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LOGIC AS CALCULUS AND LOGIC AS LANGUAGE

Answering Schröder's criticisms of *Begriffsschrift*, Frege states that, unlike Boole's, his logic is not a *calculus ratiocinator*, or not merely a *calculus ratiocinator*, but a *lingua characterica*.[1] If we come to understand what Frege means by this opposition, we shall gain a useful insight into the history of logic.

Before settling down to this task, I would like to review, or rather simply enumerate, Frege's contributions to logic, in order to provide the proper background for our discussion. These contributions are:

(1) The propositional calculus, with truth-functional definitions of the connectives, of the conditional in particular;

(2) The decomposition of the proposition into function and argument(s), instead of subject and predicate;

(3) Quantification theory, based on a system of axioms and rules of inference;

(4) Definitions of infinite sequence and natural number in terms of notions of logic.

Besides these four discoveries two more points must be mentioned:

(a) Frege was the first to present, with all the necessary accuracy, a cardinal notion of modern thought, that of formal system;

(b) Frege's philosophy is analytic, in the sense that logic has a constant control over his philosophical investigations; this marked a sharp break with the past, especially in Germany, and Frege influenced philosophers as different as Russell, Wittgenstein, and Austin.

The opposition between *calculus ratiocinator* and *lingua characterica* has several connected but distinct aspects. These various aspects, most of the time not stated by Frege, have to be brought out by a study of his work. From Frege's writings a certain picture of logic emerges, a conception that is perhaps not discussed explicitly but nevertheless constantly guides Frege. In referring to this conception I shall speak of the *universality* of logic.

This universality of Frege's *lingua characterica* is, first, the universality

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that quantification theory has in its vocabulary and that the propositional calculus lacks. Frege frequently calls Boole's logic an 'abstract logic'[2], and what he means by that is that in this logic the proposition remains unanalyzed. The proposition is reduced to a mere truth value. With the introduction of predicate letters, variables, and quantifiers, the proposition becomes articulated and can express a meaning. The new notation allows the symbolic rewriting of whole tracts of scientific knowledge, perhaps of all of it, a task that is altogether beyond the reach of the propositional calculus. We now have a *lingua*, not simply a calculus. Boole's logic, which cannot claim to be such a *lingua*, remains the study, in ordinary language, of algebraic relations between propositions. This study is carried out in ordinary language and is comparable to many branches of mathematics, say group theory. In Frege's system the propositional calculus subsists embedded in quantification theory; the opposition between *lingua* and *calculus* is, in this respect, not exclusive, and that is why Frege writes that his own logic is *not merely* a *calculus ratiocinator*.[3]

However, the opposition between *calculus ratiocinator* and *lingua characterica* goes much beyond the distinction between the propositional calculus and quantification theory. The universality of logic expresses itself in an important feature of Frege's system. In that system the quantifiers binding individual variables range over all objects. As is well known, according to Frege, the ontological furniture of the universe divides into objects and functions. Boole has his universe class, and De Morgan his universe of discourse, denoted by '1'. But these have hardly any ontological import. They can be changed at will. The universe of discourse comprehends only what we agree to consider at a certain time, in a certain context. For Frege it cannot be a question of changing universes. One could not even say that he restricts himself to *one* universe. His universe is *the* universe. Not necessarily the physical universe, of course, because for Frege some objects are not physical. Frege's universe consists of all that there is, and it is fixed.

This conception has several important consequences for logic. One, for instance, is that functions (hence, as a special case, concepts) must be defined for all objects. To take an example, the function '+' is defined not only for the natural numbers, but also for, say, the Moon and 1. What the value of the function is in that case is irrelevant here, but this value must exist for every set of arguments chosen from among the objects. When Frege has to deal with a special domain of objects, the natural numbers for example in arithmetic, he uses devices that are in fact equivalent to the method of relativization of quantifiers.

Another important consequence of the universality of logic is that nothing can be, or has to be, said outside of the system. And, in fact, Frege never raises any metasystematic question (consistency, independence of axioms, completeness). Frege is indeed fully aware that any formal system requires rules that are not expressed in the system; but these rules are void of any intuitive logic; they are 'rules for the use of our signs'.[4] In such a manipulation of signs, from which any argumentative logic has been squeezed out, Frege sees precisely the advantage of a formal system.

Since logic is a language, that language has to be learned. Like many languages in many circumstances, the language has to be learned by suggestions and clues. Frege repeatedly states, when introducing his system, that he is giving 'hints' to the reader, that the reader has to meet him halfway and should not begrudge him a share of 'good will'. The problem is to bring the reader to 'catch on'; he has to get into the language.[5]

In *Principia Mathematica* some of the aspects of the universality of logic are modified – by the introduction of types. Quantifiers now range over stratified types. But within one type there is no restriction to a specific domain, and in that sense the universality is preserved. We have a stratified universe, but here again it is *the* universe, not a universe of discourse changeable at will.

Questions about the system are as absent from *Principia Mathematica* as they are from Frege's work. Semantic notions are unknown. '⊢' is read as '... is true', and Russell could hardly have come to add to the notion of provability a notion of validity based on naive set theory. At the beginning of his 1930 paper on the completeness of quantification theory Gödel describes the axioms and the rules of inference of *Principia Mathematica* and then adds: "*Of course*, when such a procedure is followed the question *at once* arises whether the system of axioms and principles of inference initially postulated is complete, that is, whether it really suffices for the derivation of *every* true logico-mathematical proposition or whether, perhaps, true propositions (which may even be provable by means of other principles) are conceivable that cannot be derived in the system under consideration."[6] (My emphasis first two times.) Gödel wrote these lines twenty years after the publication of the first volume of *Principia*. If the question of the semantic completeness of quantification theory did not 'at once' arise, it is because of the universality – in the sense that I tried to extricate – of Frege's and Russell's logic. The universal formal language supplants the natural language, and to preserve, outside of the system, a notion of validity based on intuitive set theory, does not seem to fit into the scientific reconstruction of the language. The only question of completeness that may arise is, to use an expression of Herbrand's, an *experimental* question. As many theorems as possible are derived in the system. Can we exhaust the intuitive modes of reasoning actually used in science? To answer this question is the purpose of the Frege-Russell enterprise, to which we must adjoin, in spite of all its deficiencies, Peano's work. *Begriffsschrift, Die Grundlagen der Arithmetik,* the two volumes of *Grundgesetze der Arithmetik, Arithmetices Principia,* the various editions of the *Formulaire de mathématiques, The Principles of Mathematics,* and the three volumes of *Principia Mathematica* – each of these works can be regarded as a step in an ever-renewed attempt at establishing completeness experimentally.

In 1915 Löwenheim published a paper that contained many novel features. The system with which Löwenheim is concerned most of the time is the first-order predicate calculus with identity. He has no axioms or rules of inference. His logic is based upon naive set theory, and the notion of provability is replaced by that of validity. While the Frege-Russell approach to the foundations of logic could be called the *axiomatic*[7] approach, Löwenheim's could be called the *set-theoretic* approach. If we follow that approach, questions of validity of well-formed formulas in different domains come to the forefront. The very title of the paper, *Über Möglichkeiten im Relativkalkül,* refers to this point: if a formula is valid in a domain, it may or may not be valid in some other domain. For instance, for the singulary fragment of the first-order predicate calculus, if a well-formed formula that contains occurrences of k distinct predicate letters is valid in a domain of 2^k elements, it is valid in every domain. Or take the famous Löwenheim theorem: if a well-formed formula is valid in a denumerable domain, it is valid in every domain.[8] Several cases of the decision problem and the reduction problem are treated by the semantic method: from the validity of a well-formed formula in a domain some argument allows us to conclude to the validity of a related well-formed formula in the same domain, or to the validity of the same well-formed formula in some other domain.

These results and these methods were entirely alien to the Frege-Russell trend in logic. So alien that it is quite puzzling how Löwenheim came to think of his theorem. The explanation is perhaps as follows. From the result mentioned in the previous paragraph about the singulary fragment of the first-order predicate calculus it follows that, if a well-formed formula of that fragment is valid in every finite domain, it is valid. This does not hold for the full calculus. In fact, Löwenheim knew of formulas of that calculus that, although valid in every finite domain, are not valid in every domain. But then – since in the singulary case finite validity leads to validity – it becomes natural to raise the following question: if a well-formed formula is valid in a denumerable domain, is it valid in every domain? The answer is yes, and this is Löwenheim's theorem.

With Löwenheim's paper we have a sharp break with the Frege-Russell approach to the foundations of logic and a return to, or at least a connection with, pre-Fregean or non-Fregean logic. Löwenheim uses Schröder's logical notation, but, what is more important, with Schröder he also takes the freedom to change the universe of discourse at will and to base considerations on such changes. And just as Frege was ignored for some time because of his break with the tradition established, so Löwenheim too was ignored for some time because of his break with the new tradition established. Behind the Frege-Russell trend in logic, Löwenheim renews contact with Boole and Schröder, while making important contributions of his own to logic.

The first reaction to Löwenheim's paper was Skolem's paper of 1920 [9], which still follows the set-theoretic approach to logic. Soon, however, the opposition between the two trends in logic dissolved. During the 'twenties the work of Skolem, Herbrand, and Gödel produced an amalgamation and also a *dépassement* of these two trends. In particular, the work of Herbrand can be viewed as establishing, beside the axiomatic and the set-theoretic approaches to the foundations of logic, a third approach, that of the Herbrand expansions. But that is another story. Let me say simply, in conclusion, that *Begriffsschrift* (1879), Löwenheim's paper

(1915), and Chapter 5 of Herbrand's thesis (1929) are the three cornerstones of modern logic.

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REFERENCES

[1] Schröder's criticisms are contained in his review of *Begriffsschrift*, published in *Zeitschrift für Mathematik und Physik* **25** (1880), *Historisch-literarische Abtheilung*, 81–94. Frege's reply was an address to a learned society, delivered on 27 January 1882 and published in its proceedings, 'Über den Zweck der Begriffsschrift', *Sitzungsberichte der Jenaischen Gesellschaft für Medicin und Naturwissenschaft für das Jahr 1882* (Jena 1883), pp. 1–10, reprinted in Gottlob Frege, *Begriffsschrift und andere Aufsätze*, Hildesheim 1964, pp. 97–106. On the origin of the expression 'lingua characterica' see Günther Patzig's footnote 8, on p. 10 of Gottlob Frege, *Logische Untersuchungen*, Göttingen 1966.

[2] See, for instance, Frege's comments on Boole in 'Über den Zweck der Begriffsschrift' (mentioned in footnote 1), pp. 1–2.

[3] In 'Über die Begriffsschrift des Herrn Peano und meine eigene', *Berichte über die Verhandlungen der Königlichen Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-physische Classe* **48** (1896), 361–378, Frege writes on p. 371: "Boole's logic is a *calculus ratiocinator*, but no *lingua characterica*; Peano's mathematical logic is in the main a *lingua characterica* and, subsidiarily, also a *calculus ratiocinator*, while my *Begriffsschrift* intends to be both with equal stress." Here the terms are used with approximately the meanings given in the present paragraph: Boole has a propositional calculus but no quantification theory; Peano has a notation for quantification theory but only a very deficient technique of derivation; Frege has a notation for quantification theory and a technique of derivation.

[4] *Begriffsschrift*, § 13.

[5] Here the influence of Frege on Wittgenstein is obvious. – Frege's refusal to entertain metasystematic questions explains perhaps why he was not too disturbed by the statement 'The concept *Horse* is not a concept'. The paradox arises from the fact that, since concepts, being functions, are not objects, we cannot name them, hence we are unable to talk about them. Some statements that are (apparently) about concepts can easily be translated into the system; thus, 'the concept $\Phi(\xi)$ is realized' becomes '$(Ex)\ \Phi(x)$'. The statements that resist such a translation are, upon examination, metasystematic; for example, 'there are functions' cannot be translated into the system, but we *see*, once we have 'caught on', that there are function signs among the signs of the system, hence that there are functions.

[6] Kurt Gödel, 'Die Vollständigkeit der Axiome des logischen Funktionenkalküls', *Monatshefte für Mathematik und Physik* **37**, 349–360; English translation by Stefan Bauer-Mengelberg in J. van Heijenoort, *From Frege to Gödel, A Source Book in Mathematical Logic, 1879–1931*, Harvard University Press, Cambridge, Mass., 1967.

[7] Here 'axiomatic' is used for a method of formal derivation based on axioms and rules of inference, and this use should not be confused with broader uses, as in 'the axiomatic method in geometry'. – Let us remark that, unlike Frege, Russell never emphasized the formal aspect of logical proofs and that, in particular, the system of

Principia Mathematica does not measure up to the standards that Frege set for a formal system. (On this point see Kurt Gödel, 'Russell's Mathematical Logic', in *The Philosophy of Bertrand Russell* (ed. by Paul Arthur Schilpp), New York 1944, pp. 123–153, especially p. 126; see also W. V. Quine, 'Whitehead and the Rise of Modern Logic', in *The Philosophy of Alfred North Whitehead* (ed. by Paul Arthur Schilpp), New York 1941, pp. 125–163, especially p. 140.) The notion of formal system was again brought into the forefront by Hilbert, in the 'twenties. That is perhaps why the (in our sense) axiomatic systems of logic are called Hilbert-type systems by Kleene (*Introduction to Metamathematics*, p. 441). If the historical priority is to be respected, they should rather be called Frege-type systems.

[8] For the sake of simplicity I take the formulation of the theorem for quantification theory without identity.

[9] 'Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen', *Videnskapsselskapets skrifter, I. Matematisk-naturvidenskabelig klasse*, no. 4.