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WANG'S PARADOX*

This paper bears on three different topics: observational predicates and phenomenal properties; vagueness; and strict finitism as a philosophy of mathematics. Of these three, only the last requires any preliminary comment.

Constructivist philosophies of mathematics insist that the meanings of all terms, including logical constants, appearing in mathematical statements must be given in relation to constructions which we are capable of effecting, and of our capacity to recognise such constructions as providing proofs of those statements; and, further, that the principles of reasoning which, in assessing the cogency of such proofs, we acknowledge as valid must be justifiable in terms of the meanings of the logical constants and of other expressions as so given. The most powerful form of argument in favour of such a constructivist view is that which insists that there is no other means by which we can give meaning to mathematical expressions. We learn, and can only learn, their meanings by a training in their use; and that means a training in effecting mathematical constructions, and in recording them within the language of mathematics. There is no means by which we could derive from such a training a grasp of anything transcending it, such as a notion of truth and falsity for mathematical statements independent of our means of recognising their truth-values.

Traditional constructivism has allowed that the mathematical constructions by reference to which the meanings of mathematical terms are to be given may be ones which we are capable of effecting only in principle. It makes no difference if they are too complex or, simply, too lengthy for any human being, or even the whole human race in collaboration, to effect in practice. Strict finitism rejects this concession to traditional views, and insists, rather, that the meanings of our terms must be given by reference to constructions which we can in practice carry out, and to criteria of correct proof on which we are in practice prepared to rely: and the strict finitist employs against the old-fashioned constructivist arguments of exactly the same form as the constructivist has been

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accustomed to use against the platonist; for, after all, it is, and must necessarily be, by reference only to constructions which we can in practice carry out that we learn the use of mathematical expressions.

Strict finitism was first suggested as a conceivable position in the philosophy of mathematics by Bernays in his article 'On Platonism in Mathematics'. It was argued for by Wittgenstein in *Remarks on the Foundations of Mathematics*; but, with his staunch belief that philosophy can only interpret the world, and has no business attempting to change it, he did not propose that mathematics be reconstructed along strict finitist lines – something which evidently calls for a far more radical overhaul of mathematical practice than does traditional constructivism. The only person, so far as I know, to declare his adherence to strict finitism and attempt such a reconstruction of mathematics is Esenin-Volpin. But, even if no-one were disposed to accept the arguments in favour of the strict finitist position, it would remain one of the greatest interest, not least for the question whether constructivism, as traditionally understood, is a tenable position. It can be so only if, despite the surface similarity, there is a disanalogy between the arguments which the strict finitist uses against the constructivist and those which the constructivist uses against the platonist. If strict finitism were to prove to be internally incoherent, then either such a disanalogy exists or the argument for traditional constructivism is unsound, even in the absence of any parallel incoherence in the constructivist position.

On a strict finitist view, the conception must be abandoned that the natural numbers are closed under simple arithmetical operations, such as exponentiation. For by 'natural number' must be understood a number which we are in practice capable of representing. Clearly, capacity to represent a natural number is relative to the notation allowed, and so the single infinite totality of natural numbers, actual on the platonist view, potential on the traditional constructivist view, but equally unique and determinate on both, gives way to a multiplicity of totalities, each defined by a particular notation for the natural numbers. Such notations are of two kinds. As an example of the first kind, we may take the Arabic notation. The totality of natural numbers which we are capable in practice of representing by an Arabic numeral is evidently not closed under exponentiation; for instance, $10^{10^{10}}$ plainly does not belong to it. As an example of a notation of the second kind, we may take the Arabic numerals supplemented by the symbols for addition, multiplication and exponentiation. The totality of natural numbers determined by this notation evidently does contain $10^{10^{10}}$, and is closed under exponentiation. On the other hand, it does not have the property, which a totality determined by a notation of the first kind shares with the totality of natural numbers as traditionally conceived, that, for any number n, there are n numbers less than it: for, plainly, the totality does not contain as many as $10^{10^{10}}$ numbers. Since a totality determined by a notation of the second kind will still not be closed under all effective arithmetical operations definable over it, it possesses no great advantage over a totality of the first kind, and, for most purposes, it is better to take the natural numbers as forming some totality of this first kind.

Strict finitism is coherent only if the notion of totalities of this sort is itself coherent. My remarks will bear on strict finitism only at this point.

These preliminaries completed, consider the following inductive argument:

> 0 is small;
> If n is small, $n+1$ is small:
> Therefore, every number is small.

This is Wang's paradox. It might be urged that it is not a paradox, since, on the ordinary understanding of 'small', the conclusion is true. A small elephant is an elephant that is smaller than most elephants; and, since every natural number is larger than only finitely many natural numbers, and smaller than infinitely many, every natural number is small, i.e., smaller than most natural numbers.

But it is a paradox, since we can evidently find interpretations of 'small' under which the conclusion is patently false and the premisses apparently true. It is, in fact, a version of the ancient Greek paradox of the heap. If you have a heap of sand, you still have a heap of sand if you remove one grain; it follows, by repeated applications, that a single grain of sand makes a heap, and, further, that, by removing even that one grain, you will still have a heap. Wang's paradox is merely the contraposition of this, where 'n is small' is interpreted to mean 'n grains of sand are too few to make a heap". Another interpretation which yields a paradox is 'It is possible in practice to write down the Arabic numeral for n'.

On either of these interpretations, the predicate 'small' is vague: the word 'heap' is vague, and the expression 'possible in practice' is vague. In fact, on any interpretation under which the argument constitutes a paradox, the predicate 'small' will be vague. Now, under any such interpretation, premiss 1 (the induction basis) is clearly true, and the conclusion as clearly false. The paradox is evidently due to the vagueness of the predicate 'small': but we have to decide in what way this vagueness is responsible for the appearance of paradox. We have two choices, it appears: either premiss 2 (the induction step) is not true, or else induction is not a valid method of argument in the presence of vague predicates.

The induction step certainly seems correct, for any arbitrary *n*. One possibility is that, in the presence of vague predicates, the rule of universal generalisation fails, i.e., we are not entitled to pass from the truth, for any arbitrary *n*, of '$A(n)$', in this case of

If *n* is small, $n+1$ is small,

to that of 'For every *n*, $A(n)$', i.e., here of

For every *n*, if *n* is small, then $n+1$ is small.

But, even if we suppose this, we should still be able to derive, for each particular value of *n*, the conclusion

n is small,

even though we could not establish the single proposition

For every *n*, *n* is small.

And this does not remove the paradox, since for each suitable interpretation of 'small' we can easily name a specific value of *n* for which the proposition

n is small

is plainly false.

Let us therefore consider the possibility that induction fails of validity when applied to vague properties. Reasoning similar to that of the preceding paragraph seems to suggest that this is not an adequate solution either. If induction fails, then, again, we cannot draw the conclusion

For every *n*, *n* is small;

but it is a well-known fact that each particular instance of the conclusion of an inductive argument can be established from the premisses of the induction without appeal to induction as a principle of inference. That is, for any specific value n_0 of n, the conclusion

n_0 is small

can be established from the induction basis

0 is small

and a finite number of instances

If 0 is small, 1 is small;
If 1 is small, 2 is small;
. . . .
If m is small, $m+1$ is small;
. . . .

of the induction step, by means of a series of n_0 applications of modus ponens. Hence, just as in the preceding paragraph, it is not sufficient, in order to avoid the appearance of paradox, to reject induction as applied to vague properties.

It therefore appears that, in order to resolve the paradox without declining to accept the induction step as true, we must either declare the rule of universal instantiation invalid, in the presence of vague predicates, or else regard modus ponens as invalid in that context. That is, either we cannot, for each particular m, derive

If m is small, then $m+1$ is small

from

For every n, if n is small, then $n+1$ is small;

or else we cannot, at least for some values of m, derive

$m+1$ is small

from the premisses

If m is small, then $m+1$ is small

and

m is small.

But either of these seems a desperate remedy, for the validity of these rules of inference seems absolutely constitutive of the meanings of 'every' and of 'if'.

The only alternative left to us, short of questioning the induction step, therefore appears to be to deny that, in the presence of vague predicates, an argument each step of which is valid is necessarily itself valid. This measure seems, however, in turn, to undermine the whole notion of proof (= chain of valid arguments), and, indeed, to violate the concept of valid argument itself, and hence to be no more open to us than any of the other possibilities we have so far canvassed.

Nevertheless, this alternative is one which would be embraced by a strict finitist. For him, a proof is valid just in case it can in practice be recognised by us as valid; and, when it exceeds a certain length and complexity, that capacity fails. For this reason, a strict finitist will not allow the contention to which we earlier appealed, that an argument by induction to the truth of a statement '$A(n_0)$' for specific n_0, can always be replaced by a sequence of n_0 applications of modus ponens: for n_0 may be too large for a proof to be capable of containing n_0 separate steps.

This, of course, has nothing to do with vagueness: it would apply just as much to an induction with respect to a completely definite property. In our case, however, we may set it aside, for the following reason. Let us call n an *apodictic* number if it is possible for a proof (which we are capable of taking in, i.e. of recognising as such) to contain as many as n steps. Then the apodictic numbers form a totality of the kind which the strict finitist must, in all cases, take the natural numbers as forming, that is to say, having the following three properties: (a) it is (apparently) closed under the successor operation; (b) for any number n belonging to the totality, there are n numbers smaller than it also in the totality; and (c) it is bounded above, that is, we can cite a number M sufficiently large that it is plainly not a member of the totality. A possible interpretation of 'n is small' in Wang's paradox would now be '$n+100$ is apodictic'. Now it seems reasonable to suppose that we can find an upper bound M for the totality of apodictic numbers such that $M-100$ is apodictic. (If this does not seem reasonable to you, substitute some larger number k for 100 such that it does seem reasonable – this is surely possible – and understand k whenever I speak of 100.) Since M is an upper bound for the totality of apodictic numbers, $M-100$ is an upper bound for the totality of small numbers, under this interpretation of 'small'. Hence, since $M-100$ is apodictic, there exists a proof (which we can in practice recognise as such) containing $M-100$ applications of modus ponens whose conclusion is the false proposition that $M-100$ is small. – That is to say, an appeal to the contention that only a proof which we are capable of taking in really proves anything will not rescue us from Wang's paradox, since it will always be possible so to interpret 'small' that we can find a number which is not small for which there apparently exists a proof, in the strict finitist's sense of 'proof', that it is small, a proof not expressly appealing to induction.

We may note, before leaving this point, that the question whether Wang's paradox is a paradox for the strict finitist admits of no determinate answer. If 'natural number' and 'small' are so interpreted that the totality of natural numbers is an initial segment of the totality of small numbers (including the case when they coincide), then it is no paradox – its conclusion is straightforwardly true: but, since 'small' and 'natural number' can be so interpreted that the totality of small numbers is a proper initial segment of the totality of natural numbers, Wang's paradox can be paradoxical even for the strict finitist.

It thus seems that we have no recourse but to turn back to the alternative we set aside at the very outset, namely that the second premiss of the induction, the induction step, is not after all true. What is the objection to the supposition that the statement

For every n, if n is small, then $n+1$ is small

is not true? In its crudest form, it is of course this: that, if the statement is not true, it must be false, i.e., its negation must be true. But the negation of the statement is equivalent to:

For some n, n is small and $n+1$ is not small,

whereas it seems to us a priori that it would be absurd to specify any number as being small, but such that its successor is not small.

To the argument, as thus stated, there is the immediate objection that it is assuming at least three questionable principles of classical, two-valued, logic – questionable, that is, when we are dealing with vague statements. These are:

(1) that any statement must be either true or false;
(2) that from the negation of 'For every n, $A(n)$' we can infer 'For some n, not $A(n)$'; and
(3) that from the negation of 'If A, then B' we can infer the truth of 'A'.

However, as we have seen, in order to generate the paradox, it is sufficient to consider a finite number of statements of the form

If m is small, then $m+1$ is small.

If all of these were true, then the conclusion

n_0 is small

would follow, for some specific number n_0 for which it is evidently intuitively false. If, then, we are not to reject modus ponens, it appears that we cannot allow that each of these finitely many conditional statements is true. If we were to go through these conditionals one by one, saying of each whether or not we were prepared to accept it as true, then, if we were not to end up committed to the false conclusion that n_0 is small, there would have to be a smallest number m_0 such that we were not prepared to accept the truth of

If m_0 is small, then m_0+1 is small.

We may not be able to decide, for each conditional, whether or not it is true; and the vagueness of the predicate 'small' may possibly have the effect that, for some conditionals, there is no determinate answer to the question whether they are true or not: but we must be able to say, of any given conditional, whether or not we are prepared to accept it as true. Now, since m_0 is the smallest value of m for which we are unprepared to accept the conditional as true, and since by hypothesis we accept modus ponens as valid, we must regard the antecedent

m_0 is small

as true; and, if we accept the antecedent as true, but are not prepared to accept the conditional as true, this can only be because we are not prepared to accept the consequent as true. It is, however, almost as absurd to suppose that there exists a number which we can recognise to be small, but whose successor we cannot recognise to be small, as to suppose that there exists a number which is small but whose successor is not.

Awkward as this seems, it appears from all that has been said so far that it is the only tolerable alternative. And perhaps after all it is possible to advance some considerations which will temper the wind, which will mitigate the awkwardness even of saying that there is a number n such that n is small but $n+1$ is not. Let us approach the point by asking whether the law of excluded middle holds for vague statements. It appears at first that it does not: for we often use an instance of the law of excluded middle to express our conviction that the statement to which we apply it is *not* vague, as in, e.g., 'Either he is your brother or he isn't'. But, now, consider a vague statement, for instance 'That is orange'. If the object pointed to is definitely orange, then of course the statement will be definitely true; if it is definitely some other colour, then the statement will be definitely false; but the object may be a borderline case, and then the statement will be neither definitely true nor definitely false. But, in this instance at least, it is clear that, if a borderline case, the object will have to be on the borderline between being orange and being some other particular colour, say red. The statement 'That is red' will then likewise be neither definitely true nor definitely false: but, since the object is on the borderline between being orange and being red – there is no other colour which is a candidate for being the colour of the object – the disjunctive statement, 'That is either orange or red', will be definitely true, even though neither of its disjuncts is.

Now although we learn only a vague application for colour-words, one thing we are taught about them is that colour-words of the same level of generality – 'orange' and 'red', for example – are to be treated as mutually exclusive. Thus, for an object on the borderline, it would not be incorrect to say 'That is orange' and it would not be incorrect to say 'That is red': but it would be incorrect to say 'That is both orange and red' (where the object is uniform in colour), because 'orange' and 'red' are incompatible predicates. This is merely to say that 'red' implies 'not orange': so, whenever 'That is either orange or red' is true, 'That is either orange or not orange' is true also.

It is difficult to see how to prove it, but it seems plausible that, for any vague predicate 'P', and any name 'a' of an object of which 'P' is neither definitely true nor definitely false, we can find a predicate 'Q', incompatible with '*P*', such that the statement '*a* is either *P* or *Q*' is definitely true, and hence the statement '*a* is either *P* or not *P*' is definitely true also. And thus it appears plausible, more generally, that, for any vague statement '*A*', the law of excluded middle '*A* or not *A*' must be admitted as correct, even though neither '*A*' nor 'Not *A*' may be definitely true.

If this reasoning is sound, we should note that it provides an example of what Quine once ridiculed as the 'fantasy' that a disjunction might be true without either of its disjuncts being true. For, in connection with vague statements, the only possible meaning we could give to the word 'true' is that of 'definitely true': and, whether the general conclusion of the validity of the law of excluded middle, as applied to vague statements, be correct or not, it appears inescapable that there are definitely true disjunctions of vague statements such that neither of their disjuncts is definitely true. It is not only in connection with vagueness that instances of what Quine stigmatised as 'fantasy' occur. Everyone is aware of the fact that there are set-theoretic statements which are true in some models of axiomatic set theory, as we have it, and false in others. Someone who believed that axiomatic set theory, as we now have it, incorporates all of the intuitions that we have or ever will have concerning sets could attach to the word 'true', as applied to set-theoretic statements, only the sense 'true in all models'. Plainly he would have to agree that there exist true disjunctive set-theoretic statements neither of whose disjuncts is true.

When vague statements are involved, then, we may legitimately assert a disjunctive statement without allowing that there is any determinate answer to the question which of the disjuncts is true. And, if the argument for the validity, as applied to vague statements, of the law of excluded middle is accepted as sound, this may prompt the suspicion that all classically valid laws remain valid when applied to vague statements. Of course, the semantics in terms of which those laws are justified as applied to definite statements will have to be altered: no longer can we operate with a simple conception of two truth-values, each statement possessing a determinate one of the two. A natural idea for constructing a semantics for vague statements, which would justify the retention of all the laws of classical logic, would be this. For every vague statement, there is a certain range of acceptable ways of making it definite, that is, of associating determinate truth-conditions with it. A method of making a vague statement definite is acceptable so long as it renders the statement true in every case in which, before, it was definitely true, and false in every case in which, before, it was definitely false. Corresponding things may be said for ingredients of vague statements, such as vague predicates, relational expressions and quantifiers. Given any vague predicate, let us call any acceptable means of giving it a definite application a 'sharpening' of that predicate; similarly for a vague relational expression or a vague quantifier. Then, if we suppose that all vagueness has its source in the vagueness of certain primitive predicates, relational expressions and quantifiers, we may stipulate that a statement, atomic or complex, will be definitely true just in case it is true under every sharpening of the vague expressions of these kinds which it contains. A form of inference will, correspondingly, be valid just in case, under any sharpening of the vague expressions involved, it preserves truth: in particular, an inference valid by this criterion will lead from definitely true premisses to a definitely true conclusion.

A logic for vague statements will not, therefore, differ from classical logic in respect of the laws which are valid for the ordinary logical constants. It will differ, rather, in admitting a new operator, the operator 'Definitely'. Of course, the foregoing remarks do not constitute a full account of a logic for vague statements – they are the merest beginning. Such a logic will have to take into account the fact that the application of the operator 'Definitely', while it restricts the conditions for the (definite) truth of a statement, or the (definite) application of a predicate, does not eliminate vagueness: that is, the boundaries between which acceptable sharpenings of a statement or a predicate range are themselves indefinite. If it is possible to give a coherent account of this matter, then the result will be in effect a modal logic weaker than S4, in which each reiteration of the modal operator 'Definitely' yields a strengthened statement.

But, for our purposes, it is not necessary to pursue the matter further. It is clear enough that, if this approach to the logic of vague statements is on the right lines, the same will apply to an existential statement as we have seen to apply to disjunctive ones. When '$A(x)$' is a vague predicate, the statement 'For some x, $A(x)$' may be definitely true, because, on any sharpening of the primitive predicates contained in '$A(x)$', there will be some object to which '$A(x)$' applies: but there need be no determinate answer to the question to *which* object '$A(x)$' applies, since, under different sharpenings of the primitive predicates involved, there will be different objects which satisfy '$A(x)$'. Thus, on this account, the statement 'For some n, n is small and $n+1$ is not small' may be true, although there just is no answer to the question *which* number this is. The statement is true because, for each possible sharpening of the predicate 'small', or of the primitive notions involved in its definition, there would be a determinate number n which was small but whose successor was not small; but, just because so many different sharpenings of the predicate 'small' would be acceptable, no one of them with a claim superior to the others, we need have no shame about refusing to answer the challenge to say which number in fact exemplified the truth of the existential statement.

This solution may, for the time being, allay our anxiety over identifying the source of paradox. It is, however, gained at the cost of not really taking vague predicates seriously, as if they were vague only because we had not troubled to make them precise. A satisfactory account of vagueness ought to explain two contrary feelings we have: that expressed by Frege that the presence of vague expressions in a language invests it with an intrinsic incoherence; and the opposite point of view contended for by Wittgenstein, that vagueness is an essential feature of language. The account just given, on the other hand, makes a language containing vague expressions appear perfectly in order, but at the cost of making vagueness easily eliminable. But we feel that certain concepts are ineradicably vague. Not, of course, that we could not sharpen them if we wished to; but, rather, that, by sharpening them, we should destroy their whole point. Let us, therefore, attempt to approach the whole matter anew by considering the notions involved in a theory which takes vague predicates very seriously indeed – namely, strict finitism; and begin by examining these queer totalities which strict finitism is forced to take as being the subject-matter of arithmetic.

Let us characterise a totality as 'weakly infinite' if there exists a well-ordering of it with no last member. And let us characterise as 'weakly finite' a totality such that, for some finite ordinal n, there exists a well-ordering of it with no nth member. Then we should normally say that a weakly finite totality could not also be weakly infinite. If we hold to this view, we cannot take vagueness seriously. A vague expression will, in other words, be one of which we have only partially specified a sense; and to a vague predicate there will therefore not correspond any specific totality as its extension, but just as many as would be the extensions of all the acceptable sharpenings of the predicate. But to take vagueness seriously is to suppose that a vague expression may have a completely specific, albeit vague, sense; and therefore there will be a single specific totality which is the extension of a vague predicate. As Esenin-Volpin in effect points out, such totalities – those characterised as the extensions of vague predicates – can be both weakly finite and weakly infinite. For instance, consider the totality of heartbeats in my childhood, ordered by temporal priority. Such a totality is weakly infinite, according to Esenin-Volpin: for every heartbeat in my childhood, I was still in my childhood when my next heartbeat occurred. On the other hand, it is also weakly finite, for it is possible to give a number N (e.g., 25×10^8), such that the totality does not contain an Nth member. Such a totality may be embedded in a larger totality, which may, like the totality of heartbeats in my youth, be of the same kind, or may, like the set of heartbeats in my whole life, be strongly finite (have a last member), or, again, may be strongly infinite (that is, not finitely bounded). Hence, if induction is attempted in respect of a vague predicate which in fact determines a proper initial segment, which is both weakly finite and weakly infinite, of a larger determinate totality, the premisses of the induction will both be true but the conclusion will be false. (By a 'determinate' totality I mean here one which is either strongly finite, like the set of heartbeats in my whole life, or strongly infinite, like the set of natural numbers, as ordinarily conceived, or, possibly, the set of heartbeats of my descendants.)

Thus, on this conception of the matter, the trouble did not after all lie where we located it, in the induction step. We found ourselves, earlier, apparently forced to conclude that the induction step must be incorrect, after having eliminated all other possibilities. But, on this account, which is the account which the strict finitist is compelled to give for those cases in which, for him, Wang's paradox is truly paradoxical, the induction step is perfectly in order. The root of the trouble, on this account, is, rather, the appeal to induction – an alternative which we explored and which appeared to be untenable. Not that, on this view, induction is always unreliable. Whether it is to be relied on or not will depend upon the predicate to which it is being applied, and upon the notion of 'natural number' which is being used: we have to take care that the predicate in respect of which we are performing the induction determines a totality at least as extensive as the totality of natural numbers over which the induction is being performed.

A possible interpretation of '*n* is small' would be 'My heart has beaten at least *n* times and my *n*th heartbeat occurred in my childhood'. Now clearly the picture Esenin-Volpin is appealing to is this. Imagine a line of black dots on some plane surface; there is no reason not to take this array of dots as strongly finite, i.e., as having both a leftmost and a rightmost member. The surface is coloured vivid red (except for the dots themselves) on its left-hand half; but then begins a gradual and continuous transition through purple to blue. The transition is so gradual that, if we cover over most of the surface so as to leave uncovered at most (say) ten dots, then we can discern no difference between the shade of colour at the left-hand and at the right-hand edge. On the basis of this fact, we feel forced to acknowledge the truth of the statement, 'If a dot occurs against a red background, so does the dot immediately to its right'. The leftmost dot is against a red background; yet not all the dots are. In fact, if the dots are considered as ordered from left to right, the dots which have a red background form a merely weakly finite proper initial segment of the strongly finite set of all the dots.

This example is important; it is not merely, as might appear at first sight, a trivial variation on the heartbeat example. In examples like the heartbeat one, it could seem that the difficulty arose merely because we had not bothered, for a vague word like 'childhood', to adopt any definite convention governing its application. This is what makes it appear that the presence in our language of vague expressions is a feature of language due merely to our laziness, as it were, that is to our not troubling in all cases to provide a sharp criterion of applicability for the terms we use; and hence a feature that is in principle eliminable. Such an explanation of vagueness is made the more tempting when the question whether the presence of vague terms in our language reflects any feature of reality is posed by asking whether it corresponds to a vagueness in reality: for the notion that things might actually *be* vague, as well as being vaguely described, is not properly intelligible. But the dot example brings out one feature of reality – or of our experience of it – which is very closely connected with our use of vague expressions, and at least in part explains the feeling we have that vagueness is an indispensable feature of language – that we could not get along with a language in which all terms were definite. This feature is, namely, the non-transitivity of the relation 'not discriminably different'. The dropping of one grain of sand could not make the difference between what was not and what was a heap – not just because we have not chosen to draw a sharp line between what is and what is not a heap, but because there would be no difference which could be discerned by observation (but only by actually counting the grains). What happens between one heartbeat and the next could not change a child into an adult – not merely because we have no sharp definition of 'adult', but because human beings do not change so quickly. Of course, we can for a particular context – say a legal one – introduce a sharp definition of 'adult', e.g., that an adult is one who has reached midnight on the morning of his 18th birthday. But not all concepts can be treated like this: consider, for instance (to combine Esenin-Volpin's example with one of Wittgenstein's), the totality of those of my heartbeats which occurred before I learned to read.

A says to B, 'Stand appreciably closer to me'. If B moves in A's direction a distance so small as not to be perceptibly closer at all, then plainly he has not complied with A's order. If he repeats his movement, he has, therefore, presumably still not complied with it. Yet we know that, by repeating his movement sufficiently often, he can eventually arrive at a position satisfactory to A. This is a paradox of exactly the form 'All numbers are small'. 'n is small' is here interpreted as meaning 'n movements of fixed length, that length too small to be perceptible, will not bring B appreciably closer to A'. Clearly, 1 is small, under this interpretation; and it appears indisputable that, if n is small, $n+1$ is small.

This, at any rate, provides us with a firm reason for saying that vague predicates are indispensable. The non-transitivity of non-discriminable difference means, as Goodman has pointed out, that non-discriminable difference cannot be a criterion for identity of shade. By this is not meant merely that human vision fails to make distinctions which can be made by the spectroscope – e.g., between orange light and a mixture of red, orange and yellow light. It means that phenomenal agreement (matching) cannot be a criterion of identity for phenomenal shades. 'a has the same shade of (phenomenal) colour as b' cannot be taken to mean 'a is not perceived as of different shade from b' ('a matches b in colour'); it must mean, rather, 'For every x, if a matches x, then b matches x'. Now let us make the plausible assumption that in any continuous gradation of colours, each shade will have a distinct but not discriminably different shade on either side of it (apart of course from the terminal shades). In that case, it follows that, for any acceptable sharpening of a colour-word like 'red', there would be shades of red which were not discriminably different from shades that were not red. It would follow that we could not tell by looking whether something was red or not. Hence, if we are to have terms whose application is to be determined by mere observation, these terms must necessarily be vague.

Is there more than a conceptual uneasiness about the notion of a non-transitive relation of non-discriminable difference? I look at something which is moving, but moving too slowly for me to be able to see that it is moving. After one second, it still looks to me as though it was in the same position; similarly after three seconds. After four seconds, however, I can recognise that it has moved from where it was at the start, i.e. four seconds ago. At this time, however, it does not look to me as though it is in a different position from that which it was in one, or even three, seconds before. Do I not contradict myself in the very attempt to express how it looks to me? Suppose I give the name 'position *X*' to the position in which I first see it, and make an announcement every second. Then at the end of the first second, I must say, 'It still looks to me to be in position *X*'. And I must say the same at the end of the second and the third second. What am I to say at the end of the fourth second? It does not seem that I can say anything other than, 'It no longer looks to me to be in position *X*': for position *X* was defined to be the position it was in when I first started looking at it, and, by hypothesis, at the end of four seconds it no longer looks to me to be in the same position as when I started looking. But, then, it seems that, from the fact that after three seconds I said, 'It still looks to me to be in position *X*', but that after four seconds I said, 'It no longer looks to me to be in position *X*', that I am committed to the proposition, 'After four seconds it looks to me to be in a different position from that it was in after three seconds'. But this is precisely what I want to deny.

Here we come close to the idea which Frege had, and which one can find so hard to grasp, that the use of vague expressions is fundamentally incoherent. One may be inclined to dismiss Frege's idea as a mere prejudice if one does not reflect on examples such as these.

How can this language be incoherent? For there does not seem to be any doubt that there is such a relation as non-discriminable difference (of position, colour, etc.), and that it is non-transitive. But the incoherence, if genuine, appears to arise from expressing this relation by means of the form of words, 'It looks to me as though the object's real position (colour, etc.) is the same'. And if this language is incoherent, it seems that the whole notion of phenomenal qualities and relations is in jeopardy. (Perhaps there is something similar about preference. The question is sometimes raised whether preference is necessarily a transitive relation. It may be argued that a person will never do himself any good by determining his choices in accordance with a non-transitive preference scale: but it seems implausible to maintain that actual preferences are always transitive. But if, as is normally thought allowable, I express the fact that I prefer *a* to *b* by saying, 'I believe *a* to be better than *b*', then I convict myself of irrationality by revealing non-transitive preferences: for, while the relation expressed by 'I believe x to be better than y' may be non-transitive, that expressed by 'x is better than y' is necessarily transitive, since it is a feature of our use of comparative adjectives that they always express transitive relations.)

Setting this problem on one side for a moment, let us turn back to the question whether Esenin-Volpin's idea of a weakly finite, weakly infinite totality is coherent. It appears a feature of such a totality that, while we can give an upper bound to the number of its members, e.g. 25×10^8 in the case of heartbeats in my childhood, we cannot give the exact number of members. On second thoughts, however, that this is really a necessary feature of such totalities may seem to need some argument. Can we not conceive of quite small such totalities, with a small and determinate number of members? Suppose, for example, that the minute hand of a clock does not move continuously, but, at the end of each second, very rapidly (say in 10^{-5} seconds) moves 6 min of arc; and suppose also that the smallest discriminable rotation is 24 min of arc. Now consider the totality of intervals of an integral number of seconds from a given origin such that we cannot at the end of such an interval perceive that the minute hand has moved from its position at the origin. This totality comprises precisely four members – the null interval, and the intervals of 1, 2 and 3 seconds. The interval of 4 seconds plainly does not belong to it; it is therefore at least weakly finite. Can we argue that it is weakly infinite? Well, apparently not: because it has a last member, namely the interval of 3 seconds' duration. But would it not be plausible to argue that the totality is closed under the operation of adding one second's duration to an interval belonging to the totality?

This appears to be just the same contradiction, or apparent contradiction, as that we have just set aside. It *appears* plausible to say that the totality is closed under this operation, because, from the end of one second to the end of the next, we cannot detect any difference in the position of the minute hand. Hence it appears plausible to say that, if we cannot detect that the position of the minute hand at the end of n seconds is different from its initial position, then we cannot detect at the end of $n+1$ seconds that its position is different from its initial position. But the non-transitivity of non-discriminable difference just means that this inference is incorrect. Hence the totality of such intervals is not a genuine candidate for the status of weakly infinite totality.

In fact, from the definition of 'weakly infinite totality', it appears very clear that it *is* a necessary feature of such totalities that they should not have an assignable determinate number of members, but at best an upper bound to that number. For the definition of 'weakly infinite totality' specified that such a totality should not have a last member: whereas, if a totality has exactly n members, then its nth member is the last.

But this should lead us to doubt whether saying that a totality is closed under a successor-operation is really consistent with saying that it is weakly finite. It appears plausible to say that, if my nth heartbeat occurred in my childhood, then so did my $(n+1)$th heartbeat: but is this any more than just the illusion which might lead us to say that, if the position of the minute hand appeared the same after n seconds, it must appear the same after $(n+1)$ seconds?

The trouble now appears to be that we have shifted from cases of non-discriminable difference which give rise to vague predicates to ones which do not. That is, we assigned the non-transitivity of non-discriminable difference as one reason why vagueness is an essential feature of language, at least of any language which is to contain observational predicates. But the totality of intervals which we have been considering is specified by reference to an observational feature which is not vague (or at least, if it is, we have prescinded from this vagueness in describing the conditions of the example). The plausibility of the contention that the totality of heartbeats in my childhood is weakly infinite depends, not merely on the fact that the interval between one heartbeat and the next is too short to allow any discriminable difference in physique or behaviour by reference to which maturity is determined, but also on the fact that the criteria for determining maturity are vague. So we must re-examine more carefully the connection between vagueness and non-discriminable difference.

'Red' has to be a vague predicate if it is to be governed by the principle that, if I cannot discern any difference between the colour of *a* and the colour of *b*, and I have characterised *a* as red, then I am bound to accept a characterisation of *b* as red. And the argument was that, if 'red' is to stand for a phenomenal quality in the strong sense that we can determine its application or non-application to a given object just by looking at that object, then it must be governed by this principle: for, if it is not, how could I be expected to tell, just by looking, that *b* was not red? But reflection suggests that no predicate can be consistently governed by this principle, so long as non-discriminable difference fails to be transitive. 'Consistent' here means that it would be impossible to force someone, by appeal to rules of use that he acknowledged as correct, to contradict himself over whether the predicate applied to a given object. But by hypothesis, one could force someone, faced with a sufficiently long series of objects forming a gradation from red to blue, to admit that an object which was plainly blue (and therefore not red) was red, namely where the difference in shade between each object in the series and its neighbour was not discriminable. Hence it appears to follow that the use of any predicate which is taken as being governed by such a principle is potentially inconsistent: the inconsistency fails to come to light only because the principle is never sufficiently pressed. Thus Frege appears to be vindicated, and the use of vague predicates – at least when the source of the vagueness is the non-transitivity of a relation of non-discriminable difference – is intrinsically incoherent.

Let us review the conclusions we have established so far.

(1) Where non-discriminable difference is non-transitive, observational predicates are necessarily vague.

(2) Moreover, in this case, the use of such predicates is intrinsically inconsistent.

(3) Wang's paradox merely reflects this inconsistency. What is in error is not the principles of reasoning involved, nor, as on our earlier

diagnosis, the induction step. The induction step is correct, according to the rules of use governing vague predicates such as 'small': but these rules are themselves inconsistent, and hence the paradox. Our earlier model for the logic of vague expressions thus becomes useless: there can be no coherent such logic.

(4) The weakly infinite totalities which must underlie any strict finitist reconstruction of mathematics must be taken as seriously as the vague predicates of which they are defined to be the extensions. If conclusion (2), that vague predicates of this kind are fundamentally incoherent, is rejected, then the conception of a weakly infinite but weakly finite totality must be accepted as legitimate. However, on the strength of conclusion (2), weakly infinite totalities may likewise be rejected as spurious: this of course entails the repudiation of strict finitism as a viable philosophy of mathematics.

It is to be noted that conclusion (2) relates to observational *predicates* only: we have no reason to advance any similar thesis about relational expressions whose application is taken to be established by observation. In the example of the minute hand, we took the relational expression 'x is not in a discriminably different position from y' as being, not merely governed by consistent rules of use, but completely definite. This may be an idealisation: but, if such an expression is vague, its vagueness evidently arises from a different source from that of a predicate like 'red' or 'vertical'. If the application of a predicate, say 'red', were to be determined by observational comparison of an object with some prototype, then it too could have a consistent use and a definite application: e.g. if we all carried around a colour-chart, as Wittgenstein suggested in one of his examples, and 'red' were taken to mean 'not discriminably different in colour from some shade within a given segment of the spectrum displayed on the chart', then, at least as far as any consideration to which we have so far attended is concerned, there is no reason why 'red' should even be considered a vague term. It would not, however, in this case be an observational predicate, as this notion is normally understood.

What, then, of phenomenal qualities? It is not at first evident that this notion is beyond rescue. Certainly, if the foregoing conclusions are correct, we cannot take 'phenomenal quality' in a strict sense, as constituting the satisfaction of an observational predicate, that is, a predicate whose application can be decided merely by the employment of our sense-organs: at least, not in any area in which non-discriminable difference is not transitive. But cannot the notion be retained in some less strict sense?

One thing is beyond question: that, within some dimension along which we can make no discriminations at all, the notion of 'not phenomenally distinct' is viable and significant. For instance, light of a certain colour may be more or less pure according to the range of wavelengths into which it can be separated: if human vision is altogether incapable of discriminating between surfaces according to the purity of the light which they reflect, then here is a difference in physical colour to which no difference in phenomenal colour corresponds.

But how do things stand in respect of a dimension along which we can discern differences, but for which non-discriminable difference is not transitive? It may be thought that we know the solution to this difficulty, namely that, already mentioned, devised by Goodman. To revert to the minute hand example: we called the position which the minute hand appeared to occupy at the origin 'position *X*'; and we may call the positions which it appears to occupy at the end of 3, and of 4, seconds respectively 'positions *Y* and *Z*'. Now an observer reports that, at the end of 3 seconds, the minute hand does not appear to occupy a position different from that which it occupied at the origin: let us express this report, not by the words 'It appears still to be in position *X*', but by the words 'Position *Y* appears to be the same as position *X*'. At the end of 4 seconds, however, the observer will report both, 'Position *Z* appears to be different from position *X*', and, 'Position *Z* appears to be the same as position *Y*'. This has, as we remarked, the flavour of paradox: either we shall have to say that a contradictory state of affairs may appear to obtain, or we shall have to say that, from 'It appears to be the case that *A*' and 'It appears to be the case that *B*', it is illicit to infer 'It appears to be the case that *A* and *B*'. However, Goodman can take this apparent paradox in his stride. For him, position *Y*, considered as a phenomenal position, may appear to be identical with position *X*: it is, nevertheless, distinct, since position *Y* also appears to be identical with position *Z*, while position *X* does not. Will not Goodman's refined criterion of identity for phenomenal qualities save the notion of such qualities from the fate that appeared about to overwhelm them?

It is clear that *a* notion survives under Goodman's emendation: what is seldom observed is how unlike the notion that emerges is to the notion of phenomenal qualities as traditionally conceived. For let us suppose that space and time are continua, and let us change the example so that the minute hand now moves at a uniform rate. Let us further suppose that whether or not the minute hand occupies discriminably different positions at different moments depends uniformly upon whether or not the angle made by the two positions of the minute hand is greater than a certain minimum. It will then follow that, however gross our perception of the position of the minute hand may be, there is a continuum of distinct phenomenal positions for the minute hand: for, for any two distinct physical positions of the minute hand, even if they are not discriminably different, there will be a third physical position which is discriminably different from the one but not from the other.

This conclusion may not, at first, seem disturbing. After all, the visual field does appear to form a continuum: what is perplexing to us is not to be told that it is a continuum, but to be told that it is not, that, on the ground that we can only discriminate finitely many distinct positions, the structure of the visual field is in fact discrete. So perhaps Goodman's account of the matter, according to which there really is a continuum of distinct phenomenal positions, even though we can make directly only finitely many discriminations, may seem to be explanatory of the fact that the visual field impresses us as being a continuum. But a little reflection shows that the matter is not so straightforward: for the argument that the visual field must contain a continuum of distinct phenomenal positions is quite independent of the fineness of the discriminations that we can make. Imagine someone with a vision so coarse that it can directly discriminate only four distinct positions in the visual field (say right or left, up or down): that is, it is not possible to arrange more than four objects, big enough for this person to see, so that he can distinguish between their position. So long as non-discriminable difference of position remains for this person non-transitive, and discriminable difference of position depends for him on the physical angle of separation of the objects, the argument will prove, for him too, that his visual field, considered as composed of phenomenal positions distinguished by Goodman's criterion of identity, constitutes a two-dimensional continuum.

The argument has nothing to do with infinity. Let us consider difference of hue, as manifested by pure light (light of a single wavelength); and let us assume that the possible wave-lengths form a discrete series, each term separated by the same interval from its neighbours, so that the series is finite. And let us suppose an observer with colour-vision so coarse that he cannot distinguish more than four colours, i.e., it is not possible to show him pure light of more than four different wave-lengths so that he can discriminate directly between any two of them. If, for him, discriminable difference depends solely on the actual interval between the wave-lengths of two beams, then, again, the argument will establish that, for this observer, there are just as many phenomenal colours as physical colours. In fact, we see quite generally that, within any dimension along which we can discriminate by observation at all, and within which non-discriminable difference is non-transitive (as it surely always is), the phenomenal qualities are simply going to reflect the distinct physical qualities, irrespective of the capacities of the observer to discriminate between them. There is, of course, nothing wrong with the definition of 'phenomenal quality' which yields this result, considered merely as a definition: but what it defines is surely not anything which we have ever taken a phenomenal quality to be.

The upshot of our discussion is, then, this. As far as strict finitism is concerned, common sense is vindicated: there are no totalities which are both weakly finite and weakly infinite, and strict finitism is therefore an untenable position. But this vindication stands or falls with another conclusion far less agreeable to common sense: there are no phenomenal qualities, as these have been traditionally understood; and, while our language certainly contains observational predicates as well as relational expressions, the former (though not the latter) infect it with inconsistency.

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NOTE

* This paper was writtên in the autumn of 1970, and read at the University of New York at Buffalo and at Princeton University. It has since had some circulation in xerox form, and several people have developed the ideas in various directions; I thought, nevertheless, that it might be of interest to make the article generally accessible, particularly as others have not been so concerned with the application to strict finitism. Since I have had the benefit of seeing some of the later essays, particularly those of Dr Crispin Wright, I thought it best to leave the article in its original form, with only the most trifling stylistic changes, although I am well aware that Dr Wright's careful

exploration of the topic brings out in much more detail the differences between the various examples. The title relates to an article by Professor Hao Wang which I remember reading in an ephemeral Oxford publication many years ago. I should probably have abandoned it had I published the article sooner, since I never supposed that Professor Wang intended anything but to display the general form of a range of ancient paradoxes; but, since the name has gained some currency, I thought it better to leave it.