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EMPIRICAL LOGIC AND QUANTUM MECHANICS

I. INTRODUCTION

Our purpose in this article is to discuss some of the basic notions of quantum physics within the more general framework of operational statistics and empirical logic (as developed in Foulis and Randall, 1972, and Randall and Foulis, 1973). Empirical logic is a formal mathematical system in which the notion of an operation is primitive and undefined; all other concepts are rigorously defined in terms of such operations (which are presumed to correspond to actual physical procedures).

By a *physical operation*, we mean instructions that describe a well-defined, physically realizable, reproducible procedure and furthermore that specify what must be observed and what can be recorded as a consequence of an execution of this procedure. In particular, a physical operation must require that, as a consequence of each execution of the instructions, one and only one symbol from a specified set E be recorded as the *outcome* of this realization of the physical operation. We refer to the set E as the *outcome set* for the physical operation.

Notice that the outcome of a realization of a physical operation is merely a *symbol*; it is not any real or imagined occurrence in the 'physical world out there'. Also, observe that, if we delete or add details to the instructions for any physical operation, especially if we modify the outcome set in any way, we thereby define a new physical operation.

This definition appears to us to be the only tractable one, since the only means of settling the question of whether two individuals performed the 'same physical operation' is with a description. Since no description can be complete, then no two executions of a set of instructions can be identical in all particulars – this, of course, is a well-known source of the irregularities commonly experienced in experimental science. This point of view regarding physical operations must be adopted in some sense if experimental results are to be used to predict future events. As such, it is certainly implicit in any objective statistical analysis; here we simply propose to recognize it formally.

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Evidently, the subjective judgment of the observer is implicit in every realization of a physical operation, not only in regard to the interpretation of the instructions, but also in connection with the decision as to which symbol to record as the outcome. In our view, if a competent observer believes that he has executed a particular physical operation and obtained a certain outcome, then, in fact, the operation has been realized and the outcome in question has indeed been secured. Each realization of a physical operation is to be understood here as a 'Ding an sich', isolated, with no 'before' and no 'after'. Physical history, as it were, begins and ends with each execution of a physical operation. To put the matter in more traditional terms, the various realizations of the admissible physical operations are always to be regarded as 'independent trials'.

If physical operations are to be carried out in a 'connected sequence', then the instructions for such a *compound physical operation* must say so explicitly. When a compound physical operation is built up from more primitive physical operations by concatenating them in this manner, it is to be understood that each constituent physical operation thereby loses its identity, since it may now have temporal antecedents and consequences. In practice, the admissible physical operations will be built up by compounding appropriate primitive physical operations; a primitive physical operation is understood to be a physical operation which we cannot (or do not choose to) factor into connected sequences of more basic physical operations. (In the final analysis, the decision to regard a physical operation as being primitive must be largely subjective.)

Many of the physical operations of interest in quantum physics are compound physical operations consisting of a 'preparation operation' followed by one or more 'filtering operations' and terminated by a 'measurement operation'. In this connection, orthodox descriptions of the execution of such a compound physical operation involve the object-property idiom in that the preparation operation is construed as preparing physical objects in a certain state, the filtration operations as either passing or not passing such objects, and the measurement operations as detecting (or counting) such objects or as measuring physical parameters associated with such objects.

A physical operation having just one outcome will be called a *transformation*; the idea simply being that if such an operation has been executed, then there is just one possible outcome, namely, that the transformation in question has indeed been effected. Such a transformation might be an actual physical transformation in spacetime or it could require the adjustment of apparatus. Of special interest will be the purely temporal transformations whose instructions only require the elapse of a specified time interval.

II. MANUALS OF OPERATIONS

A well-conceived experimental program will often involve not one, but many different physical operations. The collection of all of these physical operations will be called a *physical manual*, since it could be construed as being a catalogue or manual of instructions. Often the various physical operations in such a manual will have overlapping outcome sets. Such overlaps could be produced by sheer accident on the part of the symbol chooser, but often they are intentional and reflect an attitude or belief that certain outcomes of one physical operation are 'physically equivalent' to certain outcomes of another physical operation.

It is easy to avoid accidental overlap of outcome sets corresponding to distinct physical operations in a physical manual – one simply chooses the symbols in such a way that it does not happen, that is, one sees to it that the outcome sets for the various physical operations in the manual are pairwise disjoint. If this is done, then each physical operation in the manual will correspond uniquely to its own outcome set and, therefore, can be mathematically represented by its own outcome set. Let us suppose that this has been done and that $\mathscr{D}$ is the resulting collection of disjoint outcome sets. Mathematically, $\mathscr{D}$ is just a nonempty set of pairwise disjoint nonempty sets. Each set E in $\mathscr{D}$ is the outcome set for a uniquely determined physical operation; hence, we shall refer to E as an *operation* and to $\mathscr{D}$ as a *manual* (dropping the adjective 'physical').

We are now in a position to consider just which pairs of outcomes of operations in the manual $\mathscr{D}$ we should construe as being 'physically equivalent'. Such considerations could be based on practically anything from a subjective whim to an elaborate scientific theory, but, often they are based on an appropriate 'world picture' or model. For instance, we often prefer to regard a number of outcomes of distinct physical operations as registering the same 'property' or as representing the same 'measurement'. If a voltage is measured using different instruments – or even different methods – identical numerical results are ordinarily taken to be equivalent. Frequently, the instructions for carrying out the required procedures for two different physical operations may all but dictate the indentification of certain outcomes on purely syntactic grounds.

We surely wish to avoid the necessity of taking a stand on the 'acceptability' of any such decisions concerning the 'physical equivalence' of outcomes, since we hope to keep our formalism as free as possible from ad hoc decisions. On the other hand, unrestricted identifications of outcomes of distinct physical operations seem to create rather chaotic formal systems which are mathematically intractable; hence, one is virtually forced to subject these identifications to certain mild constraints. Some of these constraints are plausible consequences of one's intuitive understanding of 'physical equivalence', while others are simply suggested by numerous concrete examples.

Thus, let us introduce a binary relation on the set Q of all outcomes of all of the operations in the manual $\mathscr{D}$ as follows: we say that outcome x is *equivalent* to outcome y and write $x \equiv y$ if we are prepared to regard these outcomes as being 'physically equivalent'. Naturally, we suppose that $\equiv$ is an equivalence relation on Q.

Suppose that $x, y \in Q$ are such that $x \neq y$ and there exists $E \in \mathscr{D}$ with $x, y \in E$. Then, we shall say that x *operationally rejects* y and write $x \perp y$, noting that whenever x is secured as a consequence of an execution of E, then y is certainly not secured as a consequence of this execution of E. Under these circumstances, it would manifestly not be appropriate to regard x and y as being physically equivalent; hence, we shall require that $x \perp y \Rightarrow x \not\equiv y$.

If $x \in Q$, we denote by $[x]$ the set of all outcomes $q \in Q$ such that $q \equiv x$. If $E \in \mathscr{D}$, we define a physical operation E' as follows: to execute E', execute the physical operation whose outcome set is E to obtain (say) the outcome x, then record the outcome of this execution of E' as $[x]$. If we define $[E]$ to be the set of all equivalence classes of the form $[x]$, as x runs through E, then the outcome set of the physical operation E' is $[E]$. Notice that whereas, given an outcome $x \in Q$, there will be exactly one physical operation in our original physical manual capable of yielding the outcome x, there may be many physical operations of the form E' capable of yielding the outcome $[x]$.

Given $E \in \mathscr{D}$, we now define a physical operation $[E]'$ as follows: to execute $[E]'$, select any $F \in \mathscr{D}$ such that $[F] \subseteq [E]$, execute F' to obtain (say) the outcome $[x]$, and record $[x]$ as the outcome of this execution of $[E]'$. Evidently, the outcome set for $[E]'$ is $[E]$, and the physical operation $[E]'$ depends only on the set $[E]$; that is, if $E, F \in \mathscr{D}$ with $[E] = [F]$, then $[E]' = [F]'$. (The latter holds because the instructions for $[E]'$ are word for word the same as those for $[F]'$; but a physical operation *is* the set of instructions that describe it.)

Suppose that $E, F \in \mathscr{D}$, that $[E] \subseteq [F]$, but that $[E] \neq [F]$. Then, there exists $x \in Q$ with $[x] \in [F]$ but $[x] \notin [E]$. Since $[x] \in [F]$, then $[x]$ is an outcome of the physical operation $[F]'$; however, the instructions for $[F]'$ permit $[F]'$ to be realized by executing E' and recording the outcome as the outcome of this realization of $[F]'$. Since $[x] \notin [E]$, then no such realization of $[F]'$ could possibly yield the outcome $[x]$, in spite of the fact that $[x]$ is one of the outcomes of $[F]'$. Although this state of affairs is not intolerable, it surely seems undesirable; hence, we shall require that if $E, F \in \mathscr{D}$, then $[E] \subseteq [F] \Rightarrow [E] = [F]$.

The above requirement is not as stringent as it might at first seem. Indeed, suppose for a moment that it fails, so that there exist $E, F \in \mathscr{D}$ with $[E] \subseteq [F]$, but $[E] \neq [F]$. Then, every outcome of $[E]'$ is an outcome of $[F]'$ and, in this sense $[E]'$ is a redundant physical operation since its outcomes are 'covered' by those of $[F]'$. Such a redundancy could be eliminated by deleting the physical operation corresponding to E from the original physical manual – that is, by deleting E from $\mathscr{D}$. In any practical situation, it is perhaps not unreasonable to suppose that after performing sufficiently many deletions of this nature, we could remove all redundancies and thus force the equivalence relation $\equiv$ to satisfy the desired condition.

A subset of the outcome set for a given physical operation will be called an *event* for this operation. Suppose that D is an event for a given physical operation. If the operation is executed to obtain a certain outcome, then we shall say that the event D *occurred* (as a consequence of this execution) precisely when the outcome in question belongs to the set D. If $x, y \in Q$ with $[x] \neq [y]$ and if $\{[x], [y]\}$ is an event for some physical operation $[E]'$, where $E \in \mathscr{D}$, then we shall say that the outcome $[x]$ *operationally rejects* the outcome $[y]$ and write $[x] \perp [y]$. Notice that if $[x] \perp [y]$ as above, then both $[x]$ and $[y]$ are possible outcomes of $[E]'$; however, an execution of $[E]'$ that yields the outcome $[x]$ will surely not yield the outcome $[y]$ and visa versa. If $x, y \in Q$, then it is clear that the relation $[x] \perp [y]$ will hold if and only if there exist $x_1, y_1 \in Q$ with $x \equiv x_1$, $y \equiv y_1$ and $x_1 \perp y_1$.

Now, suppose that D_1 is an event for the physical operation $[E_1]'$ and that D_2 is an event for the physical operation $[E_2]'$, where $E_1, E_2 \in \mathscr{D}$. If there exists $E_3 \in \mathscr{D}$ such that both D_1 and D_2 are events for $[E_3]'$, then we shall say, that D_1 and D_2 are *compatible* events. If D_1 and D_2 are compatible as above, then both D_1 and D_2 are eligible for occurrence or nonoccurrence as a consequence of a single execution of $[E_3]'$. Furthermore, if it happens that $D_1 \cap D_2$ is empty, then an occurrence of D_1 as a consequence of an execution of $[E_3]'$ will imply a nonoccurrence of D_2 as a consequence of this execution of $[E_3]'$ and vice versa. Thus, two compatible but disjoint events will be said to *operationally reject* each other.

Notice that if D_1 and D_2 are compatible but disjoint events as above, then $[x_1] \perp [x_2]$ will hold for every outcome $[x_1]$ in D_1 and every outcome $[x_2]$ in D_2. The latter condition is of some interest in its own right; thus, we shall say that D_1 and D_2 *reject* each other (deleting the adjective 'operationally') if $[x_1] \perp [x_2]$ holds for every outcome $[x_1]$ in D_1 and every outcome $[x_2]$ in D_2. Thus, to say that D_1 and D_2 reject each other is to say that each outcome favorable to the occurrence of D_1 operationally rejects each outcome favorable to the occurrence of D_2, while to say that D_1 operationally rejects D_2 is to say that such rejection is enforced by a single physical operation $[E_3]'$.

One can give simple examples of equivalence relations $\equiv$ satisfying all of the conditions imposed so far and events D_1, D_2 that reject each other, but that do not operationally reject each other. Under these circumstances, one feels intuitively that there is a 'missing physical operation' which, if adjoined to the original physical manual, would operationally enforce the mutual rejection of D_1 and D_2. To stipulate that there are no such 'missing physical operations' would be to stipulate that events which reject each other always do so operationally. The latter condition, which we have called *coherence* (Foulis and Randall, 1972) since it requires the existence of 'sufficiently many coherently related physical operations', is the final condition that we shall impose on our equivalence relation $\equiv$.

Notice that there is a one-to-one correspondence $[E]' \leftrightarrow [E]$ between the physical operations described above and their outcome sets; hence, the physical manual of all such physical operations can conveniently be represented mathematically by the collection $\mathscr{A}=\{[E] \mid E\in\mathscr{D}\}$. The conditions that have been imposed on the equivalence relation $\equiv$ confer upon $\mathscr{A}$ certain 'irredundancy' and 'coherence' properties which we shall now abstract.

A *premanual* is defined to be a nonempty set $\mathscr{A}$ of nonempty sets. An element $E\in\mathscr{A}$ is called an *$\mathscr{A}$-operation* and the set theoretic union $X=\bigcup\mathscr{A}$ is called the set of *$\mathscr{A}$-outcomes.* We shall call $\mathscr{A}$ an *irredundant* premanual provided that $E, F\in\mathscr{A}$ and $E\subseteq F$ implies that $E=F$. Two $\mathscr{A}$-outcomes $x, y\in X$ are said to be *orthogonal,* in symbols $x\perp y$, provided that $x\neq y$ and there exists $E\in\mathscr{A}$ with $x, y\in E$. A subset D of X is called an *orthogonal* set if $x\perp y$ holds for all $x, y\in D$ with $x\neq y$, while a subset D of X is called an *$\mathscr{A}$-event* if there exists an $E\in\mathscr{A}$ with $D\subseteq E$. If A and B are subsets of X, we say that A and B are *orthogonal* and we write $A\perp B$ provided that $a\perp b$ holds for all $a\in A$ and all $b\in B$. We call a premanual $\mathscr{A}$ *coherent* provided that the union of any two orthogonal $\mathscr{A}$-events is again an $\mathscr{A}$-event. A *manual* is defined to be an irredundant and coherent premanual.

If we were to confine our attention only to those manuals consisting of a single operation, we would, in effect, be adopting the Kolmogorov viewpoint as expressed in connection with the foundations of modern probability theory in Kolmogorov (1933). In a sense, this is also the view adopted in classical (statistical) mechanics where it is implicitly assumed that there exists a single 'grand canonical operation' that measures the position and momentum of every particle of a physical system. For this reason, we shall refer to a manual consisting of a single operation as a *classical* manual.

A maual in which the operations are pairwise disjoint might be regarded as a 'free union' of classical manuals; hence, such a manual will be referred to as a *semiclassical* manual. Notice that the manual $\mathscr{D}$ considered above is such a manual; consequently, we could regard a semiclassical manual as representing a physical manual of basic operations free of any outcome identification. It is perhaps worth pointing out that any manual $\mathscr{A}$ can be obtained (up to isomorphism) from a suitable semiclassical manual $\mathscr{D}$ by 'factoring out' an appropriate equivalence relation $\equiv$ as above.

III. THE COMPOUNDING OF MANUALS

In this part, let $\mathscr{A}$ be a manual with outcome set $X=\bigcup\mathscr{A}$. We shall refer to $\mathscr{A}$ as the *base manual*, and we shall regard $\mathscr{A}$ as a reservoir of primitive physical operations from which we intend to synthesize compound operations requiring the execution of the primitive operations in 'connected sequences'. Suppose, for instance, that $E_1, E_2, \ldots, E_n$ are $\mathscr{A}$-operations and that these are executed in a connected sequence (first E_1, then $E_2, \ldots$, and finally E_n) so as to obtain the sequence $x_1, x_2, \ldots, x_n$ of respective outcomes. Let us agree to record the formal product $x_1x_2\ldots x_n$ to denote the acquisition of such a sequence as a consequence of the execution of the compound operation just described. In order to be able to give such a representation to all of the outcomes of all of the possible compound operations that could be synthesized from $\mathscr{A}$, we are obliged to consider the free semigroup S over X.

The free semigroup S over X consists of all formal products $x_1x_2\ldots x_n$ with $x_1, x_2, \ldots, x_n \in X$, n running through the positive integers. The product in S of the 'word' $a=x_1x_2\ldots x_n$ and the 'word' $b=y_1y_2\ldots y_m$ is, of course, the 'word' $ab=x_1x_2\ldots x_ny_1y_2\ldots y_m$. In the following, it will be convenient to adjoin a formal identity 1 to the semigroup S so as to obtain a semigroup $X^c=S\cup\{1\}$ with identity 1, which we shall refer to as the *free monoid* over X. If $b\in X^c$ with $b\neq 1$, then b is uniquely expressible in the form $b=x_1x_2\ldots x_n$ with $x_1, x_2, \ldots, x_n\in X$; we define the *length* of the word b to be $|b|=n$. By convention, we define $|1|=0$. The elements of X^c of length one are naturally identified with the corresponding elements of X, so that $X\subseteq X^c$.

A subset A of X^c is said to be *bounded* if there is a nonnegative integer n such that $|a|\leqslant n$ holds for all $a\in A$. If A is nonempty and bounded, we define $|A|$ to be the minimum of all such nonnegative integers n, and we define $|\emptyset|=-1$. If $A, B\subseteq X^c$, we naturally define the *product* AB to be the set of all elements of X^c of the form ab with $a\in A$ and $b\in B$. If $a\in X^c$ and $B\subseteq X^c$, we define $aB=\{a\}B$.

In the following, $\{1\}$ will be regarded as representing a trivial physical operation requiring that we do nothing (other than to record the symbol 1 as the outcome). Thus, $\{1\}$ denotes the identity transformation. Of course, each basic operation $E\in\mathscr{A}$ can be regarded as a one-stage compound operation. A two-stage compound operation is formed as follows: first, select a basic operation $E\in\mathscr{A}$, an $\mathscr{A}$-event $D\neq\emptyset$ with $D\subseteq E$, and a basic operation $F_d\in\mathscr{A}$ for each outcome $d\in D$. The two-stage compound operation in question – let us call it G – is executed by first executing E to obtain (say) the outcome e; if $e\notin D$, we are done and we record the outcome of this execution of G as e, but if $e\in D$, we are obliged to execute F_e to obtain (say) the outcome $x\in F_e$ and to record the outcome of this execution of G as $ex\in X^c$.

Evidently, the outcome set for G is $(E\backslash D)\cup(\bigcup_{d\in D} dF_d)$. If we set $F_e=\{1\}$ for each $e\in E\backslash D$, then the outcome set for G is simply the set $\bigcup_{e\in E} eF_e$. Multistage compound operations can now be built up by iteration of the above procedure inductively. Below, we shall do this formally, but, informally, it should be clear just what we have in mind. Suppose that the set of all outcomes for such a compound operation G is the set $A\subseteq X^c$. We claim that the instructions for G (and hence, G itself) can be recaptured from the set A. Indeed, the set of all first letters of all of the words in A comprise the outcome set for the initial operation involved in an execution of G. Suppose that this initial operation is represented by the outcome set E and that $e\in E$. Then, the set of all second letters of those words in A that initiate with e comprise the outcome set of the second basic operation that must be executed in case the outcome of the initial execution of E is e. Proceeding inductively in this manner, we ultimately recapture the instructions for G. For this reason, we shall identify a compound operation, as described above, with its own outcome set.

We now formalize our construction of compound operations based on $\mathscr{A}$ as follows: If E, $G\subseteq X^c$ with $E\neq G$, and if there exists for each $e\in E$ a set F_e such that either $F_e\in\mathscr{A}$ or else $F_e=\{1\}$, and if $G=\bigcup_{e\in E} eF_e$, then we shall call G a *direct successor* of E. If there exists a finite sequence G_1, $G_2,\ldots, G_n$ of subsets of X^c such that G_{i+1} is a direct successor of G_i for $i=1, 2,\ldots, n-1$, then we shall say that G_n is a *successor* of G_1. We define $\mathscr{A}^c$ to be the collection of subsets of X^c consisting of $\{1\}$ and all successors of $\{1\}$.

It should be clear that if E is a nonempty bounded subset of X^c and G is a direct successor of E, then G is bounded and $|G|=|E|+1$. It follows that every $G\in\mathscr{A}^c$ is a nonempty bounded subset of X^c. A set $G\in\mathscr{A}^c$ will be called a *compound operation* over the base manual $\mathscr{A}$. Notice that – as promised – we are here identifying the compound operations with their own outcome sets. Thus, $\mathscr{A}^c$ can be regarded as the manual of all compound operations, and it becomes natural to inquire whether $\mathscr{A}^c$ is, in fact, a manual according to our official definition.

Notice that it is quite possible for the various outcome sets in $\mathscr{A}^c$ to overlap; the real question here is whether we actually intend for such overlaps to occur. This is precisely our intention and we shall now indicate why we think that it is reasonable. Suppose, for example, that $E_1 = \{x, x'\}$, $E_2 = \{y, y'\}$, and $E_3 = \{z, z'\}$ are three basic operations in the manual $\mathscr{A}$. Then, notice that $G = \{xy, xy', x'\}$ and $H = \{xy, xy', x'z, x'z'\}$ are compound operations which share the common outcome xy. If one executes G and obtains the outcome xy as a consequence, then one has executed E_1 to get the outcome x and then has executed E_2 to get the outcome y. This is exactly what one would have done if one had executed H and obtained the outcome xy. Notice that if we had executed H and obtained the outcome x' as a consequence of the initial execution of E_1, then we would have been obliged to execute E_3 as the second step of this execution of H. However, if we had executed G rather than H and obtained the outcome x' as a consequence of the initial execution of E_1, then this execution of G would terminate. Our decision to identify the outcome xy of G with the outcome xy of H amounts to the assumption that 'nature' is oblivious to our intentions and responsive only to our actions. In the end, this is the justification for the outcome identifications implicit in $\mathscr{A}^c$.

Now, it is not difficult to show that $\mathscr{A}^c$ is a manual, provided only that $\mathscr{A}$ is a manual. The orthogonality relation $\perp$ on X^c is a lexicographic extension of the orthogonality relation $\perp$ on X in the following sense: for $a, b \in \bigcup \mathscr{A}^c = X^c$, we have $a \perp b$ if and only if there exist $c, d, e \in X^c$ and there exist $x, y \in X$ with $a = cxd$, $b = cye$ and $x \perp y$.

IV. SYMMETRIES AND GROUPS

By an *operational symmetry* of the manual $\mathscr{A}$, we mean a bijective map ϕ from the set X of all $\mathscr{A}$-outcomes onto itself such that for $E \subseteq X$, $\phi(E) \in \mathscr{A}$ if and only if $E \in \mathscr{A}$. If ϕ is such an operational symmetry and $D \subseteq X$, then D is an $\mathscr{A}$-event if and only if $\phi(D)$ is an $\mathscr{A}$-event; hence, in particular, for $x, y \in X$, $x \perp y$ holds if and only if $\phi(x) \perp \phi(y)$ holds. For instance, if $\mathscr{A}$ is a set of 'measurement operations' that can be carried out on suitable 'physical systems' and if g represents an invertible 'physical transformation' on such systems, than g might be expected to induce an operational symmetry ϕ_g of $\mathscr{A}$. Here, the operation $\phi_g(E)$ would be understood to be the operation whose execution requires a preliminary transformation of the physical system by g followed by an execution of E.

We have already defined a transformation to be an operation with a single outcome; hence, a manual of transformations is by definition a manual $\mathscr{J}$ all of whose operations are of the form $\{g\}$ as g runs through the set G of all $\mathscr{J}$-outcomes. In practice, G will often form a group, where the composition gh of two elements of G is understood to be the transformation resulting from a preliminary execution of g followed by an execution of h. Under these circumstances, we shall say that the group G *acts on* the manual $\mathscr{A}$ provided that each $g \in G$ corresponds to a symmetry ϕ_g of $\mathscr{A}$ in such a way that $\phi_{gh}(x) = \phi_g(\phi_h(x))$ and $\phi_1(x) = x$ for all $g, h \in G$ and all $x \in X = \bigcup \mathscr{A}$. (Here, '1' denotes the unit element of the group G.) If G acts on $\mathscr{A}$, and no ambiguity threatens, we shall simply write gx rather than $\phi_g(x)$ for $x \in X$ and $g \in G$.

Notice that if G acts on $\mathscr{A}$, then it is quite possible to have $gx = hy$ with $g \neq h$, $x \neq y$ for $g, h \in G$ and $x, y \in X$. Such an equality would express the 'physical equivalence' of outcome x after a preliminary transformation by g with outcome y after a preliminary transformaion by h.

Suppose now that $\mathscr{J}$ is a manual of transformations whose outcome set $G = \bigcup \mathscr{J}$ forms a group as above. If $\mathscr{A}$ is a given manual, we can construct a 'free action' of G on $\mathscr{A}$ as follows: for $g \in G$ and $E \in \mathscr{A}$, we define $\{g\} \times E$ to be the operation whose execution requires a preliminary execution of the transformation g followed (in connected sequence) by an execution of the operation E to obtain (say) the outcome $e \in E$. We then are to record the outcome of this execution of $\{g\} \times E$ as the ordered pair (g, x). Notice that the outcome set for the two-stage compound operation $\{g\} \times E$ is the set of all ordered pairs of the form (g, e), e running through E; that is, the outcome set for $\{g\} \times E$ is $\{g\} \times E$ – again, we are identifying an operation with its outcome set. The collection of all $\{g\} \times E$ as g runs through G and E runs through $\mathscr{A}$ will be denoted by $G\mathscr{A}$. It is easy to verify that $G\mathscr{A}$ is a manual and that, if $\mathscr{A}$ is semiclassical, so is $G\mathscr{A}$. Clearly, there are no outcome identifications in the manual $G\mathscr{A}$ other than those that were already implicit in the manual $\mathscr{A}$, and this is the sense in which G has been made to act 'freely' on $\mathscr{A}$.

Notice that the group G acts on the manual $G\mathscr{A}$ in the following natural manner: $\phi_h((g, x))=(hg, x)$ for $g, h \in G$ and $x \in X = \bigcup \mathscr{A}$. For simplicity, we shall write $h(g, x)$ rather than $\phi_h((g, x))$, so that $h(g, x) =(hg, x)$ for $g, h \in G$, $x \in X$. As before, we may be in possession of a suitable theory, model, or 'picture of the world' that stipulates certain distinct outcomes of the manual $G\mathscr{A}$ to be 'physically equivalent', and we may wish to factor this information into the manual $G\mathscr{A}$ by dividing out a suitable equivalence relation. We now turn our attention to a particularly important instance of such a situation.

Suppose that $\mathscr{I}$ is a manual of transformations whose outcome set G forms a group, that H is a subgroup of G, and that H acts in a known way on the manual $\mathscr{A}$. Here, the action of H on $\mathscr{A}$ is not necessarily 'free'; that is, there may be nontrivial outcome identifications implicit in this action. We can, of course, allow the larger group G to 'act freely' on $\mathscr{A}$ by forming the manual $G\mathscr{A}$ as above. Suppose that $g \in G$, $h \in H$ and $E \in \mathscr{A}$. Notice that the instructions for executing $\{gh\} \times E$ are word for word the same as the instructions for executing $\{g\} \times hE$ except for the portions of these instructions pertaining to the form in which the outcomes are to be recorded. In the first case, we would record the outcome as (gh, x) and in the second case, we would record it as (g, hx), where $x \in E$. Here, we shall clearly wish to regard the outcome (gh, x) as being 'physically equivalent' to the outcome (g, hx).

With the above motivation, we define the relation $\equiv$ on the set $G \times X$ of all $G\mathscr{A}$-outcomes as follows: for $g_1, g_2 \in G$ and $x_1, x_2 \in X = \bigcup \mathscr{A}$, the relation $(g_1, x_1) \equiv (g_2, x_2)$ will hold if and only if there exists $h \in H$ with $g_1 = g_2 h$ and $x_2 = hx_1$. One verifies without difficulty that $\equiv$ is a bona fide equivalence relation on $G \times X$ and that if $(g_1, x_1) \perp (g_2, x_2)$, then $(g_1, x_1) \not\equiv (g_2, x_2)$. As usual, we define $[g, x]$, for $(g, x) \in G \times X$, to be the equivalence class in $G \times X$ consisting of all (g', x') with $(g, x) \equiv (g', x')$.

If $g \in G$ and $E \in \mathscr{A}$, we define the physical operation $[g, E]$ as follows: to execute $[g, E]$, we execute $\{g\} \times E$ to obtain (say) the outcome (g, x), but we record the outcome of this execution of $[g, E]$ as $[g, x]$. Thus, the outcome set for $[g, E]$ is the set of all equivalence classes of the form $[g, x]$ as x runs through the set of all outcomes in E. As before, we propose to identify the operation $[g, E]$ with its own outcome set, so that $[g, E] = \{[g, x] \mid x \in E\}$. It is not difficult to show that the collection $\{[g, E] \mid g \in G, E \in \mathscr{A}\}$ is again a manual, and we shall denote this manual by $G\mathscr{A}/H$. Notice that the set $\{[g, x] \mid g \in G, x \in X = \bigcup \mathscr{A}\}$ is the set of all $G\mathscr{A}/H$-

outcomes and that there is a natural action of the group G on this manual given by $g'([g, x]) = [g'g, x]$ for $g', g \in G$ and $x \in X$. One easily checks that this action is consistent with our general understanding of an action of a group of transformations on a manual. We shall refer to this action of G on $G\mathscr{A}/H$ as the *expanded action* corresponding to the original action of H on $\mathscr{A}$.

Let $\mathscr{B} = G\mathscr{A}/H$, $Z = \bigcup \mathscr{B}$ and, for every $g \in G$, put $\mathscr{B}_g = \{[g, E] \mid E \in \mathscr{A}\}$, $Z_g = \bigcup \mathscr{B}_g = \{[g, x] \mid x \in X\}$. Notice that $\mathscr{B}_g$ is itself a manual and that the original manual $\mathscr{A}$ is isomorphic to $\mathscr{B}_g$ under the correspondence taking the $\mathscr{A}$-outcome x into the $\mathscr{B}_g$-outcome $[g, x]$. Thus, $\mathscr{B}_g$ is a *submanual* of $\mathscr{B}$ in the sense that $\mathscr{B}_g \subseteq \mathscr{B}$ and $\mathscr{B}_g$ is a manual in its own right. Furthermore, $\mathscr{B}_g$ is a so-called *induced* submanual of $\mathscr{B}$ (Randall and Foulis, 1973) in the sense that two $\mathscr{B}_g$ outcomes are orthogonal with respect to $\mathscr{B}_g$ if and only if they are orthogonal with respect to $\mathscr{B}$. Evidently, for $a, b \in G$, $\mathscr{B}_a = \mathscr{B}_b$ if and only if $aH = bH$. Thus, if we denote by G/H the space of all left cosets of G modulo H, then for $A \in G/H$, we can unambigously define $\mathscr{B}_A = \mathscr{B}_a$ and $Z_A = Z_a$, where a is any element in the coset A. In this way, Z becomes decomposed into the mutually exclusive and exhaustive sets Z_A as A runs through the left coset space G/H and $\mathscr{B}$ becomes a disjoint union of the induced submanuals $\mathscr{B}_A$ as A runs through G/H. Furthermore, for $A \in G/H$ and $g \in G$, $g(Z_A) = Z_{gA}$ and $g(\mathscr{B}_A) = \mathscr{B}_{gA}$. Finally, the original action of H on $\mathscr{A}$ is evidently equivalent to the action of H on $\mathscr{B}_H$. Notice that it is only this action of H on $\mathscr{B}_H$ that can involve any physically significant outcome identifications – the 'rest' of the action of G on $\mathscr{B}$ is 'free'.

Suppose now that $\mathscr{B}$ is an arbitrary manual and that G is a group of transformations acting on $\mathscr{B}$. Part of this action may be 'free', while part of it may be 'physically significant', and we shall now turn our attention to the problem of extracting this physically significant portion. To this end, we invert the above argument. Borrowing some terminology from the classical theory of permutation groups (Burnside, 1897), we define an *operational system of imprimitivity* for the action of G on $\mathscr{B}$ to be a decomposition $Z = \bigcup_{i \in I} Z_i$ of the set $Z = \bigcup \mathscr{B}$ of all $\mathscr{B}$-outcomes by the family $\{Z_i \mid i \in I\}$ of pairwise disjoint nonempty sets such that

(i) For each $i \in I$, $\mathscr{B}_i = \{F \in \mathscr{B} \mid F \subseteq Z_i\}$ is a nonempty set and is an induced submanual of the manual $\mathscr{B}$.

(ii) The group G acts transitively on the set I in such a way that $g(Z_i)=Z_{gi}$ for all $g\in G$ and all $i\in I$.

Suppose that we have such an operational system of imprimitivity. Choose and fix one of the indices $k\in I$. Let H be the subgroup of G consisting of all elements $h\in G$ such that $hk=k$; that is, H is the isotropy group at k of the action of G on I. Let $\mathscr{A}=\mathscr{B}_k$, so that $\mathscr{A}$ is an induced submanual of $\mathscr{B}$ consisting of all of the $\mathscr{B}$-operations F such that $F\subseteq Z_k$. Evidently, we get an action of H on $\mathscr{A}$ simply by restricting the original action of G on $\mathscr{B}$ to $H\times Z_k$. As before, this action of H on $\mathscr{A}$ gives rise to an expanded action of G on $G\mathscr{A}/H$. Denote by Z' the set of all $G\mathscr{A}/H$-outcomes, so that $Z'=\{[g, z]\mid g\in G, z\in Z_k\}$. The mapping $[g, z]\rightsquigarrow gz$ provides a bijection of Z' onto Z making the manual $G\mathscr{A}/H$ isomorphic to the manual $\mathscr{B}$ in an obvious sense. Furthermore, under this isomorphism, the original action of G on $\mathscr{B}$ is equivalent to the action of G on $G\mathscr{A}/H$.

The above discussion shows that the operational systems of imprimitivity for an action of G on $\mathscr{B}$ correspond in a one-to-one fashion to the ways in which this action of G on $\mathscr{B}$ can be regarded as an expansion of an action of some subgroup H of G on some induced submanual $\mathscr{A}$ *of* $\mathscr{B}$.

In view of the above, we propose to define an action of a group G on a manual $\mathscr{B}$ to be operationally primitive provided that there are no nontrivial operational systems of imprimitivity for this action. (An operational system of imprimitivity is, of course, *trivial* if the indexing set I contains only one element.) We are inclined to regard an operationally primitive action of G on $\mathscr{B}$ to be an action involving no 'free part' and to regard the outcome identifications implicit in such an operationally primitive action to be 'physically significant'.

V. THE LOGIC OF A MANUAL

Let $\mathscr{A}$ be a manual and let X be the set of all $\mathscr{A}$-outcomes. For purposes of motivation, we shall regard $\mathscr{A}$ as the collection of all outcome sets for the physical operations in some physical manual; furthermore, we shall suppose that there is a one-to-one correspondence between these physical operations and their outcome sets. Thus, if $E\in\mathscr{A}$, we shall (by abuse of language) speak of an execution of E, when what we really mean is an execution of the physical operation whose outcome set is E.

Let us consider, for the time being, only those propositions that are operationally well defined in the sense that they are confirmed or refuted strictly in terms of evidence acquired as a consequence of the execution of $\mathscr{A}$-operations. Specifically, we define an *operational proposition* (for $\mathscr{A}$) to be an ordered pair (A, B) of subsets $A, B \subseteq X = \bigcup \mathscr{A}$. If an operation $E \in \mathscr{A}$ is executed and the outcome $e \in E$ is obtained as a consequence, we shall say that the operational proposition (A, B) is *confirmed* (by this execution of E) precisely when $e \in A$ and that it is *refuted* (by this execution of E) precisely when $e \in B$. Thus, A will be called the *confirmation set* and B the *refutation set* for the operational proposition (A, B). Since there is no requirement that $A \cup B = X$, the operational proposition (A, B) can fail to be either confirmed or refuted by an execution of E – that is, the 'truth value' assigned to (A, B) as a consequence of an execution of $E \in \mathscr{A}$ can be 'indeterminate'. If $A \cap B = \emptyset$, that is, if the operational proposition (A, B) can never be simultaneously confirmed and refuted by a single execution of an operation $E \in \mathscr{A}$, then we shall say that (A, B) is a *self-consistent* operational proposition. In the sequel, we shall consider only self-consistent operational propositions.

Notice that an operational proposition is completely described by specifying its confirmation and refutation sets and does not involve a subject-predicate (object-property) idiom in any essential way. Also, a self-consistent operational proposition is 'sharp' in the sense that it will definitely be confirmed, refuted or left indeterminate by a single execution of a given $\mathscr{A}$-operation. Although such a proposition admits three 'truth values' – confirmed, refuted, indeterminate – its 'truth values' will generally be instable in the sense that they will change from one realization of an $\mathscr{A}$-operation to another.

If (A, B) is an operational proposition, we define the *negation* of (A, B), in symbols, $(A, B)'$ to be the operational proposition (B, A). Thus, (A, B) is confirmed precisely when $(A, B)'$ is refuted and vice versa. If (C, D) is a second operational proposition, we shall say that (A, B) *implies* (C, D) and write $(A, B) \leqslant (C, D)$ provided that $A \subseteq C$ and $D \subseteq B$. Thus, if $(A, B) \leqslant (C, D)$ and if (A, B) is confirmed (by an execution of an $E \in \mathscr{A}$), then (C, D) will be confirmed (by this execution of E), while, if (C, D) is refuted (by an execution of $E \in \mathscr{A}$), then (A, B) will be refuted (by this execution of E). Observe that $(A, B) \leqslant (C, D)$ is a mathematical assertion about the two operational propositions (A, B) and (C, D) – it is not itself an operational proposition. We define the operational proposition 0 by $0=(\emptyset, X)$ and we define $1=0'=(X, \emptyset)$, noting that $0 \leqslant (A, B) \leqslant 1$ holds for all operational propositions (A, B). Notice that, if $(A, B) \leqslant (C, D)$, then $(C, D)' \leqslant (A, B)'$.

None of the above considerations involve the detailed structure of the manual $\mathscr{A}$ in any essential way, since they involve only the outcome set X. We now turn our attention to the manner in which the detailed manual structure might interact with operational propositions. An operational proposition (A, B) is said to be *testable* if there exists $E \in \mathscr{A}$, called a *test operation* for (A, B), such that $E \subseteq A \cup B$. Notice that, if E tests (A, B) as above, then each outcome of E will either confirm or refute (A, B); hence, an execution of E cannot leave (A, B) indeterminate. A collection of operational propositions is said to be *simultaneously testable* provided that there exists a single $\mathscr{A}$-operation that is a test operation for every operational proposition in the collection.

If A is any subset of X, we define $A^{\perp}=\{x \in X \mid x \perp a, \forall a \in A\}$, and we define $A^{\perp\perp}=(A^{\perp})^{\perp}$, $A^{\perp\perp\perp}=(A^{\perp\perp})^{\perp}$, etc. Evidently, $A \subseteq A^{\perp\perp}$, $A^{\perp\perp\perp}=A^{\perp}$, $A \cap A^{\perp}=\emptyset$ and if $A \subseteq B \subseteq X$, then $B^{\perp} \subseteq A^{\perp}$. If $A, B \subseteq X$ with $A \subseteq B^{\perp}$, then we say that A and B are *mutually orthogonal* and we write $A \perp B$. An operational proposition (A, B) for which $A \perp B$ is said to be *orthoconsistent*. Note that (A, B) is orthoconsistent if and only if every outcome which could confirm (A, B) operationally rejects every outcome which could refute (A, B) and vice versa. An orthoconsistent operational proposition is automatically self-consistent. Also, if (A, B) is orthoconsistent, so is its negation $(A, B)'=(B, A)$.

We define the *orthonegation* of the operational proposition (A, B), in symbols $(A, B)^{\perp}$, by $(A, B)^{\perp}=(A^{\perp}, B^{\perp})$. Notice that $(A, B)^{\perp}$ is confirmed precisely by those outcomes that operationally reject every outcome that could confirm (A, B) and that $(A, B)^{\perp}$ is refuted precisely by those outcomes that reject every outcome that could refute (A, B). We shall say that (A, B) is a *closed* operational proposition provided that its orthonegation coincides with its negation, that is, $(A, B)^{\perp}=(A, B)'$. Thus, (A, B) is closed if and only if $A^{\perp}=B$ and $B^{\perp}=A$. In particular, if (A, B) is closed, then (A, B) is orthoconsistent, $B^{\perp} \subseteq A$ and $A^{\perp} \subseteq B$. Notice, for instance, that the condition $B^{\perp} \subseteq A$ means that every outcome that rejects all of the outcomes that could refute (A, B) must actually confirm (A, B).

It is natural to associate with every $\mathscr{A}$-event D the operational proposition $(D, D^{\perp})$, since its test operations are precisely those $\mathscr{A}$-operations E

for which $D \subseteq E$. Furthermore, if such a test operation E for $(D, D^{\perp})$ is executed, then $(D, D^{\perp})$ is confirmed precisely when D occurs and it is refuted precisely when D does not occur. In general, we shall refer to a confirmation (respectively, refutation) of an operational proposition by one of its test operations as a *test-confirmation* (respectively, a *test-refutation*). In general there may be operations $E \in \mathscr{A}$ with $D \nsubseteq E$ whose execution could, nevertheless, confirm or refute $(D, D^{\perp})$ – it is the test-confirmations of $(D, D^{\perp})$ that correspond to the occurrences of the event D. Also, although $(D, D^{\perp})$ is orthoconsistent, it need not be closed. However, there is a unique closed operational proposition $p(D)$ which is confirmed by those outcomes which confirm $(D, D^{\perp})$ and refuted by those outcomes which refute $(D, D^{\perp})$, namely, $p(D) = (D^{\perp\perp}, D^{\perp})$. In this article, we shall concern ourselves only with operational propositions of the form $p(D) = (D^{\perp\perp}, D^{\perp})$ for some $\mathscr{A}$-event D and we shall define $\Pi(\mathscr{A})$ to be the set of all such operational propositions.

Notice that $0 = p(\emptyset) \in \Pi(\mathscr{A})$ and that, if E is any $\mathscr{A}$-operation, $1 = p(E) \in \Pi(\mathscr{A})$. If D_1 and D_2 are $\mathscr{A}$-events, it is easy to see that $p(D_1) \leqslant p(D_2)$ if and only if $(D_1)^{\perp\perp} \subseteq (D_2)^{\perp\perp}$, that is, if and only if $(D_2)^{\perp} \subseteq (D_1)^{\perp}$. Also, $p(D_1) \leqslant (p(D_2))' = (p(D_2))^{\perp}$ if and only if $D_1 \perp D_2$. (Caution: $(p(D_2))'$ need not belong to $\Pi(\mathscr{A})$.) If $D_1 \perp D_2$, we shall say that the operational propositions $p(D_1)$ and $p(D_2)$ are *orthogonal* to each other, and we shall write $p(D_1) \perp p(D_2)$.

The system consisting of $\Pi(\mathscr{A})$, partially ordered by the relation $\leqslant$ and carrying the relation $\perp$ of orthogonality will be called the *operational logic* (or, just the *logic*) of the manual $\mathscr{A}$. Its general properties are easily abstracted and this leads to the notion of an *orthologic* (Jeffcott, 1972). An *orthologic*, by definition, is a system $(L, \leqslant, \perp)$ consisting of a nonempty set L carrying two binary relations $\leqslant$ and $\perp$ subject to certain conditions. One thinks of the elements $p \in L$ as being 'propositions' in some sense or the other. If $p, q \in L$ with $p \leqslant q$, one interprets this as meaning that, in some sense or other, 'p implies q'. Similarly, $p \perp q$ is interpreted as meaning that 'p and q reject each other'. The conditions to which these relations are subjected are as follows:

(1) L is partially ordered by $\leqslant$ and there is a unique smallest element 0 in L and a unique largest element 1 in L, so that $0 \leqslant p \leqslant 1$ holds for every $p \in L$.

(2) The relation $\perp$ is symmetric on L and, if $p \perp p$, then $p=0$.
(3) If $p, q \in L$ with $p \perp q$, then there exists a unique element $p \oplus q \in L$ which is the least upper bound in L of p and q.
(4) If $p, q, r \in L$ and if $p \perp q, p \perp r$ and $q \perp r$ hold, then $r \perp (p \oplus q)$.
(5) If $p \in L$, there exists at least one $q \in L$ with $p \perp q$ and $p \oplus q = 1$.
(6) For $p, q \in L$, $p \leqslant q$ holds if and only if for every $r \in L$, $r \perp q$ implies that $r \perp p$.

The element $p \oplus q$ of condition (3) is, of course, interpreted as the 'disjunction', in some sense, of the propositions p and q. The element q in condition (5) is, of course, viewed as a 'complement', in some sense, of the element p.

Not only is the logic $\Pi(\mathscr{A})$ of a manual $\mathscr{A}$ an orthologic, but, conversely, every orthologic is isomorphic to the logic of some manual. Indeed, suppose that L is an orthologic whose elements (for purposes of motivation) will be thought of as being 'propositions'. Let $e_1, e_2, \ldots, e_n$ be a finite set of pairwise orthogonal nonzero elements of L such that $e_1 \oplus e_2 \oplus \cdots \oplus e_n = 1$. Such a set might be regarded as a finite exhaustive collection of propositions which are mutually exclusive in some observable sense; namely, there is a physical operation which when executed will single out one and only one of these propositions as being 'confirmed' – at least for this particular realization of the operation. Thus $\{e_1, e_2, \ldots, e_n\}$ will be the outcome set for this operation. We might expect that the collection $\mathscr{A}$ of all such finite 'partitions of the logical identity' would form a manual and, in fact, it is easy to show that it does. Moreover, $\Pi(\mathscr{A})$ is canonically isomorphic to L.

In orthodox quantum mechanics, according to von Neumann (1932), the logic is $\mathbb{P}$, the lattice of projections on a separable, complex, infinite dimensional Hilbert space $\mathscr{H}$. It is well known that $\mathbb{P}$ is a complete orthomodular lattice (Foulis, 1962) and it is clear that every orthomodular lattice is an orthologic. Furthermore, all of the proposed generalizations of $\mathbb{P}$ to so-called quantum logics have been (at least) orthomodular posets (Foulis, 1962) – and these are also orthologics. As a matter of fact, an orthologic L is an orthomodular poset if and only if orthogonal complements in L are unique; that is, if $p, q, r \in L$ with $p \oplus q = p \oplus r = 1$, then $q = r$ (Jeffcott, 1972).

It is interesting to note that the logic $\Pi(\mathscr{A})$ of a manual $\mathscr{A}$ is an orthomodular poset if and only if the manual $\mathscr{A}$ satisfies the following condition: $E\in\mathscr{A}$, $x, y\in X=\bigcup\mathscr{A}$, $E\subseteq x^{\perp}\cup y^{\perp}\Rightarrow x\perp y$ (Dacey, 1968). We call such a manual $\mathscr{A}$ a *Dacey manual* (Foulis and Randall, 1972). If $\mathscr{A}$ is a Dacey manual, then the compound manual $\mathscr{A}^c$ is also Dacey. Obviously, any semiclassical manual is Dacey. Furthermore, if the group H acts on a Dacey manual $\mathscr{A}$ and if H is a subgroup of G, then $G\mathscr{A}/H$ is again a Dacey manual.

In orthodox quantum mechanics, if $E, F\in\mathbb{P}$ with $EF-FE=0$, then E is said to *commute* with F. It is generally understood that such commuting propositions are simultaneously verifiable in some sense. Here it can be shown that E commutes with F if and only if there exist $E_1, F_1, G\in\mathbb{P}$ such that E_1, F_1 and G are mutually orthogonal and $E=E_1\oplus G$, $F=F_1\oplus G$ (Foulis, 1962). Thus, if $L$is any orthologic, and $e, f\in L$, we shall say that e *commutes* with f, and write $e\mathbb{C}f$, if and only if there exist $e_1, f_1, g\in L$such that e_1, f_1 and g are mutually orthogonal and $e=e_1\oplus g, f=f_1\oplus g$.

If $\mathscr{A}$ is a manual and if A, B are $\mathscr{A}$-events, then $p(A)$ commutes with $p(B)$ in the orthologic $\Pi(\mathscr{A})$ if and only if there exist $\mathscr{A}$-events A_1 and B_1 such that $p(A)=p(A_1)$, $p(B)=p(B_1)$ and A_1 is compatible with B_1 in the sense that there exists an operation $E\in\mathscr{A}$ with $A_1\cup B_1\subseteq E$. Consequently, two commuting operational propositions in $\Pi(\mathscr{A})$ are simultaneously testable. In general, the converse is false; however, if $\mathscr{A}$ is a Dacey manual, then two operational propositions in $\Pi(\mathscr{A})$ will commute if and only if they are simultaneously testable (Randall and Foulis, 1973).

We define the *center* of an orthologic L, in symbols $\mathbb{C}(L)$, to be the 'subset of L consisting of all elements that commute with every other element. Notice that $0, 1\in\mathbb{C}(L)$. Jeffcott (1972) has shown that $\mathbb{C}(L)$ is always a Boolean algebra. Of course, every Boolean algebra is an orthologic. A manual $\mathscr{A}$ is said to be Boolean provided that $\Pi(\mathscr{A})$ coincides with its own center. Thus, $\mathscr{A}$ is Boolean if and only if it is Dacey and any two propositions in $\Pi(\mathscr{A})$ are simultaneously testable.

The operational interpretation of the infimum and supremum of propositions in quantum logics is a difficult matter that has engaged the attention of many authors (Birkhoff, 1961, pp. 155–184; Jauch, 1968; MacLaren, 1965). In the logic $\Pi(\mathscr{A})$ of a manual $\mathscr{A}$, such an interpretation is always available for the infimum and, when the manual is Dacey, the supremum also admits an operational interpretation.

Indeed, let $\{D_j\mid j\in J\}$ be a family of events for the manual $\mathscr{A}$. If A and B

are $\mathscr{A}$-events, the notation $p(A)=\bigwedge_j p(D_j)$ (respectively, $p(B)=\bigvee_j p(D_j)$) will be understood to mean that $p(A)$ (respectively, $p(B)$) is the infimum (respectively, supremum) in $\Pi(\mathscr{A})$ of this family. The necessary and sufficient condition that $p(A)=\bigwedge_j p(D_j)$ is that $A^{\perp\perp}=\bigcap_j (D_j)^{\perp\perp}$, that is, $p(A)$ is confirmed by precisely those outcomes that confirm every proposition $p(D_j)$, $j\in J$. In conventional logic, the conjunction of a set of propositions is generally understood to be a proposition that is true if and only if each proposition in the set is true. Consequently, we feel entitled to refer to $\bigwedge_j p(D_j)$, when it exists, as the *conjunction* of the propositions in the family $\{p(D_j)\mid j\in J\}$ (Randall and Foulis, 1973). In conventional logic, the conjunction of a set of propositions is false if and only if at least one of the propositions in the set is false. Here, the analogous situation does not quite obtain; indeed, if $p(A)=\bigwedge_j p(D_j)$, any outcome refuting any one of the propositions $p(D_j)$ will refute $p(A)$. In general, there will be outcomes that refute $p(A)$ but fail to refute any one of the propositions $p(D_j)$.

An orthologic L is said to be *conjunctive* if any two elements of L have an infimum in L. Similarly, a manual $\mathscr{A}$ is called *conjunctive* if its logic $\Pi(\mathscr{A})$ is conjunctive. If L is an orthomodular poset, then L is an orthomodular lattice if and only if L is conjunctive; hence, $\Pi(\mathscr{A})$ is an orthomodular lattice if and only if $\mathscr{A}$ is a conjunctive Dacey manual. Any semiclassical manual is conjunctive Dacey. If $\mathscr{A}$ is conjunctive and Dacey, so is $\mathscr{A}^c$ (Weaver, 1971). If the group H acts on the conjunctive manual $\mathscr{A}$ and if H is a subgroup of the group G, then $G\mathscr{A}/H$ is again conjunctive.

In conventional logic, the disjunction of a set of propositions is generally understood to be a proposition which is true if and only if at least one proposition in the set is true. This suggests the following definition: a proposition $p(B)\in\Pi(\mathscr{A})$ is a *disjunction* of the propositions in the family $\{p(D_j)\mid j\in J\}$ provided that $B^{\perp\perp}=\bigcup_j(D_j)^{\perp\perp}$. It is a fact that if $p(B)$ is a disjunction of the propositions $\{p(D_j)\mid j\in J\}$, then $p(B)=\bigvee_j p(D_j)$; however, the converse is false. If $p(B)=\bigvee_j p(D_j)$ and $p(B)$ is a disjunction of the propositions $\{p(D_j)\mid j\in J\}$, we call $p(B)$ a *disjunctive supremum* of $\{p(D_j)\mid j\in J\}$ and we write $p(B)=\nabla_j p(D_j)$.

The following general distributive law holds for disjunctions in $\Pi(\mathscr{A})$: if $p(B)=\nabla_j p(D_j)$ and if $p(C)\in\Pi(\mathscr{A})$ is such that the infimum $p(D_j)\wedge p(C)$ exists for each $p(D_j)$, then the infimum $p(B)\wedge p(C)$ exists and we have $p(B)\wedge p(C)=\nabla_j p(D_j)\wedge p(C))$, An analogous distributive law for suprema which are not disjunctions is explicitly false.

Suppose that $\mathscr{A}$ is a Dacey manual, so that $\Pi(\mathscr{A})$ is an orthomodular poset. Then, $\Pi(\mathscr{A})$ is closed under the negation mapping, that is, if $p(D) \in \Pi(\mathscr{A})$, it follows that $p(D)' = p(D)^{\perp} \in \Pi(\mathscr{A})$. Furthermore, the map $p(D) \rightsquigarrow p(D)'$ is an antiautomorphism of period two on the structure $(\Pi(\mathscr{A}), \leqslant)$; hence, trivially, it converts existing infima into suprema and vice versa. Consequently, we have the conventional deMorgan laws: $(\bigvee_j p(D_j))' = \bigwedge_j p(D_j)'$ and $(\bigwedge_j p(D_j))' = \bigvee_j p(D_j)'$. It follows that the necessary and sufficient condition that $p(B) = \bigvee_j p(D_j)$ is that $p(B)$ is refuted by precisely those outcomes that refute every one of the propositions $p(D_j)$. If $p(B) = \bigvee_j p(D_j)$, then $p(B)$ will be confirmed by any outcome which confirms any one of the $p(D_j)$; however, there may be outcomes that confirm $p(B)$, yet fail to confirm any one of the $p(D_j)$. It is only when $p(B) = \nabla_j p(D_j)$ that an outcome confirming $p(B)$ will necessarily confirm at least one of the $p(D_j)$.

Of course, a *symmetry* of an orthologic L is understood to be an automorphism of L, that is, a map $\phi : L \to L$ which is bijective and has the property that both ϕ and ϕ^{-1} preserve the relations $\leqslant$ and $\perp$. A *logical symmetry* for the manual $\mathscr{A}$ is understood to be such a symmetry of its logic $\Pi(\mathscr{A})$. Suppose that ϕ is an operational symmetry for $\mathscr{A}$. It is easy to see that if A and B are $\mathscr{A}$-events with $p(A) = p(B)$, then $\phi(A)$ and $\phi(B)$ are $\mathscr{A}$-events with $p(\phi(A)) = p(\phi(B))$. It follows that every operational symmetry ϕ for $\mathscr{A}$ determines a unique logical symmetry (also denoted by ϕ) in such a way that $\phi(p(A)) = p(\phi(A))$ holds for all $\mathscr{A}$-events A. We shall refer to such a logical symmetry (corresponding to some operational symmetry) as a *regular* logical symmetry. One can give examples of logical symmetries for $\mathscr{A}$ which are not regular – again the reason for this is that two nonisomorphic manuals can give rise to isomorphic logics.

If H is a group acting on the manual $\mathscr{A}$, then for each $h \in H$ we can define a logical symmetry ϕ_h for $\mathscr{A}$ by $\phi_h(p(A)) = p(hA)$ for each $\mathscr{A}$-event A. In this way, we get a representation of the group H, $h \rightsquigarrow \phi_h$, as a group of automorphisms of the orthologic $\Pi(\mathscr{A})$. If H is a subgroup of a larger group G, then, as we have seen, we obtain an expanded action of G on the manual $G\mathscr{A}/H$ and thus a representation of G as a group of automorphisms of the logic $\Pi(G\mathscr{A}/H)$.

The operational propositions in the logic $\Pi(\mathscr{A})$ of a manual should be viewed as being phenomenological in the sense that they can only assert that the outcome of some execution of a physical operation belongs to a certain designated set of outcomes. Consequently, such propositions have no predictive power and no explanative power per se.

In empirical logic, it is essential to distinguish between an *operational proposition* $p(D)$, which may be confirmed, refuted, or left indeterminate by an execution of a physical operation, a *prediction* that such an operational proposition will be confirmed as a consequence of a particular realization of one of its test operations, and a *statistical hypothesis* that makes some claim regarding the 'long run relative frequency' with which certain operational propositions will be test-confirmed. Physical laws are ordinarily of the latter form and therefore involve 'stochastic models'. In the next part, we set up the appropriate mathematical machinery to deal with such stochastic models.

VI. WEIGHTS AND STATES – COMPLETE STOCHASTIC MODELS

By a *weight function* for a manual $\mathscr{A}$ with outcome set $X=\bigcup \mathscr{A}$, we mean a real-valued function ω defined on X, taking on its values in the closed unit interval, and such that the unordered sum $\sum_{e\in E}\omega(e)$ converges to 1 for every $\mathscr{A}$-operation E. The set of all such weight functions for $\mathscr{A}$ is denoted by $\Omega=\Omega(\mathscr{A})$. It is natural to extend an $\omega\in\Omega$ to the $\mathscr{A}$-events by defining $\omega(D)=\sum_{d\in D}\omega(d)$ for any $\mathscr{A}$-event D. It then follows that $0\leqslant\omega(D)\leqslant 1$ for all $\mathscr{A}$-events D and that ω is *finitely additive* in the sense that $\omega(\bigcup_{i=1}^{n} D_i)=\sum_{i=1}^{n}\omega(D_i)$ for any finite family $\{D_i \mid i=1,2,\ldots,n\}$ of pairwise orthogonal $\mathscr{A}$-events. In general, an $\omega\in\Omega$ need not be countably additive – such additional features as countable additivity will depend on the detailed structure of the manual $\mathscr{A}$.

A weight function $\omega\in\Omega(\mathscr{A})$ will be regarded intuitively as a possible 'complete stochastic model in the frequency sense' for the empirical situation described by the manual $\mathscr{A}$ as follows: for every $\mathscr{A}$-outcome $x\in X$, $\omega(x)$ is interpreted as the 'long-run relative frequency' with which the outcome x is secured as a consequence of the execution of operations for which x is a possible outcome (according to the stochastic model ω).

It is easy to check that if $\omega\in\Omega(\mathscr{A})$ and if C, D are $\mathscr{A}$-events with $p(C)\leqslant p(D)$, then $\omega(C)\leqslant\omega(D)$. In particular, if $p(C)=p(D)$, then $\omega(C)=\omega(D)$ and this permits us to lift ω to a function (still denoted by ω) defined on the operational logic $\Pi(\mathscr{A})$ simply by setting $\omega(p(D))=\omega(D)$ for every $\mathscr{A}$-event D. This ω, defined on $\Pi(\mathscr{A})$, will be referred to as the *regular state* induced on $\Pi(\mathscr{A})$ by the original weight function $\omega \in \Omega(\mathscr{A})$. Notice that, for any $\mathscr{A}$-outcome x, $\omega(x) = \omega(p(\{x\}))$; hence, the original weight function ω can be recaptured from the regular state that it induces.

If L is any orthologic, we define a *state* on L to be a real-valued function α defined on L, taking on its values in the closed unit interval, such that $\alpha(1) = 1$ and $\alpha(e \oplus f) = \alpha(e) + \alpha(f)$ whenever $e, f \in L$ with $e \perp f$. Clearly, every regular state on $\Pi(\mathscr{A})$ is a state; however, there may be states on $\Pi(\mathscr{A})$ that are not regular. If $\omega \in \Omega(\mathscr{A})$ and if $E \in \mathscr{A}$ is a test operation for $p(D) \in \Pi(\mathscr{A})$, where D is an $\mathscr{A}$-event, then it can be shown that $\omega(p(D)) = \omega(E \cap D^{\perp\perp})$; hence, $\omega(p(D))$ will be interpreted as the 'long-run relative frequency' with which the operational proposition $p(D)$ will be test confirmed (according to the stochastic model ω). In Randall and Foulis (1974), we showed that it is reasonable to interpret a state on $\Pi(\mathscr{A})$ as a complete and consistent assignment of betting rates for wagers on the confirmation of the propositions in $\Pi(\mathscr{A})$ as a consequence of specified realizations of test operations for these propositions.

It is surely desirable to have a lavish supply of weight functions (complete stochastic models) for a manual $\mathscr{A}$ and the physical circumstances which it describes. Any ad hoc assumption assuring such a supply of weights would necessarily be a nontrivial constraint on the manual $\mathscr{A}$, since there are large classes of conjunctive Dacey manuals that admit only one weight, or no weights at all (Greechie, 1971). Nevertheless, in any realistic situation, there always appears to be a generous supply of weight functions – in fact, the set of all weight functions is usually 'strong' in the following sense: a set Δ of states on an orthologic L is said to be *strong* provided that for $p, q \in L$ with $p \nleq q$, there exists $\alpha \in \Delta$ with $\alpha(p) = 1$ and $\alpha(q) \neq 1$. A set of weights $\Delta \subseteq \Omega(\mathscr{A})$ is said to be *strong* if the corresponding set of regular states on the orthologic $\Pi(\mathscr{A})$ is strong. If an orthologic L admits a strong set of states, then it is an orthomodular poset; hence, if a manual $\mathscr{A}$ admits a strong set of weights, it is a Dacey manual.

Any semiclassical manual $\mathscr{D}$ admits a strong set of weights. If the manual $\mathscr{A}$ admits a strong set of weights, so does the compound manual $\mathscr{A}^c$. If the group H acts on the manual $\mathscr{A}$, if $\mathscr{A}$ admits a strong set of weights, and if H is a subgroup of G, then the expanded manual $G\mathscr{A}/H$ also admits a strong set of weights. If the manual $\mathscr{B}$ admits a strong set of weights and if $\mathscr{A}$ is an induced submanual of $\mathscr{B}$, then $\mathscr{A}$ also admits a strong set of weights.

An important 'conditioning' notion is available for the weights on a compound manual $\mathscr{A}^c$. Indeed, suppose that $\omega \in \Omega(\mathscr{A}^c)$ and that a is an $\mathscr{A}^c$-outcome with $\omega(a) \neq 0$. If X^c is the set of all $\mathscr{A}^c$-outcomes, we define $\omega_a: X^c \to [0, 1]$ by $\omega_a(b) = [\omega(ab)/\omega(a)]$ for all $b \in X^c$. It can be shown (Foulis and Randall, 1974) that ω_a is again a weight function for the compound manual $\mathscr{A}^c$, and we shall refer to ω_a as the weight function obtained by *operationally conditioning* ω by the outcome a. In Foulis and Randall (1974), we showed that this notion of operational conditioning is compatible with the usual notion of 'conditioning' in quantum physics. Just as we can condition by $\mathscr{A}^c$-outcomes, so also we can condition by $\mathscr{A}^c$-events. Indeed, suppose that $\omega \in \Omega(\mathscr{A}^c)$ and that D is an $\mathscr{A}^c$-event with $\omega(D) \neq 0$. Define $\omega_D: X^c \to [0, 1]$ by $\omega_D(b) = [\omega(Db)/\omega(D)]$ for $b \in X^c$. (Note that Db is again an $\mathscr{A}^c$-event.) If we put $A = \{d \in D \mid \omega(d) \neq 0\}$, then A is a nonempty $\mathscr{A}^c$-event and we can write ω_D as a convex combination $\omega_D = \sum_{d \in A} t_d \omega_d$, where $t_d = [\omega(d)/\omega(D)]$ for each $d \in A$. Since a convex combination of weight functions is again a weight function, then $\omega_D \in \Omega(\mathscr{A}^c)$.

If ω is an extreme point of the convex set $\Omega(\mathscr{A}^c)$ – a so-called *pure weight* – then it can be shown (Foulis and Randall, 1974) that ω_a will again be an extreme point of $\Omega(\mathscr{A}^c)$. Thus, operational conditioning by $\mathscr{A}^c$-outcomes will preserve pure weights, whereas operational conditioning by $\mathscr{A}^c$-events in general will not. Furthermore, it should be noted that in general it will not be possible to condition in this operational sense by operational propositions in $\Pi(\mathscr{A}^c)$ in any naive way, since there are easy examples of $\mathscr{A}^c$-events A, B for which $p(A) = p(B)$, but $\omega_A \neq \omega_B$. In particular, even if E is an $\mathscr{A}^c$-operation, ω_E need not coincide with ω – the operation E need not be 'gentle' for the regular state ω.

If $\omega \in \Omega(\mathscr{A}^c)$ is interpreted as a complete stochastic model in the frequency sense, then $\omega_a(b)$ can be regarded as the 'long-run relative frequency' with which the outcome b will be secured (as a consequence of compound operations for which it could be secured) immediately after the execution of a compound operation for which the outcome a was secured. Here, there is a definite temporal order involved – a occurs *first*, then b. Moreover, in general, there is no temporal symmetry, as can be seen by the failure of the classical multiplication rule: $\omega_a(b)\, \omega(a)$ need not coincide with $\omega_b(a)\, \omega(b)$.

Suppose $h \in X^c$ is a transformation, that is, $\{h\} \in \mathscr{A}^c$. Then, for any

$\omega\in\Omega(\mathscr{A}^c)$, $\omega(h)=1$, so that ω_h is defined; indeed, $\omega_h(b)=\omega(hb)$ for all $b\in X^c$. If $\omega=\sum_{i\in I}t_i\omega_i$, where each $t_i\geqslant 0$ and $\sum_{i\in I}t_i=1$, then $\omega_h=\sum_{i\in I}t_i(\omega_i)_h$; that is, the map $\omega\rightsquigarrow\omega_h$ preserves convex combinations.

Suppose now that $\mathscr{J}\subseteq\mathscr{A}^c$ is a nonempty collection of transformations such that $\{1\}\in\mathscr{J}$ and $G=\bigcup\mathscr{J}$ is equipped with a binary operation $\circ$ such that $(G,\circ)$ is a group and 1 is the group identity. We understand the equality $g=g_1\circ g_2$ to mean that, in some physical sense, the transformation g corresponds to the transformation g_1 followed by the transformation g_2. Here we must distinguish between the product g_1g_2 in the free monoid X^c and the group product $g_1\circ g_2$; they are mathematically distinct. Nevertheless, the preceding interpretation forces us to regard g_1g_2 and $g_1\circ g_2$ as being equivalent; hence, the physically admissible stochastic models ω in $\Omega(\mathscr{A}^c)$ ought to belong to the set Δ of all ω for which $\omega(ag_1g_2b)=\omega(a(g_1\circ g_2)\,b)$ for all $a,b\in X^c$ and all $g_1,g_2\in G$. It can be shown that Δ is a nonempty, convex, extremal subset of the convex set $\Omega(\mathscr{A}^c)$, provided of course that $\Omega(\mathscr{A})$ is nonempty. Furthermore, if $\omega\in\Delta$ and $b\in X^c$, then $\omega_b\in\Delta$; in particular, if $g\in G$, the map from Δ into Δ defined by $\omega\rightsquigarrow\omega_g$ is a bijective map preserving convex combinations – as is its inverse $\omega\rightsquigarrow\omega_{g^{-1}}$.

In general, such a bijective affine map on a convex set will be called a *symmetry* for this convex set. If $\mathscr{M}$ is any manual and Δ is a nonempty convex subset of $\Omega(\mathscr{M})$, then a symmetry on Δ will be referred to as a *stochastic symmetry for $\mathscr{M}$ on Δ*. In particular, if $\Delta=\Omega(\mathscr{M})$, we shall simply call such a symmetry a *stochastic symmetry for the manual $\mathscr{M}$*. If ϕ is an operational symmetry of the manual $\mathscr{M}$, then ϕ defines a stochastic symmetry $\omega\rightsquigarrow\omega\circ\phi^{-1}$ for $\mathscr{M}$; such a stochastic symmetry is called a *regular* stochastic symmetry.

Now, let the group H act on a manual $\mathscr{A}$, let H be a subgroup of a larger group G, and construct the expanded manual $G\mathscr{A}/H$ as before. Then each $g\in G$ determines an operational symmetry ϕ_g on $G\mathscr{A}/H$ by $\phi_g([g',x])=[gg',x]$ for $g'\in G$, $x\in X=\bigcup\mathscr{A}$. This operational symmetry, in turn, defines a regular stochastic symmetry $\omega\rightsquigarrow\omega\circ\phi_g^{-1}$ for the expanded manual $G\mathscr{A}/H$. This, in turn, defines an action $(g,\omega)\rightsquigarrow g\omega=\omega\circ\phi_g^{-1}$ of G on $\Omega(G\mathscr{A}/H)$ and provides a representation of G by stochastic symmetries for the expanded manual $G\mathscr{A}/H$. This stochastic representation of G is formally identical to the induced representations discussed in Mackey (1968), in the following sense: define $\Delta=\{\psi\in\Omega(\mathscr{A})^G\,|$

$(\psi(gh))(x)=(\psi(g))(hx)$ for all $g\in G$, $h\in H$, $x\in X\}$, and let $\Phi:\Delta\to\Omega(G\mathscr{A}/H)$ be the affine bijection given by $(\Phi(\psi))([g,x])=(\psi(g))(x)$ for $g\in G$, $x\in X$, $\psi\in\Delta$. Under this affine bijection, the action of G on $\Omega(G\mathscr{A}/H)$ is equivalent to the natural action of G on Δ given by $(g\psi)(g')=\psi(g^{-1}g')$, $g, g'\in G$, $\psi\in\Delta$.

VII. RANDOM VARIABLES AND MACKEY OBSERVABLES

In orthodox quantum physics, the observables are represented mathematically by self-adjoint operators; that is, according to the spectral theorem, as projection-valued measures – a concept generalized by Mackey (1963) to proposition-valued measures. On the other hand, the random variables of orthodox statistics are mathematically represented by measurable functions (Kolmogorov, 1933). Both of these related concepts are readily available to us as follows:

Let $\mathscr{A}$ be a Dacey manual with logic $\Pi(\mathscr{A})$. By a block $\mathbb{B}$ in the orthomodular poset $\Pi(\mathscr{A})$, we mean a maximal Boolean suborthomodular lattice in $\Pi(\mathscr{A})$. If $\mathbb{B}$ is such a block, we denote the Stone space of $\mathbb{B}$ by $S(\mathbb{B})$. It is useful to regard $S(\mathbb{B})$ as the outcome set for a 'virtual operation' that is a 'common refinement' for all of the operations E in $\mathscr{A}$ affiliated with $\mathbb{B}$ in the sense that $p(D)\in\mathbb{B}$ for every event $D\subseteq E$. This idea can be made more precise in terms of the so-called refinement ideals discussed in Randall and Foulis (1973).

Motivated by Mackey (1963), we define a *Mackey observable* for $\mathscr{A}$ to be a mapping A sending each real Borel set Λ onto an operational proposition $A(\Lambda)$ in $\Pi(\mathscr{A})$ in such a way that the following conditions hold:

(1) $A(\emptyset)=0$ and $A(\mathbb{R})=1$.
(2) If Λ_1 and Λ_2 are disjoint, real Borel sets, then $A(\Lambda_1)\perp A(\Lambda_2)$.
(3) If $(\Lambda_n \mid n=1,2,\ldots)$ is a countable sequence of pairwise disjoint real Borel sets with $\Lambda=\bigcup_{n=1}^{\infty}\Lambda_n$, then $A(\Lambda)$ is the supremum in $\Pi(\mathscr{A})$ of the family $(A(\Lambda_n)\mid n=1,2,\ldots)$.

Note that we do not assume that $\Pi(\mathscr{A})$ is σ-complete, but only that the required suprema exist. Intuitively, we regard the operational proposition $A(\Lambda)$ as corresponding to an assertion that a 'measurement' of the observable A yields a real number belonging to the Borel set Λ. Evidently, any two operational propositions in the image of the mapping A will commute in $\Pi(\mathscr{A})$; hence, there will exist at least one block $\mathbb{B}$ of $\Pi(\mathscr{A})$ such that $A(\Lambda)\in\mathbb{B}$ for every real Borel set Λ. We call such a $\mathbb{B}$ an *A-block* in $\Pi(\mathscr{A})$.

Each operational proposition $p(D)$ in the A-block $\mathbb{B}$ corresponds to a unique compact open subset $\phi(p(D))$ of the Stone space $S(\mathbb{B})$; ϕ is an isomorphism of the Boolean algebra $\mathbb{B}$ onto the field of all compact open subsets of $S(\mathbb{B})$. We define a *$\mathbb{B}$-random variable* to be a Baire measurable real-valued function on $S(\mathbb{B})$ and we say that such a $\mathbb{B}$-random variable f *corrresponds* to the Mackey observable A if, for each real Borel set Λ, the symmetric difference of $\phi(A(\Lambda))$ and $f^{-1}(\Lambda)$ is a meager subset of $S(\mathbb{B})$. Notice that two $\mathbb{B}$-random variables f and g which correspond to the same Mackey observable A are *equivalent* in the sense that $\{s\in S(\mathbb{B}) \mid f(s)\neq g(s)\}$ is a meager subset of $S(\mathbb{B})$.

A random variable f corresponding to a given Mackey observable A can be constructed as follows: first select an A-block $\mathbb{B}$. For each real number t, let $M_t=\phi(A(-\infty, t])$, so that M_t is a compact open subset of $S(\mathbb{B})$. Let N_0 be the intersection of all of the sets M_t and let N_1 be the intersection of all of the sets $S(\mathbb{B})\backslash M_t$, as t runs through the real numbers. Put $N=N_0\cup N_1$ and $U=S(\mathbb{B})\backslash N$, so that N is a closed nowhere dense Baire subset of $S(\mathbb{B})$ and U is an open dense subset of $S(\mathbb{B})$. Define a real-valued function f on $S(\mathbb{B})$ by putting $f=0$ on N and $f(s)=\inf\{t \mid s\in M_t\}$ for $s\in U$, noting that f is a Baire measurable function on $S(\mathbb{B})$ and that f is continuous on U. Obviously, the $\mathbb{B}$-random variable f corresponds to the original Mackey observable A. If g is any $\mathbb{B}$-random variable, then g will correspond to A if and only if g is equivalent to f. Furthermore, if the Mackey observable A is *bounded* in the sense that $A(\Lambda)=1$ for some bounded open interval Λ on the real line, then $U=S(\mathbb{B})$ and f is actually continuous on $S(\mathbb{B})$.

If $\mathbb{B}$ is a block in $\Pi(\mathscr{A})$, there will (in general) exist $\mathbb{B}$-random variables that do not correspond to any Mackey observable; however, if $\mathbb{B}$ happens to be a σ-complete Boolean algebra, then any $\mathbb{B}$-random variable g will correspond to some Mackey observable A for which $\mathbb{B}$ is an A-block. Clearly, if A and B are two Mackey observables and if $\mathbb{B}$ is both an A-block and a B-block, then $A=B$ if and only if A and B correspond to equivalent $\mathbb{B}$-random variables.

Suppose now that α is a state on the orthomodular poset $\Pi(\mathscr{A})$ and that $\mathbb{B}$ is a block in $\Pi(\mathscr{A})$ with Stone space $S(\mathbb{B})$. The restriction $\alpha|_{\mathbb{B}}$ of α

to the Boolean algebra $\mathbb{B}$ is then a state on $\mathbb{B}$; hence, it induces a finitely additive probability measure α^* on the field of compact open subsets of $S(\mathbb{B})$. By Heider (1958, p. 216), α^* admits a unique extension to a countably additive probability measure α_S defined on the σ-field of Baire subsets of $S(\mathbb{B})$. Thus, if f is a $\mathbb{B}$-random variable, we can define the *expectation value* of f in the state α by $\mathbb{E}(f, \alpha) = \int_S f d\alpha_S$, $S = S(\mathbb{B})$, provided that this integral exists. If $\alpha|_{\mathbb{B}}$ is not only finitely additive, but countably additive on $\mathbb{B}$, then α_S vanishes on all meager Baire subsets of $S(\mathbb{B})$; hence, in this case, $\mathbb{E}(f, \alpha) = \mathbb{E}(g, \alpha)$ will hold for equivalent $\mathbb{B}$-random variables f and g.

If A is a Mackey observable for $\mathscr{A}$, $\mathbb{B}$ is an A-block in $\Pi(\mathscr{A})$, and α is a state on $\Pi(\mathscr{A})$, then we define the *expectation value* of A in the state α by $\mathbb{E}(A, \alpha) = \mathbb{E}(f, \alpha)$, where f is the particular $\mathbb{B}$-random variable constructed as above to correspond to A. Evidently, if α is countably additive on $\mathbb{B}$, then $\alpha \circ A$ is a Borel probability measure and $\mathbb{E}(A, \alpha)$ coincides with the expected value of the probability distribution $\alpha \circ A$.

VIII. THE PROTOTYPES – CLASSICAL PROBABILITY AND ORTHODOX QUANTUM MECHANICS

Above, we have generalized many of the basic notions of quantum physics, in particular, observables and states, within the operational context of empirical logic. Now, we shall examine these notions in more detail in connection with both classical probability theory and orthodox quantum mechanics.

Since the publication of Kolmogorov's (1933) well-known monograph, it has been generally appreciated that all classical probability questions can be cast in terms of a probability space $(X, \mathscr{F}, P)$, where X is a non-empty set, $\mathscr{F}$ is a σ-field of subsets of X, and P is a normed measure defined on $\mathscr{F}$. The classical random variables of interest can then be represented as $\mathscr{F}$-measurable functions on X. We regard X as being the set of all outcomes of *a* (perhaps idealized) physical operation. An event $D \in \mathscr{F}$ is construed as being an observable event in some appropriate sense; hence, a countable partition $X = \bigcup_{i=1}^{\infty} X_i$ of X into disjoint non-empty observable events is regarded as representing a physically accessible coarsened version of the original operation.

Thus, let $\mathscr{A}$ consist of all countable $\mathscr{F}$-measurable partitions of X. It is easy to check that $\mathscr{A}$ is a Dacey manual and moreover that the logic

$\Pi(\mathscr{A})$ is isomorphic to the σ-field $\mathscr{F}$ (a countably complete Boolean algebra) under the correspondence that associates $p(\{D\})$ with the non-empty set $D\in\mathscr{F}$ and associates $p(\emptyset)$ with $\emptyset\in\mathscr{F}$. Notice that any countable collection of propositions in $\Pi(\mathscr{A})$ is simultaneously testable. Also, the weights $\omega\in\Omega(\mathscr{A})$ are in natural one-to-one correspondence $\omega\leftrightarrow P$ with the normed measures P defined on the σ-field $\mathscr{F}$, that is, the regular states on $\Pi(\mathscr{A})$ correspond to the normed measures on $\mathscr{F}$. In general, there may exist nonregular states on $\Pi(\mathscr{A})$ corresponding to finitely additive, but not countably additive, probability measures on $\mathscr{F}$. Evidently, the probability measures concentrated on points of X provide a strong set of regular states and these states are deterministic in the sense that they can assume only the values 0 or 1.

Clearly, there is but one block $\mathbb{B}$ in $\Pi(\mathscr{A})$, namely, $\mathbb{B}=\Pi(\mathscr{A})$. Since $\mathbb{B}$ is σ-complete, then, as noted earlier, there is a natural one-to-one correspondence between Mackey observables for $\mathscr{A}$ and equivalence classes of $\mathbb{B}$-random variables. Notice that if Y is a real-valued $\mathscr{F}$-measurable function defined on X – that is, if Y is a random variable in the classical sense – then Y defines a unique Mackey observable A for $\mathscr{A}$ in such a way that, for each real Borel set Λ, $A(\Lambda)$ is the proposition in $\Pi(\mathscr{A})$ corresponding to the set $Y^{-1}(\Lambda)\in\mathscr{F}$. In this way, we obtain a one-to-one correspondence $Y\leftrightarrow A$ between $\mathscr{F}$-measurable functions Y on X and Mackey observables A for $\mathscr{A}$. Consequently, there is here a one-to-one correspondence between random variables in the classical sense defined on the measurable space $(X, \mathscr{F})$ and equivalence classes of $\mathbb{B}$-random variables for the manual $\mathscr{A}$. A classical random variable will be bounded if and only if there is a (necessarily unique) continuous $\mathbb{B}$-random variable in its corresponding equivalence class. In this way, the bounded, classical random variables are in natural one-to-one correspondence with the continuous $\mathbb{B}$-random variables.

Since the publication of the celebrated monographs of Dirac (1930) and von Neumann (1932) the customary mathematical framework for orthodox quantum mechanics has been a separable, complex, infinite dimensional Hilbert space $\mathscr{H}$. Recall that here the physical observables are presumed to correspond to self-adjoint operators on $\mathscr{H}$ and that the latter (by the spectral theorem) correspond to Mackey observables. In particular, the projection operators correspond to quantum mechanical propositions (von Neumann, 1955, p. 247) and the collection $\mathbb{P}$ of all such projections is called the quantum logic for $\mathscr{H}$. The quantum logic $\mathbb{P}$ forms a complete orthomodular lattice and the quantum mechanical states are customarily regarded as being the states on the orthologic $\mathbb{P}$ which are σ-additive. According to Gleason's (1957) theorem, these states can be represented mathematically by von Neuman density operators.

By analogy with the above classical situation, we now form the manual $\mathscr{A}$ consisting of all countable collections of pairwise, orthogonal non-zero projections in $\mathbb{P}$ that sum to the identity. Again, $\Pi(\mathscr{A})$ is naturally isomorphic to $\mathbb{P}$. In this case, a countable collection of propositions in $\Pi(\mathscr{A})$ is simultaneously testable if and only if the corresponding projections in $\mathbb{P}$ commute algebraically with one another. Moreover, the wieghts in $\Omega(\mathscr{A})$ are in natural one-to-one correspondence with the quantum mechanical states – that is, the regular states for $\Pi(\mathscr{A})$ are exactly the countably additive states – and these form a strong set of states.

Since $\Pi(\mathscr{A})$ is complete, then the blocks $\mathbb{B}$ of $\Pi(\mathscr{A})$ are complete Boolean algebras. As a consequence, if $\mathbb{B}$ is such a block, then every $\mathbb{B}$-random variable corresponds to a Mackey observable, and therefore to a self-adjoint operator whose spectral projections correspond to propositions in this block.

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