δ-functions for the distributions so as effectively to yield a description in terms of classical orbits. We shall not ourselves follow this method which is very positivistic in its approach and is not well suited to discussing symmetry in classical physics, or symmetry relations between theories in general. Next we should mention Tarski's calculus of deductive systems [62] in which theories are represented as sets of sentences closed under the operation of logical consequence. The sentences or theorems are partially ordered by the relation of implication, but this ordering is of course only partial, which makes it impossible to give precision to Post's [54] informal discussion of the horizontal slicing of a theory separating various levels of deducibility. The concept of passing from S to S^* by stripping off a dispensable, not independently confirmed, superstructure is an important one, which we shall analyse carefully in Section 7, but Tarski's approach to the description of theories, which is essentially the one we used in Section 4, cannot help us here, nor with our problem of the analysis of symmetry.

In order to discuss our particular problem most conveniently we shall seek to represent the canonical theory as a unary relation on a generalized function space. This approach will have a number of advantages. Firstly we keep close to classical physics. The method effectively generalizes such familiar examples of physical laws as Maxwell's equations, Newtonian dynamics or equations of state in thermodynamics. Secondly we shall find that arbitrary theories can be represented in a simple diagrammatic fashion and intertheory relations and symmetry properties can be exhibited in a direct and comprehensive way. Thirdly quantum mechanics can easily be handled by using the Heisenberg picture and regarding the range of our functions as linear operators in a Hilbert space. We will in fact follow closely here Heisenberg's original view [31] of quantum mechanics as retaining the dynamical laws of classical mechanics but altering the interpretation of mechanical quantities from real numbers to (effectively) matrices. We follow essentially Wigner's approach [68], that laws in physics serve to establish correlations between events – as Wigner puts it "The laws of nature permit us to foresee events on the basis of the knowledge of other events". We shall represent an event by the value $\psi(\xi)$ of a field ψ, whose argument ξ denotes in general position in space and time and also characterizes the tensor/spinor component[15] and type of field we are concerned with in any particular theory. For example, to fix our ideas, we might consider Maxwell's equations in which case $\psi(\xi)$ would be the value of some component of the electric or magnetic field at a particular space-time point and the value of ξ would give use the information as to which space-time point we are referring to and also tell us whether we are considering the electric or the magnetic field and which vector component is being evaluated. To express our ideas more precisely we introduce a generalized function $\psi(\xi)$[16] which is in general a many-one mapping from a set $\{\xi\}$, the domain of the function, into a set $\{\psi\}$, the possible range of the function. We denote the function space of all functions from $\{\xi\}$ into $\{\psi\}$ by $\Gamma=\{\psi(\xi)\}$. In general $\{\xi\}$ is a set of ordered quintuples, so we write $\{\xi\}=\{x, y, z, t; i; k\}$ where $xyzt$ denote coordinates in space-time, i is a tensor or spinor index and k a label for the type of field. Thus for the case of Maxwell's equations i runs from 1 to 3 giving the three vector components, and k takes values 1 or 2 distinguishing the electric and magnetic fields. So, for example, $E_x(x, y, z, t) \rightarrow \psi(x, y, z, t; 1; 1)$ and $H_y(x, y, z, t) \rightarrow \psi(x, y, z, t; 2; 2)$. In this case $\{\xi\}=R^4 \times \{3\} \times \{2\}$ and $\{\psi\}=R$ where R denotes the real line and $\{n\}$ an n-element index set. To take two other simple examples from classical physics, for the dynamical motion of a particle in one dimension, taken to be the x-axis, $\{\xi\}=R$, $\{\psi\}=R$ and $\psi(\xi) \rightarrow x(t)$, while for the equation of state of a gas $\{\xi\}$ is a 3-element index set $\{1, 2, 3\}$, $\{\psi\}=R^+$ and we identify $\psi(1)=P$, $\psi(2)=T$, $\psi(3)=V$, where P, T and V are the pressure, absolute temperature and volume of the gas. In the case of quantum mechanics $\{\psi\}$ will no longer comprise a set of real numbers but is now a set of linear operators in a Hilbert space, the space of possible state vectors for the system. The function of the commutation relations in quantum mechanics is to determine the possible set of such operators which may be associated with a particular observable.

We are led then to distinguish two kinds of law in physics. In the first place there are *constitutive laws* which are laws governing the structure of the sets $\{\xi\}$ and $\{\psi\}$. In the example of Maxwell's equations the transitive law for the strength of the field which tells us that if field A is stronger than field B and field B than field C, then field A is stronger than field C, would be a constitutive law in our sense. Similarly the commutation relations for operators in quantum mechanics belong to this category. Then in contrast there are *correlative laws* which tell us how different events are correlated, i.e., how one event determines another. This is the familiar sense in which Wigner uses the term law of nature to tell us how initial or boundary conditions determine the values of $\psi(\xi)$ for all values of ξ, that is to say the correlative laws tell us the possible forms for the function $\psi(\xi)$. This is just the function of Maxwell's equations in electromagnetism or the equation of state in thermodynamics. From the analogy with the example of Newtonian mechanics we shall refer to correlative laws collectively as *generalized equations of motion.* Now the great advantage of introducing the function space Γ is apparent. Take the case of the equation of state. This is a 3-place relation between the sets of possible values for P, T and V, but in our notation this 3-place relation is replaced by a unary relation (property or subset) defined on the function space Γ with $\{\xi\}=\{1, 2, 3\}$ as index set, i.e., each allowed set of concurrent (correlated) values of P, T and V permitted by the equation of state is represented by a point in the function space, and the totality of such points is a subset of the function space, which we may conveniently term the *solution space* γ associated with the correlative law in question. In general we see that γ is simply the set of all possible 'solutions' of our generalized equations of notion, while Γ is the set of all possible functions, whether they satisfy the equations of motion or not. All we have done in fact is to employ the general process well-known to mathematicians whereby one can exhibit an n-place relation on a set by a unary relation on a function space defined over an n-point index set. Thus quite generally if $R(X_1, X_2,\dots, X_n)$ is an n-place relation between sets $X_1, X_2,\dots, X_n$, i.e. a set of ordered n-tuples $x=(x_1, x_2,\dots, x_n)$ where $x_i \in X_i$, then we take an index set $I=\{1, 2,\dots, n\}$ and identify x with a function on I such that $x(i)=x_i \in X_i$ and the relation R is then represented simply by a subset of all possible functions from I into $X_1 \cup X_2 \cup \dots = \bigcup i\, X_i$. (See for example Simmons [58], p. 24.)

There are basically three ways of representing a particular function $\psi(\xi)$: (1) we can illustrate its graph, i.e., the set of all pairs of values for ξ and $\psi(\xi)$, (2) we can view the function as a vector whose components are the values of ψ for each value of ξ, i.e., the coordinate axes are labelled by the value of ξ, (3) we can simply represent a particular $\psi(\xi)$ by a point in the function space Γ. This third method will provide us with a simple diagrammatic method of representing theories. A theory is simply to be identified with the solution space γ of a suitable function space Γ as illustrated in Figure 4.

We are now in a position to discuss very simply what is meant by a symmetry of a theory and to distinguish mathematical and physical symmetries, but first we note some ways in which we may need to extend our analysis of the canonical theory in order to meet with the full range of examples presented by theoretical physics. In the first place, referring to the dependence of ψ on the index k specifying the type of field, we may

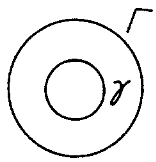

Fig. 4.

have to allow for the possibility that $\psi(k)$ takes values in a set X_k dependent on k. In this case the range of our function space $\{\psi\}$ may be identified with $\bigcup_k X_k$ or in some cases it may be appropriate to use the product space $P_k X_k$. This latter possibility is what is done in the quantum mechanics of many-particle systems in which the state vector space is formed as the tensor product of spaces relating to the individual particles. In the second place we might want to consider the case of nonlocal fields in quantum field theory, i.e., fields which are not diagonal in the coordinate representation of space-time. Such fields are functions of two sets of space-time coordinates, and this is very easily handled in our notation by including an extra 2-valued index j for labelling the domain of our function space, which is used to distinguish these two sets of space-time coordinates.[17] Finally we note that we may extend Γ to comprise all functions from subsets of $\{\xi\}$ into $\{\psi\}$, i.e. Γ comprises all possible functions $\psi(\xi)$ and all restrictions of such functions. γ as usual denotes the set of all physically allowable members of this enlarged function space. This extension is important if we want to consider a restricted domain of validity for a theory. A suitable defined restriction of a particular function may be valid as a law even if the unrestricted function is not valid. Our account of inconsistent correspondence between theories in Section 7 will involve this point. None of these extensions requires any modification in our diagrammatic representation of theories and we believe that our method of exhibiting theories in physics as unary relations on generalized function spaces is in fact a comprehensive one.

6. The distinction between mathematical and physical symmetries

Consider an arbitrary unary relation R defined on Γ, i.e. R is an arbitrary subset of Γ. We define an *R-automorphism* of Γ as a one-to-one mapping θ of points of Γ onto itself which preserves the relation R, i.e., if under θ $x \rightarrow \theta(x)$ where x, $\theta(x) \in \Gamma$ then $R(\theta(x))$ iff $R(x)$. Thus θ maps points of R onto points of R and points not in R onto points not in R. In the previous section we have already introduced the unary relation γ, the solution space corresponding to the generalized equations of motion. We now define a *physical symmetry* of a theory as a γ-automorphism of the generalized function space Γ associated with the theory. A *mathematical symmetry* of a theory is any R-automorphism of Γ for a mathematical relation R that may be of interest ot us. The concept of a mathematical symmetry need not be restricted to the case where R is a unary relation, the concept of an R-automorphism can clearly be immediately extended to an arbitrary n-place relation. But in practice the unary relations, which single out a subset of possible solutions (i.e. functions) for our theory are the ones we shall be concerned with in the examples which we shall give.

Furthermore, our discussion so far has been confined to a physical theory and its related isomorphic structure M in the notation of Section 4. But if we regard M as embedded in a wider structure M', as described in Section 4, then we can extend the concept of mathematical symmetry further to include an R-automorphism of M' where R is now on arbitrary relation defined on the set of mathematical objects which constitute the universe of discourse for the mathematical theory M'. It is interesting to note that the set of all mathematical symmetries which preserve a relation R itself possesses in general the structure of a group, to which the concept of symmetry can again be applied (i.e. automorphisms of the group of symmetry transformations) and indeed this process can be continued in an infinite regress of symmetries of symmetries of symmetries,... (cf. Weyl [65], p. 145).

We can represent symmetries diagramatically as follows. We indicate the motion of a point in Γ-space under a purported symmetry transformation by an arrow. A physical symmetry would be represented as in Figure 5(a) where the tip of the arrow originating at a typical point in γ remains in γ.

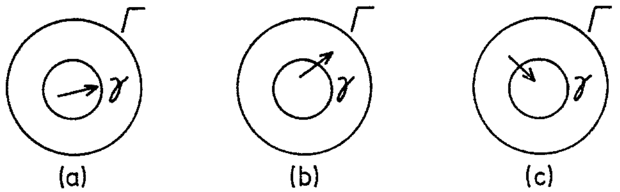

Fig. 5.

But if the tip of the arrow moves outside γ for some point originally in γ, then the physical symmetry is broken by the transformation, as illustrated in Figure 5(b). We have already noted that since the mapping is supposed to be one-one, then in the case of a transformation which is a physical symmetry arrows must lie wholly inside or wholly outside γ. The situation illustrated in Figure 5(c) is clearly impossible for the case of a physical symmetry.

In passing we may note that our approach to physical symmetries corresponds to the second active view in the classification of Fonda and Ghirardi [22] in that we look at different systems (solutions) from the point of view of a single observer. This is in fact the most general approach as stressed in [22] since the other approaches, the passive in which the same system is viewed by different observers, and the first active in which different systems and different observers are in the same relative state, both involve changing the observer, which may not, in general, have physical significance. Houtappel, van Dam and Wigner [33] refer to the point of view that universal geometrical symmetries are limited to those for which the concept of a changing observer is valid but we by no means wish to confine our discussion of symmetry to this particular class.

We note also that we do not require that a particular solution (point in γ) should itself be invariant under the symmetry transformation. In general particular solutions are singled out by specifying boundary (including initial) conditions. If these boundary conditions are themselves invariant under the symmetry transformation then the particular solution $\psi(\xi)$ uniquely determined by these invariant boundary conditions must be itself invariant, i.e. $\psi(\xi) \rightarrow \psi'(\xi) = \psi(\xi)$. The particular case where we are dealing with the invariance of one particular solution is the one originally studied by Curie in the formulation of his principle (see Curie [14] and Chalmers [11]). The boundary conditions are to be regarded as the cause and the solution the effect. Symmetry in the former is transmitted to the latter, but lack of symmetry in boundary conditions does not necessitate an asymmetric solution[18]. As Chalmers has stressed Curie never directly discussed or called in question the symmetry of laws as distinct from the symmetry of boundary conditions.

We may notice further that a symmetry transformation must map the range of the function space onto itself and hence preserve the constitutive laws which determine the structure of this range. For example in quantum mechanics physical symmetries preserve not only the form of the equations of motion of the operators in the Heisenberg picture, but also preserve the commutation relations between the operators.[19]

Reverting to the distinction between mathematical symmetries and physical symmetries it is clearly possible to have mathematical symmetries which are not physical symmetries, the case of contact transformations in Hamiltonian dynamics would furnish an example, but of course every physical symmetry is a mathematical symmetry. We have already noticed in Section 2 that the class of physical symmetries is much wider than the philosophically more interesting class of heuristic physical symmetries. If we start from an arbitrary point in γ we can always define a physical symmetry which moves this point (solution) to any other selected point (solution) or in other words given one solution of the theory we can generate all other solutions if we know the set of all physical symmetries (cf. Houtappel, van Dam and Wigner [33]). But the majority of such symmetries are entirely concerned with the particular theory in question and have no universal validity or heuristic potential – we referred to them in Section 2 as being accidental. In Section 8 we shall consider the question of delimiting the heuristic physical symmetries but in the next section we shall analyse intertheory relations in terms of the more general class of physical symmetries, although clearly everything we say about this class will apply also to the more interesting subclass of heuristic physical symmetries.

Firstly however we shall conclude this section by giving a number of simple mathematical examples which should serve to clarify the preceding rather abstract discussion:

(i) Consider a function space $\Gamma = \{y(x)\}$ with $\{x\} = \{y\} = R$ and we take for γ the solution space (or *primitive* as it is usually called in this context) of the differential equation $(\mathrm{d}y/\mathrm{d}x) - x = 0$. This equation is invariant under the symmetry transformation $x \to -x$ which induces in Γ the transformation $y(x) \to y'(x) = y(-x)$. The general solution of the differential equation is of course $y = A + \frac{1}{2}x^2$ for arbitrary values of the parameter A, and the primitive is simply a set of parallel parabolas displaced along the y-axis by an amount A. Each individual solution (or *characteristic*) is itself clearly invariant under our symmetry transformation (i.e. $y'(x) = y(x)$).

(ii) To illustrate the more general situation in which a symmetry transformation of a differential equation maps any solution not onto itself but onto some other solution we can take the example $(\mathrm{d}^2y/\mathrm{d}x^2) - y = 0$ with primitive $y = Ae^x + Be^{-x}$ and consider again the transformation $x \to -x$ which, for example, carries e^x into e^{-x}, i.e. one solution into another.

(iii) We give now a very simple mathematical example of Curie's Principle. The equation $(\mathrm{d}y/\mathrm{d}x) - y = 0$ lacks symmetry under the transformation $x \to -x$. The general solution $y = Ae^x$ also lacks symmetry but there exists a particular solution, viz. $y = 0$, which is invariant with respect to the transformation we are considering, so we have here the situation that a particular solution of a differential equation may be more symmetric than the equation itself[20] which is the 'cause'[21] of that solution, and this exemplifies the situation envisaged in Curie's Principle which allows that an effect may be more symmetric than its cause.

(iv) We now give a simple example of a physical symmetry which is not a symmetry of a mathematical structure in which the physical theory is embedded. Consider a physical theory which says that two variables x and y are linked by an equation

$$(1) \qquad x^2 + y^2 = 1.$$

We now 'explain' this relation (a circle in the xy plane) as the intersection of two surfaces in a three-dimensional space. For example we can consider the circle as the intersection of a cylinder and a plane

$$(2) \qquad \begin{cases} x^2 + y^2 = 1 \\ \quad\;\; z = 0. \end{cases}$$

Under a rotation about the z-axis both the physical law (1) and the mathematical representation (2) are clearly invariant. But in algebraic geometry we can consider a circle as the intersection of two surfaces in many different ways. Suppose for example we represent the circle (1) as the intersection of the two surfaces

$$(3) \qquad \begin{cases} x^2+y^2+2xyz=1 \\ z=0. \end{cases}$$

The mathematical representation (3) no longer possesses symmetry under rotations about the z-axis, the sections of the first surface by planes perpendicular to the z-axis being conic sections which clearly lack rotational symmetry except in the plane $z=0$.

(v) Finally we give an example of a mathematical symmetry which is not a physical symmetry. The example is constructed as a simple soluble analogue of the more complicated case of contact transformations in Hamiltonian and Lagrangian dynamics. Consider a single coordinate q which satisfies an equation of motion

$$(4) \qquad \dot{q}=1/q.$$

The solution of this equation is clearly

$$(5) \qquad q=\pm\sqrt{2(t+c)}$$

where c is an arbitrary constant. Consider a transformation

$$(6) \qquad q \to q'=\alpha q$$

where α is a scale factor unequal to unity. This transformation is clearly not a symmetry of Equation (4). Indeed q' satisfies the different equation

$$(7) \qquad \dot{q}'=\alpha^2/q'$$

Now we set up a new mathematical formulation to describe our equation of motion. Introduce a function $H(q)$ such that

$$(8) \qquad \frac{\mathrm{d}H}{\mathrm{d}q}=1/\dot{q}.$$

Clearly the choice $H(q)=\frac{1}{2}q^2$ yields Equation (4). Now suppose that H is a scalar function under the transformation (6) so that H transforms thus, $H'(q')=H(q)$. Equation (8) is now easily seen to be invariant in form under the transformation (6), i.e H' satisfies the identical Equation (8) with q replaced by q'. So we have a mathematical symmetry which, as we have seen, is not a physical symmetry. To pursue our analogue a little further, if we had introduced a function $L(q)$ such that

$$\frac{\mathrm{d}L}{\mathrm{d}q}=\dot{q} \tag{9}$$

for which the choice $L(q)=\ln q$ yields the equation of motion (4), then this formulation is easily seen to lack symmetry under the transformation (6). Indeed $L'(q')$ now satisfies the equation

$$\frac{\mathrm{d}L'}{\mathrm{d}q'}=\frac{\dot{q}'}{\alpha^2}. \tag{10}$$

So the behaviour of our two formulations under the transformation (6) is clearly analogous to the case of Hamiltonian and Lagrangian dynamics under contact transformations. In passing we may notice that a true physical symmetry of our equation of motion would be given by $q\to q'=-q$, which simply exchanges the two solutions with the positive and negative signs as given in Equation (5). We can say if we like that the two possibilities represented by the ambiguity in sign in Equation (5) 'generate' each other under the symmetry transformation $q\to -q$.

7. Q CONDITITIONS AND THE CURIE-POST PRINCIPLE

We are now going to consider two theories L and S and investigate their possible relations of correspondence and the relations between their respective symmetries. In order to achieve correspondence we will have in general to introduce appropriate restrictions on the L-theory which we term Q conditions, and also possibly restrictions on the S-theory, which we distinguish as P conditions.

In order to analyse these ideas we begin by introducing the concept of a *derived function space*. Consider a set of functionals $\mathscr{F}^j$ each of which maps the function space Γ into a set we denote by $\{\phi\}$ and we define Λ to be a derived function space with respect to Γ, where Λ is the set of all functions $\phi(j)$ defined on the index set $\{j\}$ which carry j into $\mathscr{F}^j(\psi(\xi))$, for all the functions $\psi(\xi)$ in Γ. We note that j may well be itself a continuous variable. There is clearly in general a many-one mapping from points of Γ onto points of Λ which carries $\psi(\xi)$ to $\phi(j)$.

As a very simple example take again the case where $\{\xi\}$ is a 3-element index set $\{1, 2, 3\}$ and write $\mathscr{F}^1(\psi(\xi))=\psi(1)$, $\mathscr{F}^2(\psi(\xi))=\psi(2)$. Then the mapping of Γ onto Λ carries $\psi(\xi)$ in Γ into $\phi(j)$ in Λ where $\phi(1)=\psi(1)$, $\phi(2)=\psi(2)$. Thus our formal device has effectively contracted the function space Γ by projecting parallel to the 3-axis. In the example of the equation of state we have gone from the phase space of P, T and V to the contracted phase space of P and T alone.

To take another example we can describe the transition from the N-particle function space of the statistical mechanics of an ideal gas to a derived function space of thermodynamic variables by defining temperature T as the functional $T=(1/3N\ell)\sum_{i=1}^{N} m(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2)$ where ℓ is Boltzmann's constant and m the mass of a molecule. Similarly we take for the pressure P the functional $P=(1/3V)\sum_{i=1}^{N} m(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2)$ where the functional V is itself just the volume of the subset of R^3 defined by the range of $\psi(\xi)$.

We notice that the generalized equations of motion which define the solution space in γ in Γ now induce, under our mapping of Γ onto Λ, a restriction on the permitted solutions for the correlations described by the $\phi(j)$ to a subset λ of Λ. That is to say the equations of motion determine both a solution space γ in Γ and an associated solution space λ in Λ.

We now revert to the case of the two theories L and S which we identify as solution spaces of appropriate generalized function spaces which we conveniently denote by L' and S'. We attempt to match these two theories, i.e. to establish a relation of correspondence between them, in a series of stages. If we fail to achieve a match at one stage we must proceed to the next and we shall in fact distinguish three different stages or levels of correspondence at which we may expect a match to occur. Firstly we choose appropriate derived spaces with respect to L' and S' which we denote by $(L|Q_1)'$ and $(S|P_1)'$ with corresponding solution spaces $(L|Q_1)$ and $(S|P_1)$. By appropriate we will certainly mean that $(L|Q_1)'$ and $(S|P_1)'$ can be placed in one-to-one correspondence so that the derived theories may be seen to be talking about the same objects under an appropriate scheme of translation. We now see if we can match $(L|Q_1)$ and $(S|P_1)$ by this translation, i.e. if the correspondence between $(L|Q_1)'$ and $(S|P_1)'$ maps $(L|Q_1)$ onto $(S|P_1)$. If this is possible for some choice of the derived spaces obtained by the mapping processes designated by the symbols Q_1 and P_1, then we have achieved *correspondence at the first level.* But this matching may by no means be possible at this stage. If not we introduce restrictions designated by Q_2 and P_2 on the spaces L' and S' which restrict $(L|Q_1)$ to a subset $(L|Q_1|Q_2)$ and $(S|P_1)$ to a subset $(S|P_1|P_2)$. If we can now match $(L|Q_1|Q_2)$ and $(S|P_1|P_2)$ for some choice of Q_1, Q_2, P_1, P_2 we have achieved *correspondence at the second level.* If we have correspondence at the first or second level we shall speak of *consistent correspondence.* If consistent correspondence fails we may yet notice the following situation. Points in $(L|Q_1|Q_2)$ and $(S|P_1|P_2)$ may be found to agree under appropriate translation in a certain subregion but to diverge in a controlled way as we move to other regions of these function spaces. (We here assume the function spaces to be extended so as to include restrictions of functions as described at the end of Section 5.) Now we may be able to introduce restrictions Q_3 and P_3 on L' and S' in such a way that the corresponding restrictions on the spaces $(L|Q_1|Q_2)$ and $(S|P_1|P_2)$ which we denote by $(L|Q_1|Q_2|Q_3)$ and $(S|P_1|P_2|P_3)$ satisfy

$$(L \mid Q_1 \mid Q_2 \mid Q_3) \overset{\text{approx}}{\approx} (S \mid P_1 \mid P_2 \mid P_3)$$

where $\overset{\text{approx}}{\approx}$ indicates that the mapping between the two spaces is approximately valid in a sense which could be made precise by introducing an appropriate metric in the function spaces, but such a refinement appears to be inessential to our analysis. This situation we describe as *correspondence at the third level* or *inconsistent correspondence* following the terminology of Post [54]. In passing we notice an important special case of correspondence, viz. *reduction.* This occurs when we have consistent correspondence with no P_1 or P_2 conditions so that $(L|Q_1|Q_2)$ 'explains' the whole of S.[22]

Informally we may say that the Q_1 condition restricts what the L-theory talks about, while the Q_2 condition specifies in the language of L the conditions under which the S-theory is intended to apply to phenomena comprehended under $(L|Q_1)$, and the Q_3 condition specifies conditions under which $(L|Q_1|Q_2)$ will give approximately the same account of a common group of phenomena as the S-theory. Similarly *mutatis mutandis* for the S-theory and the varieties of P condition. The different levels of correspondence can be illustrated in the following simple way. We represent the solution spaces $(L|Q_1)$ and $(S|P_1)$ by a collection of red and green dots in a single space, when the function spaces $(L|Q_1)'$ and $(S|P_1)'$

have been identified under a translation $\mathscr{T}$. If all the different coloured dots coincide with their appropriate partners then we have correspondence at the first level. But correspondence at the second level is achieved if by applying Q_2 and P_2 conditions we can eliminate all pairs of dots which do not match leaving only matching pairs. Correspondence at the third level arises when after all attempts at the first and second level have failed, nevertheless the dots coincide very closely in some region of the space and diverge on a controlled way as we depart from this region.

If we assume the absence of Kuhn losses then $(S|P_1|P_2|P_3)$ will correspond to S^* the well-confirmed part of S (or in the presence of Kuhn losses to S^{**}) in the terminology of Post [54]. Our analysis of the passage from S to S^* in terms of P conditions replaces the informal analysis of Post in terms of horizontal slicing, which we have already had occasion to criticise in Section 5. In point of fact P_2 conditions do not appear to be required in standard historical examples. But Q_2 conditions on the L-theory are often of paramount importance in achieving correspondence. We shall give a detailed example later in this section. But first we illustrate the general situation by the diagram shown in Figure 6.

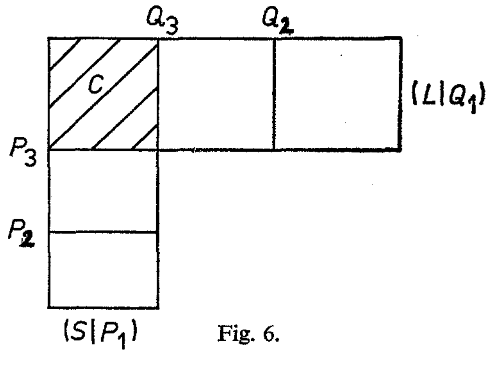

Fig. 6.

The region of overlap C represents the set of solutions in the derived spaces $(L|Q_1)$ and $(S|P_1)$ which are in (in general approximate) correspondence.

We can now easily assess the effect of symmetry transformations performed on the L or S theories. In the first place we note that a symmetry transformation on the function space Γ induces a corresponding symmetry transformation in the derived space Λ so that a physical symmetry of L' is also a physical symmetry of $(L|Q_1)'$, similarly for symmetries of

S' and $(S|P_1)'$. But with regard to the Q_2 and Q_3 conditions the situation is quite different. A physical symmetry of $(L|Q_1)$ may or may not break either the Q_2 or the Q_3 condition. This arises because an automorphism of a mathematical structure is by no means necessarily an automorphism of a substructure. This is the reason why the Curie Principle cannot be applied directly to our problem. In sufficient generality for this purpose we take the case of reduction, and consider the situation depicted in Figure 7.

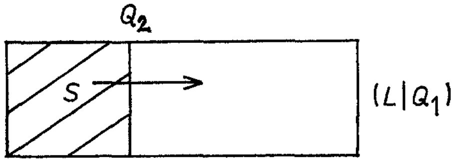

Fig. 7.

The symmetry of $(L|Q_1)$ (and hence of L) is not a symmetry of S since the Q_2 condition is broken by the indicated symmetry transformation. This is a clear counterexample to the Curie-Post Principle which asserts that in cases of reduction the L-theory cannot be more symmetric than the S-theory.

We proceed to give a simple example to clarify the analysis we have given. For the L-theory we take the set of Maxwell's equations for the electromagnetic field in free space, viz.

$$\left.\begin{aligned} \operatorname{curl}\mathbf{H} &= \partial\mathbf{E}/\partial t \\ \operatorname{div}\mathbf{H} &= 0 \\ \operatorname{curl}\mathbf{E} &= -\,\partial\mathbf{H}/\partial t \\ \operatorname{div}\mathbf{E} &= 0. \end{aligned}\right\}$$

For the S-theory we take the equations governing an electrostatic field, viz.

$$\left.\begin{aligned} \partial\mathbf{E}/\partial t &= 0 \\ \operatorname{curl}\mathbf{E} &= 0 \\ \operatorname{div}\mathbf{E} &= 0. \end{aligned}\right\}$$

To achieve the reduction of the S-theory to the L-theory we first contract the function space involved in Maxwell's theory by projecting out the magnetic field and arriving at a derived function space involving only the electric field. This is our Q_1 condition. We then apply the restriction $\mathbf{H}=0$ to arrive at the electrostatic equations. This is the Q_2 condition which clearly has no effect on the function space $(L|Q_1)'$ which is here the space of all functions $\mathbf{E}(x, y, z, t)$, but which does restrict the solution space to just the class of electrostatic fields.[23] Consider now a pure Lorentz transformation which is of course a symmetry of the L-theory. This induces a new electric field which may no longer satisfy the equations of electrostatics since it is in general time-dependent. Thus a Lorentz transformation may break the symmetry of the S-theory. But the Q_2 condition has now been broken since the magnetic field is no longer in general zero after a pure Lorentz transformation. For example if the original electric field is the field of a stationary point charge then the new field will be the electric field of a moving point charge which no longer satisfies the equations of electrostatics since it is time-dependent and is also associated with a non-zero magnetic field arising in part from the convection current represented by the moving charge, in the new frame, and in part from the displacement current produced by the time-variation of the electric field.

It should now be clear that if we want to restore the validity of the Curie-Post Principle we can revise the formulation in either of two ways. In the first place we could further restrict the type of correspondence envisaged so as exclude cases which involve a Q_2 condition. Thus we arrive at the formulation

P2′: The L-theory cannot be more symmetric than S^* in the case where L and S stand in a relation of consistent correspondence which does not involve a Q_2 condition.

Alternatively we can restrict the allowable symmetry transformations to those which do not violate the Q_2 condition. However in the case of inconsistent correspondence those symmetry transformations which demonstrate an L-theory which is more symmetric than S^* include ones which break the Q_3 condition.[24] Thus if we permit ourselves to restrict allowable symmetries we can obtain a new version of the Curie-Post Principle which applies both in cases of consistent and inconsistent correspondence, namely

P3: The L-theory cannot be more symmetric than S^* provided the symmetry transformations considered do not break the Q conditions used in formulating the correspondence relation.[25]

Clearly P2′ is included as a special case of P3.

We have now gone as far as we can in analysing the formal relationships between the symmetries of successive theories. In the concluding section we shall return to discussing the heuristic rôle of symmetries, and how one might classify heuristic physical symmetries.

8. Heuristic symmetries

We may be content to follow Houtappel, van Dam and Wigner [33] when they refer to the point of view that those physical symmetries which possess heuristic potential comprise "all transformations which leave the known laws of nature invariant and the simplicity of which suggest their universal validity" although we would want to replace the qualification 'universal' by something like 'wide-ranging'. We are confronted here with essentially an aesthetic consideration. But we may prefer to adopt a more metaphysical approach and the explication of symmetry in terms of non-observability already referred to in Section 1 may be useful here (cf. Lee [42]). But in fact any metaphysical speculation or intuition regarding the essentially non-observable features of a physical situation is notoriously unreliable. The discovery that in the weak interactions of elementary particles nature provides an ultimate distinction between right-handed and left-handed reference frames reminds us that the superlaws of symmetry, as Wigner calls them, are as liable to empirical revision as other laws of physics having a less obviously intuitive character.

In these circumstances we shall employ a purely taxonomic approach, and using the empirical-historical method, attempt a simple classification of the heuristic physical symmetries already revealed to us by the history of theoretical physics. We note the limitation on the objective – we make no claim to classify physical symmetries so as to reveal those with heuristic potential, we are concerned with the classification of the heuristic symmetries themselves so as to introduce some order into their bewildering variety. If we represent our canonical theory as usual in terms of a space of possible functions $\psi(\xi)=\psi(x, y, z, t; i; k)$ where $xyzt$ are coordinates in space-time, i is a tensor or spinor index and k a label for the type of field, then a purported heuristic symmetry transformation will induce a transformation in this function space carrying $\psi(\xi)$ into $\psi'(\xi)$, where we distinguish three different categories or classes for the transformed field $\psi'(\xi)$.

Class 1: $\psi'(\xi)$ describes the same event at a different location in space-time, and suitably 'rotated' in the space of the tensor/spinor index i.

By the same type of event we mean that the transformation leaves the k-label unaffected. Examples in this class are Wigner's geometrical symmetries, notably spatial rotations, pure Lorentz transformations, displacements in space and time and also reflections and time inversions.

Class 2: $\psi'(\xi)$ describes different events at the same location.

Examples would include charge conjugation and the SU(2) (isospin) and SU(3) symmetries of particle physics.[26] Permutational symmetry for identical particles also belongs to this class.[27]

Class 3: $\psi'(\xi)$ describes a different event at a different location.

Examples of this class include the mixing of class 1 and class 2 symmetries in a non-trivial way as in Wigner's SU(4) symmetry in nuclear physics [66] or the SU(6) symmetry introduced by Gürsey and Radicati [29] and by Sakita [56] in particle physics. Another example of a rather different sort is crossing symmetry[28] in particle physics in which space-time location is replaced by relativistically invariant parameters (the so-called Mandelstam parameters) in terms of which scattering cross-sections for various related processes can be expressed.

We must also note the interesting possibility that a mathematical symmetry which is trivially a physical symmetry (see remark at end of Section 3) may have heuristic potential. We may thus introduce a fourth category of heuristic symmetries.

Class 4: $\psi'(\xi)$ is a redescription of the same event at the same location.

The most familiar example of such a redescription is the case of gauge transformations of the first and second kind [49] in quantum electrodynamics. What we are actually concerned with here is a transformation affecting the surplus structure of the mathematical system used for describing the theory, and the consideration of such a class of symmetry principles underlines the importance of surplus structure and its physical repercussions. As is well known the requirement of gauge invariance essentially constrains the type of interaction that is possible between a charged particle and an electromagnetic field. General covariance also appears to belong to this class, in view of the analogy with a gauge symmetry suggested by Utiyama [64] (for further discussion see Wigner [69]).

With regard to class 4 we note an important distinction between the familiar gauge invariance of electrodynamics and the more recent non-

Abelian gauge symmetries (compare for example Yang and Mills [71]). These latter symmetries are usually introduced as a straightforward generalization of the Abelian symmetries derived from U(1), but in our classification the non-Abelian symmetries, being related to isospin (SU(2)), belong to class 2 rather than class 4. We may also note the point of view of Melvin [46] who seeks to describe all symmetries in terms of redescriptions, but the general applicability of this passive approach to symmetry has been effectively criticised by Houtappel, van Dam and Wigner [33] and Fonda and Ghirardi [22] (see also Section 6 above).

In conclusion we should wish to emphasize the point that heuristic symmetries are in general constraints on new theories. They may just possibly constrain them so strongly as to characterize them completely, but in general other principles, such as unitarity and analyticity in particle physics, may be required. Indeed if we knew a final complete theory symmetry principles would be seen merely as interesting properties derivable from it. This attitude to symmetry has also motivated the attempts such as Cutcosky's [15] to bootstrap the SU(3) symmetry in particle physics, i.e. to derive the symmetry from other self-consistency conditions imposed on the theory. But in the absence of a knowledge of such a theory heuristic physical symmetries of the various sorts we have discussed have provided very fruitful guidelines in our suggested methodology of heuristics.

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NOTES

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[1] The attributive metaphysical may be taken in two quite distinct senses as applied to the concept of symmetry. In the first place, as stressed by Wigner [68] symmetry principles operate on a meta-level as compared with ordinary laws of physics. They serve to correlate laws in the same way that laws correlate events. (Note that Wigner here uses the term law not as a synonym for a theory, but as referring to some particular correlation or coordination of events.) The significance of Wigner's remarks will become clear as we proceed. But we may also take the more usual sense of metaphysics in the philosophy of science as being concerned with the analysis of concepts and ultimate presuppositions. This is the sense we shall employ in this section.

[2] We can regard our definition of symmetry as a Carnapian explication (see [10], Chapter I) in which the concept is sharpened so as to express just those features which

are of importance in science, the vaguer aesthetic connotations being eliminated. See also Popper ([51], p. 9ff.) for a relevant discussion of the rôle of definitions in philosophy of science.

[3] For the particular case of temporal change this problem of identity is of course an ancient philosophical one, and underlies the distinction between essential and accidental attributes of an object. Two answers have traditionally been given to the question, can essential attributes, the candidates for invariants, change? One is typified by Greek atomism, for which the atoms are immutable, only their accidental configurations change. The other sort of answer is the view of Aristotle that essences can alter in the sense that they can be actualized but notice in this case it is the potentiality of their presence which plays the rôle of the invariant, so the twofold distinction of invariant and transformation in fact underlies all philosophical analysis of change. The concept of symmetry has very wide applications. For example the object of interest may be simply a mathematical structure and one can raise the question for a given set of invariants what is the widest set of transformations for which the invariants do not change? Or conversely for a given set of transformations can one list a complete set of invariants such that two systems which have the same values for all the invariants must be connected by some element of the transformation set? The first question leads naturally to the consideration of sets of transformations which possess a group structure, and hence to the prevalence of group-theoretic methods in discussions of symmetry.

[4] Reservations about linking symmetry to conservation laws are discussed in [67] and [28].

[5] Sideways motion is of course dealt with by a similar argument.

[6] The reader is warned that the term dynamical symmetry is used in quite a different sense from ours by other authors. See, for example, Pais [48].

[7] Wigner contrasts geometrical and dynamical symmetries, using the latter term in a different sense from ours. See note 6 above.

[8] Cf. Yang and Mills [71], Utiyama [64] and Sakurai [57].

[9] A notable exception is Hanson who writes, for example, [30] "We must attend as much to how scientific hypotheses are caught, as to how they are cooked".

[10] See [53] Chapter I, p. 31.

[11] This point of view is contrary to that of Feyerabend ([19] and [20]) who has expounded the incommensurability of theories. For a reply to Feyerabend see for example Post [54], Koertge [37] or Achinstein [1].

[12] The use of the term correspondence in this context is due to Popper [52] by analogy with Bohr's Correspondence Principle in the quantum theory.

[13] Kuhn losses refer to those well-confirmed parts of the S-theory which are not comprehended under the L-theory, i.e. the loss of explanatory power that may occur in replacing the S-theory by the L-theory. See [39], p. 169.

[14] The existence of such an isomorphism has been challenged by Körner ([38], Chapter VIII) who describes the relationship between mathematics and physics in terms of the replacement of the inexact concepts of the latter discipline by exact concepts of the former, rather than in terms of the identification of concepts. In our account the admitted idealization of mathematical concepts reflects the existence of surplus structure in the mathematical description of the physical theory as discussed below in the text.

[15] The possibility that the field supports a non-linear realization of a space-time symmetry would not affect our subsequent analysis.

[16] We use the same notation for a function and one of its values, which is usual in

physics and should cause no confusion as the meaning of our notation will always be clear from the context.

17 A more general concept of nonlocal field has recently been discussed [36] in which, in our description, the index j would itself label a function space, representing the 'shape' of the nonlocal objects subsumed under the field.

18 Chalmers gives the very nice example of an irregular block of ice melting in a spherical container to give a sperically symmetric distribution of gas.

19 In quantum mechanics there is another sort of correlative law quite distinct from that expressed by the equations of motion for operators, viz. the correlation between states, represented by a density operator W, observables Q and expectation values $\langle Q \rangle$ subsumed under the formula $\langle Q \rangle = \mathrm{Tr}\ WQ$. The solution space of correlated values of $\langle Q \rangle$, W and Q is clearly invariant with respect to transformations of the operators Q which define the physical symmetry in the sense of preserving the equations of motion for observables and their commutation relations.

20 This is a clear counterexample to Bunge [8] who in discussing time reversal writes "If a process is T-invariant then its laws are T-invariant".

21 We use the word 'cause' here in a different sense from Curie (see discussion above) but the example is I believe a cogent one.

22 For a discussion of reduction see Nagel [47].

23 Clearly we may often choose an appropriate Q_2 condition in a number of different ways. For example we could apply the condition $\partial \mathrm{E}/\partial t = \partial \mathrm{H}/\partial t = 0$, which yields the same set of electric fields, although a stationary magnetic field is now permitted. Notice how with this choice of Q_2 condition both $(L \mid Q_1)'$ and $(L \mid Q_1)$ are subject to restriction.

24 We are assuming continuity of the symmetry transformations with respect to the suggested metric in terms of which approximate, inconsistent, correspondence is defined. For transformations which do not break the Q_3 condition S^* will be approximately symmetric in a sense that could be specified in terms of the metric.

25 Notice that symmetry transformations of L never break the Q_1 condition.

26 An excellent account of these examples is given by Gasiorowicz [26]. For SU(3) see also the collection of, and commentary on, the original papers by Gell-Mann and Ne'eman [27].

27 The closely related symmetry under place permutations (see [16], Chapter IX) would belong to class 1.

28 For a clear discussion of crossing the text by Martin and Spearman [45] may be consulted.

BIBLIOGRAPHY

[1] Achinstein, P., *Concepts of Science: A Philosophical Analysis*, Baltimore, Maryland, The Johns Hopkins Press, 1968.

[2] Archimedes, *On the Equilibrium of Planes*, English translation in T.L. Heath (ed.), *The Works of Archimedes*, Cambridge, Cambridge University Press, 1897.

[3] Aristotle, *On the Heavens*, English translation by W. K. C. Guthrie, London, Heinemann, 1939.

[4] Bateman, H., 'The Transformation of the Electrodynamical Equations', *Proceedings of the London Mathematical Society* **8** (1910), 223–264.

[5] Bernouilli, J., *Ars Conjectandi*, Brussels, Culture et Civilisation, 1968.

[6] Birkhoff, G. and von Neumann, J., 'The Logic of Quantum Mechanics', *Annals of Mathematics* **37** (1936), 823–843.

[7] Brereton, M. G., 'Symmetry in Physics', *Physics Bulletin* **25** (1974), 95–99.

[8] Bunge, M., 'Physical Time: The Objective and Relational Theory', *Philosophy of Science* **35** (1968), 355–388.
[9] Bunge, M., 'Problems Concerning Intertheory Relations', in Weingartner and Zecha (eds.), *Induction, Physics, and Ethics,* Dordrecht, Holland, Reidel, 1970, 285–315.
[10] Carnap, R., *Logical Foundations of Probability,* Chicago, University of Chicago Press, 1950.
[11] Chalmers, A. F., 'Curie's Principle', *British Journal for the Philosophy of Science* **21** (1970), 133–148.
[12] Chew, G. F., *The Analytic S Matrix,* New York, Benjamin, 1966.
[13] Cunningham, E., 'The Principle of Relativity in Electrodynamics and an Extension Thereof', *Proceedings of the London Mathematical Society* **8** (1909), 77–98.
[14] Curie, P., 'Symétrie dans les Phénomènes Physiques', *Journal de Physique* **3** (1894), 393–415.
[15] Cutcosky, R. E., 'A Mechanism for the Induction of Symmetries among the Strong Interactions', *Physical Review* **131** (1963), 1888–1890.
[16] Dirac, P. A. M., *The Principles of Quantum Mechanics,* 4th ed., Oxford, Clarendon Press, 1958.
[17] Dyson, F. J., *Symmetry Groups in Nuclear and Particle Physics,* New York, Benjamin, 1966.
[18] Einstein, A., 'Zur Electrodynamik bewegter Körper', *Annalen der Physik* **17** (1905), 891–921. English translation by W. Perrett and G. B. Jeffery in A. Einstein *et al., The Principle of Relativity,* London, Methuen and Co., 1923. Reprint ed., New York, Dover Publications, 1951.
[19] Feyerabend, P. K., 'Explanation, Reduction and Empiricism', in Feigl and Maxwell (eds.), *Minnesota Studies in the Philosophy of Science* III, pp. 28–97, Minneapolis, University of Minnesota Press, 1962.
[20] Feyerabend, P. K., 'Problems of Empiricism', in R. C. Colodny (ed.), *Beyond the Edge of Certainty,* pp. 145–260, Englewood Cliffs, New Jersey, Prentice-Hall, 1965.
[21] Fock, V., 'Zur Theorie der Wasserstoffatoms', *Zeitschrift für Physik* **98** (1935), 145–154.
[22] Fonda, L. and Ghirardi, G. C., *Symmetry Principles in Quantum Physics,* New York, Marcel Dekker, 1970.
[23] Fourier, J., *The Analytical Theory of Heat,* English translation by A. Freeman, Cambridge, Cambridge University Press, 1878. Reprint ed., New York, Dover Publications, 1955.
[24] Galilei, G., *Dialogue Concerning the Two Chief World Systems – Ptolemaic and Copernican,* English translation by Stillman Drake, Berkeley, University of California Press, 1953.
[25] Galilei, G., *Dialogues Concerning Two New Sciences,* English translation by H. Crew and A. de Salvio, New York, MacMillan, 1914. Reprint ed., New York, Dover Publication, 1954.
[26] Gasiorowicz, S., *Elementary Particle Physics,* New York, John Wiley and Sons, 1966.
[27] Gell-Mann, M. and Ne'eman, Y., *The Eightfold Way,* New York, Benjamin, 1964.
[28] Greenberger, D. M., 'The Scale Transformation in Physics', *Annals of Physics* (N.Y.) **25** (1963), 290–308.
[29] Gürsey, F. and Radicati, L. A., 'Spin and Unitary Spin Independence of Strong Interactions', *Physical Review Letters* **13** (1964), 173–175.

[30] Hanson, N. R., 'Is there a Logic of Scientific Discovery?', in Feigl and Grover (eds.), *Current Issues in the Philosophy of Science*, pp. 20–35, New York, Holt Rinehart and Winston, 1961. Reprinted in B. A. Brody (ed), *Readings in the Philosophy of Science*, Englewood Cliffs, New Jersey, Prentice-Hall, 1970.

[31] Heisenberg, W., 'Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen', *Zeitschrift für Physik* **33** (1925), 879–893. English translation in B. L. van der Waerden (ed.), *Sources of Quantum Mechanics*, Amsterdam, North Holland Publishing Co., 1967.

[32] Hilbert, D., von Neumann, J., and Nordheim, L., 'Über die Grundlagen der Quantenmechanik', *Mathematische Annalen* **98** (1928), 1–30.

[33] Houtappel, R. M. F., van Dam, H., and Wigner, E. P., 'The Conceptual Basis and Use of the Geometric Invariance Principles', *Reviews of Modern Physics* **37** (1965), 595–632.

[34] Jauch, J. M. and Hill, E. L., 'On the Problem of Degeneracy in Quantum Mechanics', *Physical Review* **57** (1940), 641–645.

[35] Jeans, J., *The Mysterious Universe*, Cambridge, Cambridge University Press, 1930.

[36] Kaku, M. and Kikkawa, K., 'Field Theory of Relativistic Strings, I. Trees', *Physical Review* **D 10** (1974), 1110–1133.

[37] Koertge, N., *A Study of Relations between Scientific Theories: A Test of the General Correspondence Principle*, Doctoral Thesis, University of London, 1969.

[38] Körner, S., *The Philosophy of Mathematics*, London, Hutchinson and Co., 1960.

[39] Kuhn, T. S., *The Structure of Scientific Revolutions*, Chicago, University of Chicago Press, 1962.

[40] Lakatos, I., 'History of Science and its Rational Reconstructions', in Buck and Cohen (eds.), *Boston Studies in the Philosophy of Science* VIII, pp. 91–136, Dordrecht, Holland, Reidel, 1972.

[41] Laplace, P. S. Marquis de, *A Philosophical Essay on Probabilities*, English translation by F. W. Truscott and F. L. Emory, New York, John Wiley and Son, 1902. Reprint ed., New York, Dover Publications, 1951.

[42] Lee, T. D., 'Symmetry Principles in Physics', in A. Zichichi (ed.), *Elementary Processes at High Energies*, Part A, pp. 306–319. New York, Academic Press, 1971.

[43] Leibniz, G. W., *The Monadology*, English translation by M. Morris in *Leibniz, Philosophical Writings*, London, Dent, 1934.

[44] Lorentz, H. A., 'Electromagnetic Phenomena in a System Moving with any Velocity smaller than that of Light', *Proceedings of the Royal Academy of Amsterdam* **6** (1904), 809–831. Reprinted in A. Einstein *et al.*, *The Principle of Relativity*, London, Methuen and Co., 1923.

[45] Martin, A. D. and Spearman, T. D., *Elementary Particle Theory*, Amsterdam, North-Holland Publishing Co., 1970.

[46] Melvin, M. A., 'Elementary Particles and Symmetry Principles', *Reviews of Modern Physics* **32** (1960), 477–518.

[47] Nagel, E., *The Structure of Science*, London, Routledge and Kegan Paul, 1961.

[48] Pais, A., 'Dynamical Symmetries in Particle Physics', *Reviews of Modern Physics* **38** (1966), 215–255.

[49] Pauli, W., 'The Connection between Spin and Statistics', *Physical Review* **58** (1940), 716–722.

[50] Poincaré, H., 'Sur la Dynamique de l'Electron', *Rendiconti del Circolo Matematico di Palermo* **21** (1906), 129–176.

[51] Popper, K. R., *The Open Society and its Enemies*, Volume II, London, Routledge and Kegan Paul, 1945.
[52] Popper, K. R., 'The Aim of Science', *Ratio* **1** (1957), 24–35. Reprinted in K. R. Popper, *Objective Knowledge*, Oxford, Clarendon Press, 1972.
[53] Popper, K. R., *The Logic of Scientific Discovery*, London, Hutchinson and Co., 1959.
[54] Post, H. R., 'Correspondence, Invariance and Heuristics', *Studies in History and Philosophy of Science* **2** (1971), 213–255.
[55] Redhead, M. L. G., 'Some Philosophical Aspects of Particle Physics', forthcoming.
[56] Sakita, B., 'Supermultiplets of Elementary Particles', *Physical Review* **136B** (1964), 1756–1760.
[57] Sakurai, J. J., 'Theory of Strong Interactions', *Annals of Physics* (N.Y.) **11** (1960), 1–48.
[58] Simmons, G. F., *Introduction to Topology and Modern Analysis*, New York, McGraw-Hill, 1963.
[59] Stevin, S., *The Principal Works*, Volume I, edited by E. J. Dijksterhuis, Amsterdam, C. V. Swets and Zeitlinger, 1955.
[60] Stoll, R. R., *Sets, Logic and Axiomatic Theories*, San Francisco, Freeman and Co., 1961.
[61] Strauss, M., 'Intertheory Relations', in Weingartner and Zecha (eds.), *Induction, Physics, and Ethics*, pp. 220–284, Dordrecht, Holland, Reidel, 1970.
[62] Tarski, A., 'Grundzüge des Systemenkalkül', *Fundamenta Mathematicae* **25** (1935), 503–526, **26** (1936) 283–301. English translation by J. H. Woodger in *Logic, Semantics, Metamathematics*, Oxford, Clarendon Press, 1956.
[63] Tisza, L., 'The Conceptual Structure of Physics', *Reviews of Modern Physics* **35** (1963), 151–185.
[64] Utiyama, R., 'Invariant Theoretical Interpretation of Interaction', *Physical Review* **101** (1956), 1597–1607.
[65] Weyl, H., *Symmetry*, Princeton, N.J., Princeton University Press, 1952.
[66] Wigner, E. P., 'On the Consequences of the Symmetry of the Nuclear Hamiltonian on the Spectroscopy of Nuclei', *Physical Review* **51** (1937), 106–119.
[67] Wigner, E. P., 'Conservation Laws in Classical and Quantum Physics', *Progress of Theoretical Physics* **11** (1954), 437–440.
[68] Wigner, E. P., 'Events, Laws of Nature and Invariance Principles', in *Nobel Lectures; Physics;* 1963–1970, pp. 6–17, Amsterdam, Elsevier, 1972. Reprinted in [70].
[69] Wigner, E. P., 'Symmetry and Conservation Laws', *Proceedings of the National Academy of Sciences* **51** (1964), 956–965. Reprinted in [70].
[70] Wigner, E. P., *Symmetries and Reflections*, Bloomington, Indiana University Press, 1967.
[71] Yang, C. N. and Mills, R. L., 'Conservation of Isotopic Spin and Isotopic Gauge Invariance', *Physical Review* **96** (1954), 191–195.