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A PIAGETIAN PERSPECTIVE ON MATHEMATICAL CONSTRUCTION*

ABSTRACT. In this paper, we offer a Piagetian perspective on the construction of the logico-mathematical schemas which embody our knowledge of logic and mathematics. Logico-mathematical entities are tied to the subject's activities, yet are so constructed by reflective abstraction that they result from sensorimotor experience only via the construction of intermediate schemas of increasing abstraction. The 'axiom set' does not exhaust the cognitive structure (schema network) which the mathematician thus acquires. We thus view 'truth' not as something to be defined within the closed 'world' of a formal system but rather in terms of the schema network within which the formal system is embedded. We differ from Piaget in that we see mathematical knowledge as based on social processes of mutual verification which provide an external drive to any 'necessary dynamic' of reflective abstraction within the individual. From this perspective, we argue that axiom schemas tied to a preferred interpretation may provide a necessary intermediate stage of reflective abstraction en route to acquisition of the ability to use formal systems in abstracto.

1. INTUITIVE STRUCTURES AND FORMALIZED MATHEMATICS

Our claim is that a proper understanding of mathematical knowledge requires us to focus on the tension between a formal system (of axioms and rules of proof) and the broader network of knowledge we bring to bear when we "do" mathematics. To start, consider that, in providing a proof in geometry, we might use a phrase such as 'Take a triangle ABC', yet must avoid recourse to properties of a triangle too special for the study at hand. This led Locke (*An Essay Concerning Human Understanding*, Book IV, ch. 7, section 9) to introduce the idea of the general triangle, which would be neither obtuse, nor right-angled, nor equilateral, nor isosceles, nor scalene. Yet we may have a specific figure of a triangle before us as we follow the proof. Beth asks (MEP, p. 8)[1]

(1) Why do we introduce into the demonstration of a universal mathematical proposition an intermediate phase which relates to a particular object (for example, a triangle)?

(2) How can an argument which introduces such an intermediate phase nevertheless give rise to a universal conclusion?

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Beth soon provides a partial answer:

> We might imagine two disputants, one of whom asserts "that the three angles of a triangle are equal". The other refutes this assertion by constructing a right-angled triangle . . . . However, a certain mathematical background will enable the mathematician to anticipate, to some extent, the "counter examples" which his adversary might use . . . [and thus] avoid hasty generalizations.
>
> Let us note that the structure of such an anticipation is transferred from the level of discussion to that of formal reasoning. If we introduce a deductive argument with the words "Let ABC be any triangle" . . . it is because the choice of this triangle is left to an imaginary opponent. (MEP, p. 10)

This fits well the notion of mutual verification espoused by Piaget (MEP, pp. 289–90):

> . . . Owing to language . . . the general coordination of actions ceases to be uniquely intrapersonal as it may be in the animal or the very young child, to become interpersonal and contribute to an objectivity of which the individual is himself doubtless incapable, at least at a certain level . . . . The very coordination of interpersonal actions, that is, cooperation as opposed to the constraints of opinion, in fact constitutes a system of operations carried out in common all by cooperation, and, . . . this is then a question of the same operations as those of intra-individual coordination: combinations, overlappings, correspondence, reciprocities etc.; for communication is only the setting up of a correspondence between individual operations, this correspondence being yet another operation . . . But these operations in common require a mutual verification of a higher level than self-verification, so that the laws of coordination become normative laws regulating intellectual intercourse between people . . . .

We thus have a view of mathematical truth as not being purely contained within a formal framework. 'Truth' is subject to continual testing by the individual both against her own experience and against critiques provided by peers. This personal experience provides a stock of intuitions which provide the "self-evident truths" against which other statements may be tested. But such 'truths' may not stand the test of time. In a related vein, Beth (MEP, p. 125) quotes with approval the following statement by Bernays:

> We often think that we must either accept an absolute self-evidence or renounce entirely the contribution of self-evidence to the sciences. Instead of resigning ourselves to this "All or None" it seems more appropriate to formulate a conception of self-evidence as acquired. Man masters self-evidence as he learns to walk or as the bird learns to fly. In this way we arrive at the Socratic recognition that, in principle, we know nothing in advance. In the theoretical domain, we can only experiment with opinions and points of view, and thus eventually achieve an intellectual success.

We have clearly entered the Piagetian domain, a constructivist or developmental theory of knowledge. We take two lessons from all this for the notion of schema (the unit of knowledge central to our 'Piagetian perspective'):

(1) The schemas which enter into our mathematics will be both 'intuitive' and 'formal'. In general, such schemas will be acquired rather than innate.

(2) The acquisition of such a schema in no way guarantees its 'infallibility'.

In Piaget's theory, 'errors' in the application of a schema may be 'assimilated' to extend its applicability. Beth gives a forcible analog of such 'testing' when he states (MEP, p. 72) "the Fundamental Criterion of Demonstrative Force: An argument has demonstrative force if it admits no counterexample". The problem, of course, is to determine when we may confidently accept the lack of a counterexample in acting upon a given schema. ('Real life' schemas seldom offer the unequivocal true or false criterion of logical argument.) Beth offers the method of semantic tableaux as a way to systematically search for counterexamples: "then, by verifying the failure of the search, we can be assured of the non-existence of an appropriate counterexample and of the demonstrative force of the argument" (MEP, p. 73). What this statement does not make explicit is that, in many mathematical situations, even if a search strategy will always yield a counterexample if it exists, we may or may not be able to "solve the halting problem" to verify that no counterexample exists. As much to the point for our later discussions is that no exhaustive search procedure may be accessible to us when what we are exploring is some 'external reality' rather than a formal structure of statements with computable truth values.

When discussing the sensory-motor experiences of the young child, Piaget would wish to talk of an environment in which objects have external reality: rocks are heavy to lift, objects when released will fall to the ground unless supported in some way. In the case of mathematics, the situation would seem to be rather different. The concept of i, the square root of -1, need not be seen as being possessed of a reality external to human constructions. However, this does not deny that such a concept has a social reality external to any individual when learning it, as distinct from the initial act of invention/discovery. The concept first arose when some human constructed it to provide a link in a chain of mathematical reasoning and, having constructed it, was able to communicate it through the use of language. Through the process of 'mutual verification' it does take on a reality which can influence the development of other mathematicians, as we have already suggested, but this reality is a social reality, and is itself the fruit of earlier constructions. In other words, from a synchronic viewpoint, a mathematical construct may be part of the reality which can causally drive the cognitive development of a student, but viewed diachronically, it is definitely a construct of the cognitive activity of (possibly many) people.

The construction of a schema is the fruit of assimilation and accommodation, processes based on the comparison of expectation and result of actions (Neisser 1976). We thus do well to distinguish the schema of an object or action (Arbib 1981) – which is indeed constructed by each human anew – from whatever it may be in the environment, whether it be physical or social, which provides the aliment (to continue the Piagetian metaphor) which is to be assimilated. It is the process of constructing a schema within each individual that Piaget would refer to as "the genetic sense" of construction, and he would explicitly distinguish this from the "mathematical sense" of construction proceeding within a formal system and unconstrained by anything that is not already within that system. Notice, then, the very strong contrast between the mathematician seeking to construct objects within a given formal system from the child constructing schemas (not necessarily consciously) to enable it to assimilate a world which extends far beyond the schemas already available to the child at any given stage.

Piaget asserts that

> If the truth of $2 + 2 = 4$ is not a factual datum but a logical demonstration, it nonetheless remains true that the epistemological problem is not solved when we show why the demonstration is valid: we still have to know what 2, 4, + and = "are" or "designate", and what the subject does to comply with the normative necessity of this demonstration. (MEP, p. 153)

This distinction is similar to the crucial one, reiterated below, between axiom systems like Peano's axioms where we feel that we understand theorems by reference to a specific interpretation, (i.e., a preexisting network of schemas) such as the "counting numbers", from structures like that of groups, where we accept a statement as being valid within a formal system without requiring the cognitive underpinning of a preferred interpretation.

We also stress the distinction between mathematical properties and 'real world' properties. If, as mathematicians, we accept the Peano axioms as characterizing certain properties of the numbers, then we do not have to go back to the original interpretation to make new discoveries about the numbers: we can do this by proving theorems within the formal system. In "everyday cognition of the real world", we certainly can and do make valid inferences from what we already know, but we can also learn about an object not from the schema that we have constructed but by carrying out new operations of observation and action on objects in the world recognized as falling under the schema. When we use a name, whether it be a generic name or a proper name, we have to have a critical set of recognition criteria before we can use the name, but then there are other criteria which constitute further knowledge. With time, these criteria can migrate, as when we go from the individual recognition of our own mother to a sophisticated adult notion of "mother" based on biological and kinship relations.

We shall reiterate the distinction between those formal structures which "have a life of their own" and those which are anchored in some preferred or unique interpretation. One would want to give an account of how axioms are accommodated to fit examples, and how examples are assimilated to the axioms. In the actual practice of mathematics, or in everyday cognition, one works not simply within a given formal structure or set of informal schemas, but strives rather to bring the appropriate "tools" to bear (cf. our discussion of artificial intelligence approaches to proof construction in section 3). But whether we are in the "mathematical" or "everyday" realm, we come back to Piaget's view of genetic epistemology: to show how something can be constructed, or rather how a schema for it is constructed, is to answer the question of how it can be, or is, known.

With this perspective we may come to understand how it is that "the result of original work in the mathematical field is called sometimes a creation or invention, sometimes a construction or discovery" and agree with Beth that "the fact that [for many mathematicians] Platonism expresses a psychical reality clearly does not prove that Platonism, or even the Platonist conception of mathematics, contains the truth" (MEP, pp. 99–100).

2. REFLECTIVE ABSTRACTION AND MATHEMATICAL DEVELOPMENT

In the previous section, we have argued for the relevance of a developmental approach to logico-mathematical systems which embeds them within the broader schema network of everyday knowledge and experience. When mathematics "gets into someone's head", what has she grasped, what has she constructed? In MEP, Piaget argues that when the mathematician constructs the set of formal axioms which characterize some mathematical structure, she does so within a web of not necessarily consistent intuitions (by which we do not mean that the thought is intuitionistic in the technical sense).

Piaget is careful to distinguish formal structures from 'natural thought'. Formal systems are 'artificial' and may not faithfully reflect the processes of thought from which they are abstracted. Natural thought might even bring about an inconsistent system.[2] We suggest that the situation be understood in the following way. Our mental development equips us with a stock of schemas which may be seen as embodying examples and skills more often than logical principles. In a given situation, we mobilize a stock of relevant schemas to approximate a solution which will guide our action. The criterion which guides schema change, then, is not one of consistency from occasion to occasion, but rather of continuing viability. Since our observations are sporadic rather than rigorous, we may elevate a frequently observed regularity to a universal law. This may lead to inconsistencies when the law is embedded within a formal theory, yet be perfectly serviceable as a guide to action. With this realization, we may understand how it was that Frege's formal system of arithmetic was formed by reflective abstraction, and proved to be inconsistent. Frege's axiom of comprehension allowed Russell to define the set R of all those sets which do not belong to themselves, and this yielded the contradiction that R could belong to itself if and only if it did not belong to itself. The theory of types was one attempt to limit a claimed universality that did not survive a particular type of logical test.

This very talk of consistency and inconsistency shows us how far we have come from discussions at the sensory-motor stage. For a child interacting with objects in its environment, there is no strict criterion for consistency of action. Moreover, the separation by the observer of the child's behavior into a number of separate schemas may well be arbitrary. By contrast, when we speak of a mathematical theory, we ask that the various axioms and rules of inference be explicitly formulated, so that we may open up for inspection the corpus of theorems associated with that structure. Only then can we properly talk of consistency.

Two basic results, due to Tarski and Gödel, restrict the semantics of (a certain broad family of) formal systems of logic. Tarski showed that any formal system which contains its own truth predicate must be inconsistent. In other words, if the system is consistent, then it would be impossible to define 'truth' over the declarative sentences of that system. On the other hand, Gödel has shown that any formal system which is non-contradictory, and contains elementary arithmetic, must be incomplete in that there is a formula of the system which can be neither proved nor refuted. By contrast, an approach to the semantics of natural language from the broad Piagetian viewpoint sketched in Hill and Arbib 1984, would seem to escape the strictures of Tarski's and Gödel's results. Such a system is never entirely consistent and is in a continual state of Piagetian becoming. We are able to speak of 'truth' in our own language, and we are able to hone that discussion when we concentrate on specific formal systems, but we do not envisage any complete characterization of 'truth'.

The point we are making is akin to that which Arbib and Hesse 1986 make when they assert that "all language is metaphorical". Where some philosophers of language see literal interpretation as the touchstone of semantics, with metaphor as aberrant, they would see metaphor as the norm, with literal meaning attainable as a limiting case where social convention (including the norms of a community of scientists) blocks the rich variety of alternative explorations of an individual's schema network that can be initiated by, and constitute the interpretation of, most sentences. We may accept a sentence as true either because we assimilate it to our existing schemas, or (remember the point about the unreliability of "self-evident truths" in section 1) by accommodating our schemas to yield ones that can provide an acceptable interpretation. The criterion is local coherence rather than global consistency. A schema network is not a static formal system whose consistency is to be established or denied, but is rather a shifting approximation to a reality defined by social interaction (mutual verification) as well as by other tests of experience. We do not deny the importance of consistency, but we do suggest that it can only be attained in formal and limited systems that do not exhaust our knowledge of that (as defined by an extended schema network, and the environmental interactions that it supports) which the system is designed to represent.

Piaget's approach to cognition is a constructive one, and we shall now relate it to an analysis of the development of mathematical thought in the individual mathematician, with numerical and geometric intuition providing the basis for the acquisition of formal axiom systems. Piaget analyzes the three basic structures of the Bourbaki school – group, relational structure and topological structure – and seeks to causally explain their development from schemas embodying the way an individual interacts with the world. The child learns how to "undo" an action, and so manifests the schema for reversibility, which may be seen to underlie the notion of group structure. Such a notion of reversing actions ties in with our ability to find our way about the environment. Even by two or three years of age, a child has many such operations at her disposal, and over time these operations yield structures on which, Piaget suggests, a process of reflective abstraction can produce the formal structures of the mathematician.

Piaget explicitly notes that the ability of Bourbaki to build mathematics from the three basic structures is a reversal of historical order. While concepts such as reversibility are basic to cognitive development, the abstract concept of a group (as distinct from notions of reversibility) was not basic to the historic development of mathematics. The Euclidean geometers certainly studied operations of translation and rotation, but the formalization of the fact that the collection of translations or of rotations in the plane could be characterized as interpreting the formal definition of a group did not come until the nineteenth century.

Here, then, is the fundamental observation at the core of our analysis: A mathematician's knowledge goes far beyond ability to parrot the axioms and rules of inference of a formal system. Thus the reductive program of Bourbaki is not compelling as an account of logico-mathematical knowledge. The understanding of "basic structures" provides a vanishingly small proportion of the understanding of later constructs. The "axiom set" does not exhaust the cognitive structure which the mathematician has acquired, and mathematical understanding of such a structure normally embeds it in a web of examples, interpretations, powerful results, and "hooks" which allow the mathematician to retrieve useful theorems and examples as they are needed. If we try to give a developmental account of formal structures which is pedagogically sound, it must be very different from one based on the formal structures alone (compare Rissland's 1978 'Understanding Understanding Mathematics' for the importance of the ability to construct, recall, and apply examples both positively and as counterexamples).

It is worth stressing that our "Piagetian perspective" would seem to go beyond Piaget's, for Piaget does seem to impute a necessity to the formal structures as set forth in Bourbaki where we would at most impute a necessity to more restricted operations which find their later reflective abstraction in these notions. By taking the Bourbaki structures as being in some sense inevitable, Piaget seems to be close to Chomsky in his insistence on the importance of certain linguistic universals – but where Chomsky speaks of "innateness", Piaget sees inevitability within a constructive process:

> The construction of pure mathematics starts from a system of schemas of action, the roots of which must undoubtedly be sought in the nervous and biological organization of the subject; and the construction only exhibits itself in the field of conscious thought by being forced to integrate the initial relationships included in the schemas. . . . the successive constructions obey directional laws, not because everything is given in advance, but because the need for integration itself involves a continuity which is only perceived retrospectively, but which nonetheless imposes itself. (MEP, p. 238)

Our rejection of "inevitability" or "directional laws" follows in part from the fact that we are perhaps more concerned than Piaget with the importance of instruction (socio-linguistic interaction) in the third of the following levels of cognitive development: (i) implicit use of reversibility; (ii) the use of addition and subtraction and the study of movements in the plane; and (iii) the ability after suitable instruction to reflect on the abstract notion of a group. This process of instruction provides a content which replaces "directional laws" by "historical contingency". We may contrast (a) a process of induction whereby from examples we come to create for ourselves a structure which subsumes those examples, with (b) a process of verification whereby we test that a structure explicitly presented to us does indeed subsume a specified set of examples (and note the assumption that we possess a language in which the description can be presented).

In this sense, we would distinguish the psychological status of Euclid's axioms and Peano's axioms from those of group theory. The former have preferred interpretations. We think of Euclid's axioms as being about line drawings which we can make on a piece of paper, subject to certain idealizations about line thickness, etc.; we think of Peano's axioms as being about numbers with which we can count, compare classes, etc. With propositional logic, we still have a preferred interpretation, namely recourse to truth tables, but make a somewhat interesting transition in that the actual construction of formal proofs is usually guided by criteria of symbol manipulation rather than by intuition about the interpretation. With group theory, however, there is no "standard" model at all. We may use particular interpretations both to suggest possible truths which we may strive to establish as lemmas, and we can also use such interpretations to provide counterexamples to our conjectures (recall our discussion in section 1). But truth in any particular model has no logical force with respect to group theory as it does in the study of numbers or geometry. We can only appeal to the axioms.

What we have said in no way invalidates the Piagetian notion that reflective abstraction underlies the evolution of more and more abstract structures. Let us make explicit Piaget's four stages of reflective abstraction:

(a) Operational relationships are abstracted from an antecedent structure.
(b) The relationships thus abstracted are reflected onto a new plane, carried forward in a more abstract structure.
(c) The new operations form the novelty of the derived construct.
(d) The new operations, together with the original operations, cohere into new wholes.

Piaget's view of the role of reflective abstraction in the development of logico-mathematical thought may be summarized as follows:

> A schema of action is, in fact, only the form of a series of actions which take place successively without a simultaneous perception of the whole. Reflective abstraction, on the other hand, upgrades it to the form of an operational schema, that is, of a structure such that, when one of the operations is used, its combination with others becomes deductively possible through a reflection going beyond the momentary action . . . and these operations can sooner or later be carried out symbolically without any further attention being paid to the objects which were in any case "any whatever" from the start. (MEP, pp. 237–38)

With this quote, I think we are in a position to better understand our difference with Piaget. The quote on "directional laws" in mathematics (MEP, p. 238) comes directly after the above quote on reflective abstraction in MEP. But reflective abstraction is seen by Piaget as an internal process operating on a stock of schemas to provide more abstract schemas "on a new plane". The motor for this process is the induction of regularities and relationships within the stock of schemas available within the mind of the individual. It is worth noting that the social dimension plays no role in this argument, and that Piaget's discussion of mutual verification (which we quoted in section 1) does not come until fifty pages later in MEP. It is perhaps because he does not integrate the social perspective into his analysis of logico-mathematical knowledge that Piaget sees a "necessary dynamic" which downplays the role of instruction and historical context.

We do not deny that the child's cognitive development includes the intrapersonal development of schemas, nor that we may liken this intrapersonal process to the induction of axioms from a variety of examples. However, we have stressed that much of mathematical development involves rather a process of mutual verification, as does the acquisition of membership in the community playing a given language game. As the new player joins the game, the structures are already "in place", and the new player must come to acquire these structures.

Certainly, great inventions arise from the efforts of individuals, but the ability of others to use these inventions need not imply spontaneous re-invention. It was a great invention of the Greeks to realize that much knowledge about geometry could be captured by logical inference from a set of axioms. But the Greeks thought that these axioms embodied a set of necessary and self-evident truths about the world. It was thus an equally great invention of nineteenth-century Europeans, in the wake of the discovery of non-Euclidean geometry, that axioms could be seen as formal statements divorced from any specific "world". We have thus distinguished the abstraction of axioms from a specific "model" which would seem to underlie the acquisition of axioms for number and plane geometry from the acquisition of such formal systems as group theory where one must work with the axioms rather than with the examples.

What we would add to Piaget's account of the mathematical development of the individual is the suggestion that the earlier form of axiomatization (that systematizing a preferred interpretation) provides the understanding of the axiomatic method which makes the acquisition of these later mathematical constructs possible, and that these must both (save for the rare genius) be provided by explicit instruction. Reflective abstraction may still be operative, but it involves the assimilation of externally supplied generalizations. We may thus offer the following synthesis of reflective abstraction with mutual verification, operating as a supplement to the "internally driven" reflective abstraction described above.

(a) Patterns of relationships are pointed out with respect to an antecedent structure.
(b) Extant schemas are re-organized in the light of these patterns, but serve to enrich these patterns by connecting them into the schema network.
(c) The new schemas form the novelty of the derived construct.
(d) The new operations, together with the original operations, cohere into new wholes.

3. CAUSAL GENESIS AND THE CONCEPT OF PROOF

We may consider three aspects regarding any mathematical demonstration:

1. There is the question (which we have already addressed) of how the mathematical structures themselves, i.e., the objects in terms of which the proof proceeds, arise for consideration in the mathematician's mind.
2. There is the actual process of constructing, or discovering, a proof.
3. There is the question of what it is that allows the mathematician, once a proof has been constructed, to re-examine it and ensure that each step constitutes a valid inference.

While it is up to the logician (or mathematician) to specify what validly follows from what, it is up to the psychologist (perhaps, as we shall see, aided by the concepts of artificial intelligence [AI]) to give a causal explanation of what makes this discovery possible. This is the problem we address in the present section.

We should not try to give a causal account of why the logician moved from step N to step N + 1 of the proof. In the first place, the rule of inference which created step N + 1 must, in general, call upon the fruits of several earlier steps of the proof. More importantly, if we look at AI approaches to theorem proving, we see that the "state" of the construction of a proof at any time is not a single line of a proof, but is rather a large search tree or graph, with each node corresponding to a line of a possible proof, connected to those other lines which would justify it (Nilsson 1980, chapters 5 and 6). At each stage in the proof, new nodes are added to the graph, but after a period of construction, it may turn out that no cumulative progress has been made in the last few additions. In such a case, backtracking occurs to an earlier stage, and alternative paths are to be explored. Thus, it is this overall search graph that determines what is to be constructed next; no one node determines its immediate successor. And only when one finally reaches a node which counts as a solution is the search graph then reduced to yield the path from axioms to result which counts as the formal proof. Beth (MEP, p. 22) distinguishes a phase of enquiry from the phase of arrangement which takes the solution found in the enquiry phase and arranges it in the "correct" format for an argument.

Most AI programs for proof construction lay no claim to psychological validity. They are just algorithms which constrain search through a space whose nodes are well-formed formulas in such a way as to eventually generate a proof. Nonetheless, considerations of the type of search space required, the use of different search techniques, and the way in which reference to conceptual structures (e.g., the classic Gelernter 1959 program which used geometric constructions to help prove theorems from the Euclidean axioms) would seem to be necessary for a psychologist's account of causal genesis of a proof. We might also note the notion of hierarchical planning and of subgoal generation, showing that we may come to "entertain" a well-formed formula without yet knowing whether or not it is obtainable by valid inference from the available theorems to date. In fact, the usual proof generated by a mathematician does not involve the careful application of a specifically formalized rule of inference, but rather involves a somewhat large jump from statement to statement based both on formal techniques and on intuitions about the subject matter at hand. The psychologist must understand why it is that such jumps are compelling, even though these jumps may not always prove to be valid when subjected to microanalysis.

Continuing, we should note that any premise has many implications. Different techniques have been developed in AI for systems which attempt to prove (or disprove) a specific statement, which then serves as the goal which constrains the search process proceeding outward from the given premises. Other systems have been developed which generate a number of inferences from given premises, and then try to evaluate them to find those which are worth adding to the stock of premises as a basis for further search, thus generating a range of interesting theorems (Davis and Lenat 1982). We might say that the task of the logician is to understand that a step is valid, while the job of the psychologist is to understand how the step is taken. The current efforts to construct theorem proving systems in AI (reviewed in, e.g., Nilsson 1980) may be seen as first steps to showing what the shape of such a causal explanation of the 'why' might look like.

Beth argues that acceptance of the possibility of a "thinking machine" capable of "replacing" the logician or mathematician "would force us to deny all originality to logical and mathematical thought, and it would thus be incompatible with our experience according to which the solution of mathematical problems, in particular, requires original thought" (MEP, p. 114). He then asserts that

> The proper function of intelligence consists in solving problems, and to solve a problem is equivalent to finding means which are adequate in relation to a certain end. If the means are never inevitably determined by the end in view, then it will always be necessary to have recourse to intelligence to find means adequate to the proposed end. This consideration excludes the possibility of constructing a machine capable of solving any problem whatever. (MEP, p. 118)

It may be objected that intelligence does not reduce to problem-solving: Does it really illuminate what the artist does to label the painting of a picture as a "problem to be solved"? But here our task is simply to criticize Beth's arguments against the eventual emulation of "mathematical creativity" without asserting that AI is capable of such emulation now or in the near future. Beth is too limited by his view that "the construction of a 'thinking machine' presupposes the solution of a decision problem". While a full critique of this viewpoint is beyond the scope of this paper, we can briefly note (i) that the type of search and means-end analysis embodied for Beth in his method of semantic tableaux is achieved by AI programs which prove theorems by the resolution technique (Nilsson 1980, chapter 5 for a review); and (ii) that a thinking machine, from our Piagetian perspective, would be a learning machine using approximate (and thus not necessarily consistent) knowledge to guide searches whose successful completion is no more guaranteed than is that of a human mathematician trying to prove a theorem.

NOTES

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[1] Part I of Mathematical Epistemology and Psychology (Beth and Piaget, 1966), hereinafter referred to as MEP, was written by the Dutch mathematical logician Ewert W. Beth. I am most grateful to Barry Richards, whose lectures on the semantic implications of Piaget's work stressed the fruitfulness of close study of MEP and provided the spur for the present paper.

[2] Much of current work in distributed problem solving in artificial intelligence – such as the HEARSAY approach to speech understanding [Erman and Lesser, 1980] – provides techniques for the formal treatment of inconsistencies, where hypotheses are given weighted values, and computation proceeds until some hypothesis receives a weight which is sufficiently well above threshold.

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