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Entropy Jump Across an Inviscid Shock Wave

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Abstract. The Shock jump conditions for the Euler equations in their primitive form are derived by using generalized functions. The shock profiles for specific volume, speed, and pressure and shown to be the same, however, density has a different shock profile. Careful study of the equations that govern the entropy shows that the inviscid entropy profile has a local maximum within the shock layer. We demonstrate that because of this phenomenon, the entropy propagation equation cannot be used as a conservation law.

1. Introduction

A consequence of the nonlinearity of the equations of motion is the steeping of compression waves into a shock wave. Within the shock layer, the gradients of velocity and temperature become large, and irreversible thermodynamic processes caused by friction and heat conduction become dominant. At high Reynolds numbers, the shock-layer thickness is of the order of several mean free paths; for all practical purposes, the shock layer can be represented as a mathematical abstraction that corresponds to a surface across which the flow variables experience a sudden jump. Away from this discontinuity surface, viscous and heat conduction effects are usually negligible and the inviscid equations of motion model the flow well. Remarkably the information needed to account for the final outcome of the irreversible processes that take place within the shock layer is contained in the inviscid equations.

The study of shock waves is 150 years old. The jump conditions satisfied by the conservation of mass and momentum were discovered by Stokes [11] in the middle of the 19th century. Stokes' excitement at making this discovery is evident in his paper: "These conclusions certainly seem sufficiently startling; yet a still more extraordinary result... the result, however, is so strange..." The shock jump condition associated with the conservation of energy was implicit in an investigation conducted by Rankine [8] in 1870; however, a precise exposition was not made until the work of Hugoniot [6] in 1889. The increase in entropy across a shock was a more difficult concept to grasp. The leading fluid dynamicists in England (Stokes, Kelvin, and Rayleigh) questioned the validity of the shock discontinuity because it violated the conservation of entropy. The correct principles were not well understood, and did not appear in their present form until around 1915.

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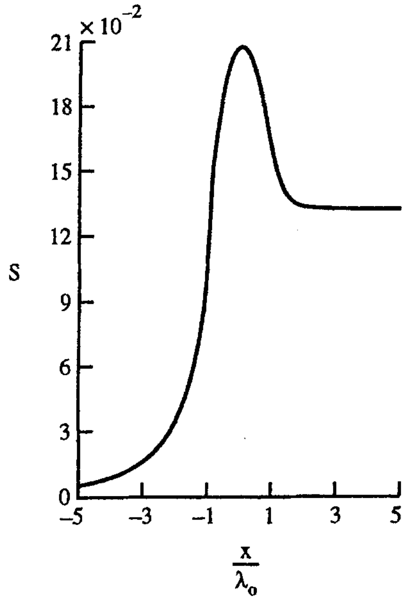

Figure 1. Entropy profiles through viscous shock layer.

Detailed studies of the viscous shock layer emerged several years later with the work of Becker [1], who solved exactly the one-dimensional equations of a real fluid. In a related study, Morduchow and Libby [7] found the exact entropy distribution across the shock layer of a viscous heat-conducting gas. Morduchow and Libby observed, see Figure 1, that the entropy, unlike the other flow variables that behave monotonically, increases through the shock layer until it reaches a maximum at the center of the layer and then decreases to its expected value on the other side of the shock. Morduchow and Libby explained this phenomenon as follows: "It may at first sight appear that the recovery of mechanical energy on the downstream side of the center of the wave thus indicated by this solution would violate the second law of thermodynamics... However, the second law applies to an entire system—that is, to the end points—and permits energy recovery in separated sections thereof. The negative entropy gradient here might also be interpreted as physical effects that are not taken into account by the governing equations..."

Today, the shock jump conditions are obtained for the inviscid equations by casting them in their integral conservation form. A brief derivation, based on this standard procedure, is given Section 2. However, the purpose of this work is to show that the shock jump conditions can be derived from the primitive differential form of the equations. The genesis of the analysis presented here was contained in an unpublished work of Gino Moretti written in the early 1970s. The significance of this work is primarily that it demonstrates how to obtain the shock jump conditions for equations that cannot be cast in a conservation integral. Similar work has been presented by Colombeau in [2]. An interesting consequence of this exposition is a better understanding of why the entropy equation does not yield the proper jump.

2. Standard Shock Jump Analysis

The derivation of the jump conditions across a shock associated with the Euler equations is well known. See, for example, [10]. The derivation, which is valid for the equations expressed in conservative form, is included here so that it can be contrasted with the "nonstandard" approach introduced in Section 3.

The one-dimennsional conservation laws for the inviscid flow of a perfect gas are

$$\begin{gathered}\int [\rho_t + (\rho u)_x]\, dx = 0, \\ \int [(\rho u)_t + (\rho u^2 + p)_x]\, dx = 0, \\ \int [(\rho E)_t + (\rho u H)_x]\, dx = 0,\end{gathered} \tag{1}$$

where ρ is the density, p is the pressure, E is the specific total energy, H is the specific total enthalpy such that

$$H = E + \frac{p}{\rho},$$

and u is the velocity of the gas. In the standard analysis for the shock jump conditions, we weaken the usual smoothness requirements associated with the classical notion of a function by introducing the concept of a weak or generalized solution. Basically, the integrand of (1) is multiplied by a test function that is at least C^1 smooth and has compact support. Then we integrate over space and time in the neighborhood of the shock and use integration by parts to move differentiation from the discontinuous fluid variables onto the smooth test function. Thus let (1) be symbolically represented by

$$\int (U_t + F_x)\, dx = 0, \tag{2}$$

where

$$U = (\rho, \rho u, \rho E)^T$$

and

$$F = (\rho u, \rho u^2 + p, \rho u H)^T,$$

and let the initial conditions be given by

$$U(x, 0) = U_0(x).$$

Let φ be a test function that is continuously differentiable and has compact support. Consider the domain D around the shock Σ defined in the rectangle $0 \le t \le t_1$ and $a \le x \le b$, see Figure 2. Let φ be zero outside of D and on its boundary. Multiply the integrand of (2) by φ, integrate over x and t, and use integration by parts to obtain

$$\int_D \int_{t \ge 0} (U\varphi_t + F\varphi_x)\, dx\, dt = 0. \tag{3}$$

We say that U is a weak solution of the initial value problem

$$U_t + F_x = 0$$

with initial data U_0 if (3) holds for all differentiable test functions φ with compact support.

Let D_l be the subset of D on the left of Σ and let D_r be the subset on the right of Σ as in Figure 2. Assume that U is differentiable everywhere except across Σ; hence, on D_l with the divergence theorem, we find

$$\int_{D_l} \int_{t \ge 0} (U\varphi_t + F\varphi_x)\, dx\, dt = \int_{D_l} \int_{t \ge 0} [(U\varphi)_t + (F\varphi)_x]\, dx\, dt = \int_{\partial D} \varphi(-U\, dx + F\, dt) \tag{4}$$

and similarity for D_r. Because φ is zero on the boundary of D, the line integrals are only nonzero along the

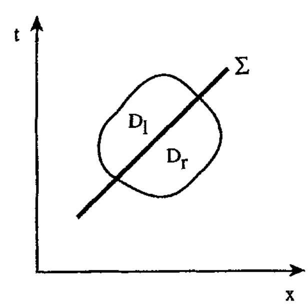

Figure 2. Domain of integration.

shock Σ. Let the shock be defined by $x(t)$ and let U_{l} be the value of U on the left of the shock; similarly, let U_{r} be the value of U on the right side of the shock. Then by using (4) and the equivalent expression on the right of Σ, we obtain

$$\int_{\Sigma} \varphi(-[U]\,dx + [F]\,dt) = 0,$$

where $[U] = U_{\mathrm{r}} - U_{\mathrm{l}}$ and $[F] = F(U_{\mathrm{r}}) - F(U_{\mathrm{l}})$. Because φ is arbitrary.

$$c[U] = [F] \tag{5}$$

along Σ, where $c = dx/dt$ is the speed of the shock. Equation (5) results in the following Rankine–Hugoniot (RH) jump conditions:

$$\begin{aligned} [\rho(c-u)] &= 0, \\ [\rho u(c-u)] + [p] &= 0, \\ [\rho E(c-u)] + [pu] &= 0. \end{aligned} \tag{6}$$

One solution to (6) corresponds to no mass flow across the discontinuity and leads to the conditions across a slip line. The other solution results in jumps in pressure, density, and velocity. After some manipulation, the RH jumps can be expressed as

$$\begin{aligned} \rho_{\mathrm{l}}\tilde{u}_{\mathrm{l}} &= \rho_{\mathrm{r}}\tilde{u}_{\mathrm{r}}, \\ p_{\mathrm{l}} + (\rho\tilde{u}^2)_{\mathrm{l}} &= p_{\mathrm{r}} + (\rho\tilde{u}^2)_{\mathrm{r}}, \\ \frac{1}{\gamma-1}a_{\mathrm{l}}^2 + \tfrac{1}{2}\tilde{u}_{\mathrm{l}}^2 &= \frac{1}{\gamma-1}a_{\mathrm{r}}^2 + \tfrac{1}{2}\tilde{u}_{\mathrm{r}}^2, \end{aligned} \tag{7}$$

where $\tilde{u} = u - c$, a is the speed of sound, and γ is the ratio of specific heats. The above relations indicate that

$$\tilde{u}_{\mathrm{r}}\tilde{u}^{\mathrm{l}} = \frac{p_{\mathrm{r}} - p_{\mathrm{l}}}{\rho_{\mathrm{r}} - \rho_{\mathrm{l}}}, \tag{8}$$

which is known as Prandtl's relation, and

$$\frac{\tilde{u}_{\mathrm{l}}}{\tilde{u}_{\mathrm{r}}} = \frac{(\gamma+1)p_{\mathrm{r}} + (\gamma-1)p_{\mathrm{l}}}{(\gamma-1)p_{\mathrm{r}} + (\gamma+1)p_{\mathrm{l}}}. \tag{9}$$

These results imply that the entroyp jumps across a shock. Its jump is given by

$$[S] = \ln\frac{p_{\mathrm{r}}}{\rho_{\mathrm{r}}^{\gamma}} - \ln\frac{p_{\mathrm{l}}}{\rho_{\mathrm{l}}^{\gamma}}. \tag{10}$$

Although the entropy propagation equation can be expressed in the form of a conservation law as

$$\int [(\rho S)_t + (\rho u S)_x]\,dx = 0, \tag{11}$$

it cannot be used to obtain the correct entropy jump across a shock wave.

3. Nonstandard Shock Jump Analysis

The shock jump conditions can be derived without relying on the integral conservation laws. The importance of this method is threefold. First, this method provides a means for determining the jump conditions for physical laws that cannot be expressed in conservation form. Second, it will ultimately lead to an understanding of the nature of the entropy structure across an inviscid shock. Thirdly, it may suggest how to derive shock-capturing algorithms with proper jumps from the nonconservation form of the Euler equations.

Consider the following system of equations:

$$v_t - vu_x + uv_x = 0, \tag{12}$$

$$u_t + uu_x + vp_x = 0, \tag{13}$$

$$p_t + up_x + \gamma pu_x = 0, \tag{14}$$

where $v = 1/\rho$. The reason for using v instead of ρ will become clear in Section 4.

We look for solution to v, u, and p of the form

$$v = v_1 + [v]H(\xi), \tag{15}$$

$$u = u_1 + [u]K(\xi), \tag{16}$$

$$p = p_1 + [p]L(\xi), \tag{17}$$

where for now we only require that

$$H, K, L = \begin{cases} 0 & \text{for } x \to -\infty, \\ 1 & \text{for } x \to \infty, \end{cases} \tag{18}$$

$\xi = x - ct$, and $[w] = w_r - w_l$. The functions H, K, and L provide a description of the shock profile or structure with end conditions w_l at $x = -\infty$ and w_r at $x = \infty$, where w stands for v, u, and p.

Consider (12) and introduce (15) and (16):

$$-c[v]H' - (v_1 + [v]H)[u]K' + (u_1 + [u]K)[v]H' = 0. \tag{19}$$

We can then rewrite (19) as

$$\frac{dH}{dK} - \frac{H}{a+K} = \frac{v_1}{[v](a+K)}, \tag{20}$$

where

$$a = \frac{u_1 - c}{[u]}. \tag{21}$$

By integrating (20) we obtain

$$H = \frac{v_1}{[v]} - b(a+K), \tag{22}$$

where b is a constant of integration. Now, for $x \to -\infty$, both H and $K \to 0$; therefore, the constant of integration is

$$b = \frac{v_1}{[v]a} \tag{23}$$

and

$$H = \frac{v_1}{[v]}\frac{K}{a}. \tag{24}$$

In the same way as for $x \to \infty$, both H and $K \to 1$; therefore,

$$\frac{v_1}{[v]a} = 1. \tag{25}$$

This last relation, together with (21), gives (with $\tilde{u} = u - c$)

$$\frac{\tilde{u}_r}{\tilde{u}_1} = \frac{\rho_1}{\rho_r}, \tag{26}$$

which is the RH jump for the conservation of mass equation found in Section 2 (equation (6)). By using (24) and (25), we find that

$$H = K. \tag{27}$$

Now consider (13); if we introduce equations (15)–(17), then

$$-c[u]K' + (u_1 + [u]K)[u]K' + (v_1 + [v]H)[p]L' = 0. \tag{28}$$

With (27) and (25), we can rewrite (28) as

$$\frac{dL}{dK} + \frac{[u]^2}{[v][p]} = 0. \tag{29}$$

We can integrate

$$L + \frac{[u]^2}{[v][p]} K = d, \tag{30}$$

where d is a constant of integration. As $K \to 0$, $L \to 0$, we can conclude that $d = 0$. Hence, for $K \to 1$, $L \to 1$,

$$\frac{[u]^2}{[v][p]} = -1. \tag{31}$$

This reduces to Prandtl's relation:

$$\tilde{u}_r \tilde{u}_l = \frac{p_r - p_l}{\rho_r - \rho_l}. \tag{32}$$

If we use (31) in (30), we get

$$K = L. \tag{33}$$

From the first two of (12)–(14), we find that the functions H, K, and L must be the same to obtain solutions as in (15)–(17).

Now consider (14) and introduce (16) and (17):

$$[p](u_1 - c)L' + \gamma p_1[u]K' + [u][p](KL' + \gamma LK') = 0. \tag{34}$$

However, we see from (33) that $K = L$ such that

$$[p](u_1 - c)K' + \gamma p_1[u]K' + [u][p](\gamma + 1)KK' = 0. \tag{35}$$

We integrate (35) from $x = -\infty$ to $x = \infty$ as

$$\{[p](u_1 - c) + \gamma p_1[u]\} \int_0^1 dK + [u][p](\gamma + 1) \int_0^1 K\, dK = 0. \tag{36}$$

Equation (36) yields

$$(p_r - p_l)\tilde{u}_l + \gamma p_l(\tilde{u}_r - \tilde{u}_l) + (\tilde{u}_r - \tilde{u}_l)(p_r - p_l)\frac{\gamma + 1}{2} = 0, \tag{37}$$

which can be reduced to the final RH jump condition:

$$\frac{\tilde{u}_l}{\tilde{u}_r} = \frac{(\gamma + 1)p_r + (\gamma - 1)p_l}{(\gamma - 1)p_r + (\gamma + 1)p_l}. \tag{38}$$

This analysis has shown that if $H = K = L$ (i.e., if the shock profiles for v, u, and p are identical) then the full set of correct RH jumps conditions can be recovered from the equations in primitive form.

4. Multiplication of Discontinuous Solutions

Consider (35) rewritten as

$$\{[p](u_1 - c) + \gamma p_1[u] + [u][p](\gamma + 1)K\}K' = 0. \tag{39}$$

If K belongs to the set of C^1 functions, then the equation above only admits $K = \text{constant}$ as a solution. If K is allowed to be a discontinuous solution, then K' cannot be factorized from (39). Here we follow the mathematical construction of generalized functions proposed by Colombeau in [2] and Colombeau and Le Roux in [4]. The main advantage in using this construction is that most of the operations admissible with smooth functions can be defined for discontinuous functions, including differentiation. (See Appendix A). We restrict our attention to the Heaviside function and its derivative, the Dirac delta function.

The Heaviside function is such that

$$H(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad x > 0. \end{cases} \tag{40}$$

The Dirac delta function $\delta(x)$ is 0 in $[-\infty, 0[\cup]0, \infty]$ and is such that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Let $C_0^\infty(\Omega)$ denote the set of all C^∞ functions on Ω with compact support. Given $G_1(x)$, $G_2(x)$, and the test function $\Psi(x) \in C_0^\infty(\Omega)$, if

$$\int_\Omega [G_1(x) - G_2(x)]\Psi(x)\, dx = 0$$

for all $\Psi(x)$, then we say that $G_1(x)$ and $G_2(x)$ are associated and write $G_1(x) \sim G_2(x)$. According to the definition above, $G_1 \sim G_2$ does not imply that $G_3 G_1 \sim G_3 G_2$, where G_3 is some other function. Consider, for example, H^n (the nth power of the Heaviside funcion H). We can show that $H^n \sim H$, but $H^n H'$ is not associated with HH'. In fact, we have

$$H^{n-1}H' \sim \frac{1}{n} H'. \tag{41}$$

Note that if we replace the associative symbol $\sim$ in (41) with the equal sign, then we obtain, for example, $HH' = \frac{1}{2}H'$ and multiplication by H yields $H^2H' = \frac{1}{2}HH'$. Now by substituting again an equal sign into (41) we get $\frac{1}{3}H' = \frac{1}{2}\frac{1}{2}H'$, which is absurd.

In conclusion, if we replace the equal sign in (35) with the symbol $\sim$, then the subsequent integration is fully justified.

In another example, we present a case in which the Heaviside functions describing the shock are not equal because of their behavior at zero. Consider the mass conservation equation

$$\rho_t + \rho u_x + u\rho_x = 0. \tag{42}$$

If we seek a solution for ρ of the form

$$\rho = \rho_1 + [\rho]I(\xi), \tag{43}$$

substituting in (42), we obtain

$$\frac{dI}{dK} + \frac{I}{a + K} + \frac{\rho_1}{[\rho]}\frac{1}{a + K} = 0. \tag{44}$$

The solution to this equation is

$$I = \frac{k}{a + K} - \frac{\rho_1}{[\rho]}, \tag{45}$$

where k is the constant of integration. For $x \to -\infty$ both K and $I \to 0$, therefore

$$k = \frac{\rho_1}{[\rho]} a. \tag{46}$$

For $x \to \infty$ both K and $I \to 1$, so that

$$a + 1 = -\frac{\rho_1}{[\rho]}, \tag{47}$$

(21) with the above equation reduces to the jump condition expressed by (26). Finally,

$$I = \frac{a+1}{a+K} K \tag{48}$$

is obtained. Equation (48) is significant because it shows that the microscopic behavior of the Heaviside function that describes the ρ jumps is not the same as that for p or u. For example, consider K defined as

$$K(\xi) = \lim_{\alpha \to \infty} \left(\frac{\tanh(\alpha\xi) + 1}{2} \right), \tag{49}$$

$K(0)$ is defined as

$$\lim_{\xi \to 0} K(\xi),$$

therefore it follows that $K(0) = \frac{1}{2}$. For I it follows from (48) that

$$I(0) = \lim_{\xi \to 0} \frac{a+1}{a + K(\xi)} K(\xi), \tag{50}$$

hence

$$I(0) = \frac{a+1}{2a+1} \neq \frac{1}{2}. \tag{51}$$

5. Entropy Structure in a Shock Wave

As pointed out in Section 2, although a "conservation law" can be written for entropy, this law does not lead to the correct jump. Although this fact is well known, the reason why is not well understood. In this section we show that the shock profile that corresponds to the entropy cannot be represented by a Heaviside function; hence, the entropy propagation equation does not yield the correct jumps.

Consider (28) and (34). Multiply (28) by γ and then divide by v; divide (34) by p and add the two equations. We obtain the following after simplifying:

$$\gamma(p_1 + [p]L)[v]H' + (v_1 + [v]H)[p]L' = 0. \tag{52}$$

Because $H = L$ we have $H'L \sim HL' \sim \frac{1}{2}H'$, we get the RH jump,

$$v_r[p_1(\gamma - 1) + p_r(\gamma + 1)] = v_1[p_r(\gamma - 1) + p_1(\gamma + 1)]. \tag{53}$$

This shows that the equation

$$\frac{p_t}{p} + \gamma\frac{v_t}{v} + u\left(\frac{p_x}{p} + \gamma\frac{v_x}{v}\right) = 0 \tag{54}$$

has a valid jump. However, (54) may be rewritten as

$$S_t + uS_x = 0, \tag{55}$$

because $dS = dp/p + \gamma\, dv/v$. Now we look for a solution for S of the form

$$S = S_1 + [S]T(\xi), \tag{56}$$

where T is a Heaviside function. If we substitute (56) and (16) into (55), then we get

$$\{-cT' + (u_1 + [u]K)T'\}[S] = 0, \tag{57}$$

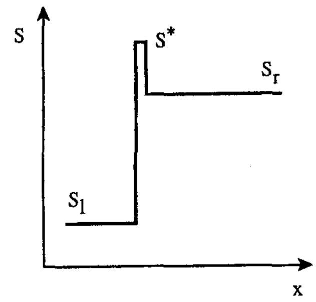

Figure 3. Entropy structure across inviscid shock wave.

either the expression within the braces must be zero or the jump $[S]$ must be zero. In general, because no relation exists between T and K, the expression within the braces is not zero, hence, we conclude that (57) gives the wrong jump, namely $[S]=0$. In actuality, the problem lies elsewhere. In going from (54) to (55), we have gone from an equation with two jumps $[p]$ and $[v]$ to an equation with a single jump $[S]$. By combining the two equations, we have lost some information; furthermore, the assumption that the solution can be expressed as in (56) is incorrect, which will be shown below.

Consider the following. Without loss of generality, let $v_l=1$ and $p_l=1$ and take S_l as the reference state for entropy. Because S by definition is

$$S=\ln p+\gamma \ln v \tag{58}$$

we obtain, given (15) and (17) and the fact that $L=H$,

$$\frac{dS}{dH}=\frac{(1+[v]H)^{\gamma-1}\{(\gamma+1)[p][v]H+[p]+\gamma[v]\}}{pv^{\gamma}}. \tag{59}$$

Because the jump $[p]$ is

$$[p]=-\frac{2\gamma[v]}{(\gamma+1)[v]+2}, \tag{60}$$

we can obtain the following by substituting and simplifying:

$$\frac{dS}{dH}=\frac{\{1+[v]H\}^{\gamma-1}}{pv^{\gamma}}(1-2H)\frac{\gamma(\gamma+1)[v][v]}{(\gamma+1)[v]+2}. \tag{61}$$

When $H=0$ we have $dS/dH>0$, while if $H=1$ we have $dS/dH<0$; hence, S has a maximum when H takes values from zero to one. As a result, we conclude that S cannot be described by a simple Heaviside function but rather by the sum of two Heaviside functions

$$S=S_l+(S^*-S_l)T(\xi)+(S_r-S^*)N(\xi), \tag{62}$$

where S^* is the value of S at the maximum. (See Figure 3.) The figure shows clearly that by using a single equation such as (55) for the entropy, we cannot determine the two jumps that actually represent the structure of the entropy at the shock. A more technical proof of the fact that entropy cannot be represented by a single step function is given in Appendix B.

6. Computational Remarks

In this section we discuss some aspects related to the design of a shock-capturing scheme for systems of equations written in quasi-linear form, see also [3]. Consider the equation

$$u_t+\left(\frac{u^2}{2}\right)_x=0 \tag{63}$$

and its quasi-linear form

$$u_t+uu_x=0, \tag{64}$$

where

$$u = u_l + [u]\,Y(x - ct), \tag{65}$$

Y being a step function and $[u] = u_r - u_l$. It is well known that a conservative discretization of (63) leads to a shock-capturing scheme. In contrast, if a characteristic scheme is used for (64), the speed of the shock is not properly computed.

According to the theory of multiplication of distribution, we discretize the product uu_x in a way that mimics the behavior of the product YY' at the shock.

Substituting (65) into (63) and considering (41), we obtain

$$uu_x = \frac{u_r + u_l}{2}[u]\,Y'.$$

If we discretize $[u]\,Y'$ as $(u_r - u_l)/\Delta x$, and we apply a characteristic scheme in which the derivative u_x is upwinded according to the speed $(u_r + u_l)/2$, we obtain the same results that we would obtain using a conservative scheme for integrating (63).

In extending this method to the Euler equation we encounter the following difficulty: the characteristic variables are quantities that cannot be modeled with a step function, for the same reason as for entropy. Therefore a shock-capturing characteristic scheme for the Euler equations cannot be devised by a direct extension of these ideas. A way to overcome this difficulty is presently under investigation.

7. Conclusions

The shock jump conditions for the Euler equations in their primitive form were derived using generalized functions. It was shown that the structure of the shock profile is not the same for all variables. A study of the entropy propagation equation showed that if the shock structure for the entropy is represented by a single Heaviside function, then the wrong entropy jump is obtained. It was then shown that the proper representation of the entropy profile requires two Heaviside functions, but not all the information required to specify this profile can be obtained from the entropy propagation equation.

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Appendix A. Some Definitions About Generalized Functions

Here we briefly discuss generalized functions. For more details on this subject, see [9] and [5].

A *test funcion* $\varphi(x)$ exists such that:

1. $\varphi(x)$ is C^∞.
2. $\varphi(x)$ has a compact support (i.e., $\varphi(x)$ vanishes outside of some compact interval $[a, b]$).

Furthermore, a sequence $\varphi_n(x)$ of test functions converges to 0 if:

1. For each k, the sequence of the kth derivative $\varphi_1^{(k)}(x)$, $\varphi_2^{(k)}(x), \ldots$ converges uniformly to zero.
2. Every $\varphi_n(x)$ vanishes outside a given interval $[a, b]$.

A *generalized function* T is a mapping from the set $\mathscr{D}$ of all test functions into the real or complex numbers such that if $\langle \cdot, \cdot \rangle$ is the interval product operator, then we have:

1. $\langle T, a\varphi(x) + b\psi(x) \rangle = a\langle T, \varphi(x) \rangle + b\langle T, \psi(x) \rangle$.
2. If $\varphi_n(x)$ converges to zero in the manner defined above, then $\langle T, \varphi_n(x) \rangle$ convergence to zero.

Generalized functions are useful because their derivatives are always well defined. In fact, we have the following definition:

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle.$$

Appendix B. Entropy Structure Proof

Consider (58) and substitute (12) and (14). Because $H = L$, we have

$$S = \ln(1 + [p]H) + \gamma \ln(1 + [v]H).$$

The equation above can be written as

$$S = \ln(p_r)F + \gamma \ln(v_r)G,$$

where F and G are two Heaviside functions defined as

$$F = \frac{\ln(1 + [p]H)}{\ln(p_r)}, \qquad G = \frac{\ln(1 + [v]H)}{\ln(v_r)}.$$

We now show that $F \sim G$, although they are not identical in the sense that FH' is not associated with GH'. We take a test function $\varphi(x)$ (defined in Appendix A) and compute the following integrals:

$$\int_{-\infty}^{\infty} F\varphi(x)\,dx = \int_{-\infty}^{\infty} \frac{\ln(1 + [p]H)}{\ln(p_r)} \varphi(x)\,dx = \int_{0}^{\infty} \varphi(x)\,dx,$$

$$\int_{-\infty}^{\infty} G\varphi(x)\,dx = \int_{-\infty}^{\infty} \frac{\ln(1 + [v]H)}{\ln(v_r)} \varphi(x)\,dx = \int_{0}^{\infty} \varphi(x)\,dx,$$

from which we conclude $F \sim G$ in accordance with the definition of association. To verify whether $FH' \sim GH'$, we have

$$\int_{-\infty}^{\infty} FH'\,dx = \int_{0}^{1} \frac{\ln(1 + [p]H)}{\ln(p_r)}\,dH = 1 + \frac{1}{[p]} - \frac{1}{\ln(p_r)},$$

$$\int_{-\infty}^{\infty} GH'\,dx = \int_{0}^{1} \frac{\ln(1 + [v]H)}{\ln(v_r)}\,dH = 1 + \frac{1}{[v]} - \frac{1}{\ln(v_r)}.$$

Thus, we conclude that FH' and GH' are not associated. Therefore, F and G must be considered as two locally different Heaviside functions, although they are associated.

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