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FOUR VALUED SEMANTICS AND THE LIAR*

1. INTRODUCTION

The present paper interweaves various themes. Two main themes are four valued logic and the Liar Paradox. Each main theme divides into two interconnected subthemes: four valued logic into valuation schemes and structure theory; the Liar into iterations and access to structures on the one hand, and comparison of structures on the other hand.

Four valued logic has values * (undefined), *T*, *F* and *TF* (overdefined). A valuation scheme is a method of evaluating formulas of an essentially classical language (say of propositional logic or first order predicate logic) in four valued models, in such a way that when the relevant inputs for evaluating a formula are classical, then its evaluation is classical. If the valuation scheme is truthfunctional one can simply say that it tells us how to extend the definitions of the classical connectives to the four values. Of course one could also consider non-classical vocabularly or the possibility that some classical connectives turn out to be ambiguous when extended. Many of the results of the paper will work also for this more general case.

A basic constraint on valuation schemes is monotonicity (see Section 2). This constraint follows more or less directly from the basic explanation of what the truth values are.

I consider two valuation schemes: one truthfunctional; one non-truthfunctional. The first, extended Strong Kleene, is well known from the literature. The second, a generalization of Van Fraassen Supervaluations, is introduced in the paper.

The schemes considered have certain important properties, like selfduality in the case of Strong Kleene and overlap preservation in the case of Van Fraassen Valuations (see Section 2). These properties are studied in Section 2 and extensively used in Section 3. It would be nice to have a further or alternative explanation of the semantics that would turn, e.g., selfduality into a constraint on schemes, but I know of no such explanation.

One attractive feature of four valued logic for the study of the Liar

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Paradox is the possibility of making certain intuitive distinctions *within one single model*. Consider the Liar and the Samesayer, "This sentence is true". In Section 3 I present various models in which the Liar is both true and false and the Samesayer neither true nor false. The intuitive idea here is that the Liar *must* be true, *must* be false; the Samesayer *need not* be true, *need not* be false. It is instructive to compare this way of viewing things to Kripke's (dual) way of drawing these distinctions: for Kripke the Liar *cannot* be true, *cannot* be false; the Samesayer *can* be true, *can* be false.

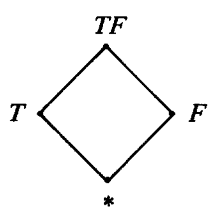
Another advantage of four valued logic is technical convenience. The structures naturally associated with the four values are certain complete lattices. To work with these is in many respects simpler than to work with, for example, the structures naturally associated with partial valued logic (i.e., ccpo's, see Section 2).

A further point is that there are methods of four valued access to partial valued fixed points, like Kripke's maximal intrinsic fixed point (see Section 3). Thus four valued logic may hold some attraction even for the staunch believer that the ultimate form of a 'solution' of the Liar is a Kripkean partial valued fixed point.

I do not offer any definite 'solution' of the Liar, rather my aim is first to show how to obtain 'solutions' by combining the possibilities of four valued semantics with recent ideas of Herzberger, Gupta, Belnap and more generally to study the problem of access to various salient structures. Secondly I wish to compare the various structures so obtained with each other, with some Kripkean partial valued fixed points and with certain structures associated with the Gupta/Belnap processes.

2. FOUR VALUED LOGIC AND ITS STRUCTURES

The values of four valued logic are structured:



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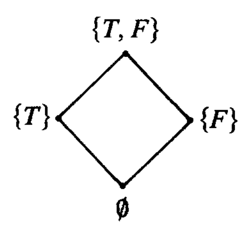


We will call the above complete lattice $\mathbb{T} = \langle\{T, F, *, TF\}, \leqslant\rangle$. *Part* of what it means to say that the values are structured is that we will only regard valuation schemes monotonic in $\mathbb{T}$ as acceptable.

The value $*$ stands for 'underdefined', 'underspecified', TF for 'overdefined', 'overspecified'. The intuitive picture behind this nomenclature is that Reality + Semantics produce, impose, accumulate truth-values. This is in contrast to the picture where Reality + Semantics put constraints on admissible truthvalues. In the second picture TF for 'underdefined' or 'T and F are both admitted', $*$ for 'overdefined' or 'T and F are both excluded' are more apt.

Four valued logic based on $\mathbb{T}$ is an extensive subject, where many basic questions are still unanswered. Pioneering studies are Dunn [5, 6] and Belnap [1]. Here I just want to concentrate on two basic aspects: the description of the structures, relevant for the study of the paradoxes, that can be generated from $\mathbb{T}$ and the construction of two four valued valuation schemes corresponding to Strong Kleene and Van Fraassen Supervaluations in the partial (three valued) case. For reasons of presentation I treat first 'Strong Kleene', then 'Structures', then 'Van Fraassen'.

2.1. *The Strong Kleene Scheme*

The proper generalization of the Strong Kleene Scheme is well known from the literature – as is the corresponding notion of validity. One pleasant way of introducing it is in terms of truth values as subsets of $\{T, F\}$:

$$T \in [\phi \wedge \psi] \quad \text{iff} \quad T \in [\phi] \quad \text{and} \quad T \in [\psi],$$

$$F \in [\phi \wedge \psi] \quad \text{iff} \quad F \in [\phi] \quad \text{or} \quad F \in [\psi],$$

$$T \in [\phi \vee \psi] \quad \text{iff} \quad T \in [\phi] \quad \text{or} \quad T \in [\psi],$$

$$F \in [\phi \vee \psi] \quad \text{iff} \quad F \in [\phi] \quad \text{and} \quad F \in [\psi],$$

$$T \in [\neg \phi] \quad \text{iff} \quad F \in [\phi],$$

$$F \in [\neg \phi] \quad \text{iff} \quad T \in [\phi],$$

$$T \in [\forall x \phi(x)] \quad \text{iff for all} \quad m \in M, T \in [\phi(\mathbf{m})],$$

$$F \in [\forall x \phi(x)] \quad \text{iff for some} \quad m \in M, F \in [\phi(\mathbf{m})],$$

$$T \in [\exists x \phi(x)] \quad \text{iff for some} \quad m \in M, T \in [\phi(\mathbf{m})],$$

$$F \in [\exists x \phi(x)] \quad \text{iff for all} \quad m \in M, F \in [\phi(\mathbf{m})].$$

Here M is a given domain, and we assume for convenience that constants $\mathbf{m}$ for all m in M are in the language.

Note that above interpretation differs from the obvious alternative, where we take, e.g., $[\phi \wedge \psi]_1 := \{\wedge(\tau, \tau') \mid \tau \in [\phi]_1, \tau' \in [\psi]_1\}$. Here $\wedge$ is the usual classical truthfunction. The last interpretation is Weak Kleene with respect to $\emptyset$, Strong Kleene with respect to $\{T, F\}$.

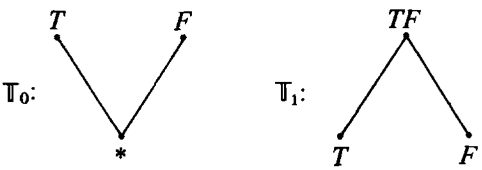
As usual we will not treat truth simpliciter but truth at an assignment or truth relative to a model. Assignments map atomic ϕ into $\{T, F, *, TF\}$. We write '$[\psi]f$' for the Strong Kleene Valuation of ψ at f. Let $\leqslant$ be the pointwise induced ordering of $\mathbb{T}$ on assignments. As is easily checked: if $f \leqslant g$ then $[\psi]f \leqslant [\psi]g$ or: Strong Kleene is *monotonic* in $\leqslant$.

Define $(\hat{\ })$: $\{T, F, *, TF\} \to \{T, F, *, TF\}$ as: $\hat{T} := T$, $\hat{F} := F$, $\hat{*} := TF$, $\widehat{TF} := *$. Define for assignments f: $\hat{f}(\phi) := \widehat{f(\phi)}$ (ϕ atomic). By an easy induction: $[\psi]\hat{f} = \widehat{[\psi]f}$. In other words: Strong Kleene is *self dual*.

Note that $[\,.\,]_1$ is monotonic but not self dual. $[\,.\,]_1$ *does* satisfy: $[\psi]_1\hat{f} \leqslant \widehat{[\psi]_1 f}$. (In Peter Woodruff's terminology this means that $[\,.\,]_1$ preserves overlap.)

2.2. *Structures*

$\mathbb{T}$ contains the substructures $\mathbb{T}_0$ and $\mathbb{T}_1$:



$\mathbb{T}_0$ is the usual structure for partial valued logic, $\mathbb{T}_1$ the structure for overdefined valued logic. We will need a number of basic facts about structures generated from $\mathbb{T}_0$, $\mathbb{T}_1$. Clearly $\mathbb{T}_1$ is $\mathbb{T}_0$ on its head (or $(\hat{\ })$ is a dual isomorphism between $\mathbb{T}_0$ and $\mathbb{T}_1$) so we may concentrate on $\mathbb{T}_0$.

The treatment of 'partial structures' is quite brief; I hope to publish a fuller treatment elsewhere.

2.2.1. *Complete Coherent Partial Orders*

2.2.1.1. DEFINITION. Let $D = \langle D, \leqslant \rangle$ be a po (partial order).

(a) A subset X of D is *consistent* if for all x, y in X there is a z in D such that $x \leqslant z$, $y \leqslant z$.

(b) D is a ccpo (complete, coherent partial order) if every consistent subset X of D has a supremum $\sqcup X(D)$ (or if no confusion is possible: $\sqcup X$).

2.2.1.2. FACT. Let D be a ccpo. The following are easily verified:

(a) $\mathbb{T}_0$ is a ccpo.

(b) D has a bottom $*$.

(c) Every *non empty* subset X of D has an infinimum $\sqcap X(D)$.

(d) For every d in D there is a maximal d' in D such that $d \leqslant d'$.

2.2.1.3. DEFINITION. Let $D = \langle D, \leqslant \rangle$ be a po.

(a) a subset X of D is *closed* under $\sqcup$ if for every subset Y of X for which $\sqcup Y$ exists, $\sqcup Y \in X$.

(b) Let X be a subset of D, we write: '$\langle X, \leqslant \rangle$' for $\langle X, \leqslant \restriction X \rangle$.

(c) Let f: $D \to D$. Define:
Fix $(D, f) := \{d \in D \mid f(d) = d\}$,
Fix $(D, f) := \langle \text{Fix}\,(D, f), \leqslant \rangle$.

(d) Let A be any set. Define:
$D^A := \langle D^A, \leqslant \rangle$ where $f \leqslant g$ if for all d in D, $f(d) \leqslant g(d)$.
In other words $\leqslant$ is the pointwise induced ordering on D^A.

2.2.1.4. FACT. Let $D = \langle D, \leqslant \rangle$ be a ccpo.

(a) Let X be a subset of D, closed under $\sqcup$ then $\langle X, \leqslant \rangle$ is a ccpo.

(b) Let f be monotonic on D, then Fix (D, f) is a ccpo.

(c) Let A be any set, then D^A is a ccpo.

2.2.1.5. DEFINITION. Let $D = \langle D, \leqslant \rangle$ be a ccpo. d in D is called *intrinsic* (in D) if for all d' in D $\{d, d'\}$ is consistent (or: $\sqcup\{d, d'\}$ exists.)

2.2.1.6. FACT. Let $D = \langle D, \leqslant \rangle$ be a ccpo, let I be the set of intrinsic points, M the set of maximal points. We have: there is a maximal intrinsic point i and $i = \sqcup I = \sqcap M$.

Proof. Let d be intrinsic, m maximal. Then $\sqcup\{d, m\}$ exists and $\sqcup\{d, m\} = m$. Hence $d \leqslant m$. Conclude $d \leqslant \sqcap M$. On the other hand $\sqcap M$ is intrinsic: let e be any element of D. For some maximal m, $e \leqslant m$, hence $e \leqslant m$, $\sqcap M \leqslant m$ so $\{e, \sqcap M\}$ is consistent. Trivially: $\sqcap M = \sqcup I$. □

2.2.1.7. REMARK. The notion of intrinsicness was introduced by Kripke [11] and independently in the context of Computer Science/Recursion Theory by Manna and Shamir; see, e.g., [12].

After this excursion into partial valued structures we return to our four values. Just the ordering gives too little information, so we consider structures that are a bit richer. A consequence is that a new definition of $\mathbb{T}$ is necessary.

2.2.2. DEFINITION

(a) $\mathbb{T} := \langle\{T, F, *, TF\}, (\hat{\ }), \leqslant\rangle$, where $\leqslant$ is as usual and $(\hat{T}) := T$, $(\hat{F}) := F$, $(\hat{*}) := TF$, $(\widehat{TF}) := *$.

(b) $\boldsymbol{E} := \langle E, (\hat{\ }), \leqslant\rangle$ is a *complete selfdual lattice* or *csl* if $\langle E, \leqslant\rangle$ is a complete lattice and $(\hat{\ })$ is a selfdual isomorphism of $\boldsymbol{E}$, i.e., for all x, y in E, $x \leqslant y \Rightarrow \hat{y} \leqslant \hat{x}$ and for all x in E, $\hat{\hat{x}} = x$.

(c) Let $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant\rangle$ be a csl. Define:
$E{-}U := \{x \in E \mid x = \hat{x}\}$ ('U' for: Unique Valued),
E-under $:= \{x \in E \mid x \leqslant \hat{x}\}$,
$\boldsymbol{E}$-under $:= \langle E\text{-under}, \leqslant\rangle$,
E-over $:= \{x \in E \mid \hat{x} \leqslant x\}$,
$\boldsymbol{E}$-over $:= \langle E\text{-over}, \geqslant\rangle$.
For x in E:
x-under $:= \{y \in E\text{-under} \mid y \leqslant x\}$,
$\boldsymbol{x}$-under $:= \langle x\text{-under}, \leqslant\rangle$,
x-over $:= \{y \in E\text{-over} \mid y \geqslant x\}$,
$\boldsymbol{x}$-over $:= \langle x\text{-over}, \geqslant\rangle$.

(d) $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant\rangle$ is a rich csl or rcsl iff $\boldsymbol{E}$ is a csl and:
d_1: for every x in E-under there is a u in $E{-}U$ with $x \leqslant u$.

d_2: for every y in E, $y = \sqcup y$-under$(\boldsymbol{E})$.

(e) Let $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant\rangle$ be a csl, A a set, then $\boldsymbol{E}^A := \langle E^A, (\hat{\ }), \leqslant\rangle$, where $(\hat{f})(a) := \widehat{f(a)}$, and $\leqslant$ is the pointwise induced ordering.

(f) Let $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant\rangle$ be a csl, f monotonic on $\boldsymbol{E}$ and selfdual on $\boldsymbol{E}$ (i.e., $f(\hat{x}) = \widehat{f(x)})$, then $\text{Fix}(E, f) := \{x \in E \mid f(x) = x\}$ and $\text{Fix}(\boldsymbol{E}, f) := \langle \text{Fix}(E, f), (\hat{\ }), \leqslant\rangle$.

2.2.3. FACT

(a) $\mathbb{T}$ is a rcsl.

(b) Let $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant\rangle$ be a csl, A a set. Then $\boldsymbol{E}^A$ is a csl. If $\boldsymbol{E}$ is a rcsl then $\boldsymbol{E}^A$ is a rcsl.

(c) Let $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant\rangle$ be a csl, f monotonic and selfdual on $\boldsymbol{E}$. Then $\text{Fix}(\boldsymbol{E}, f)$ is a csl.

Proof. The only problematic point is to show that $\text{Fix}(\boldsymbol{E}, f)$ is a complete lattice. One takes, e.g., for X a set of fixed points of f: $\sqcup X(\text{Fix}(\boldsymbol{E}, f)) := (\sqcup X(\boldsymbol{E}))_f$. Here $(\ )_f$ is defined in Section 3.3.3. $(\sqcup X(\boldsymbol{E}))_f$ will be the least fixed point above the elements of X □

2.2.4. FACT. Let $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant\rangle$ be a csl. Then

(a)
$y \in E$-under and $x \leqslant y \Rightarrow x \in E$-under,
$y \in E$-over and $x \geqslant y \Rightarrow x \in E$-over,
$u, v \in E - U$ and $u \leqslant v \Rightarrow u = v$.

(b) $\boldsymbol{E}$-under, $\boldsymbol{E}$-over, are ccpo's.

(c) Let $x \in E$, then $\boldsymbol{x}$-under, $\boldsymbol{x}$-over are ccpo's.

Proof. (a) is routine, (c) follows from (b).

(b) Let X be consistent in $\boldsymbol{E}$-under. We show $\sqcup X(\boldsymbol{E}) \leqslant \widehat{\sqcup X(\boldsymbol{E})} = \sqcap \hat{X}(\boldsymbol{E})$, i.e., for every x in X, y in $\hat{X}$, $x \leqslant y$. Consider x in X, y in $\hat{X}$, then $x, \hat{y} \in X$. X is consistent in $\boldsymbol{E}$-under, so there is a z in E-under such that $x \leqslant z$, $\hat{y} \leqslant z$. Hence $x \leqslant z \leqslant \hat{z} \leqslant y$. As is easily seen: $\sqcup X(\boldsymbol{E}\text{-under}) = \sqcup X(\boldsymbol{E})$. □

2.2.5. DEFINITION. Peter Woodruff made the nice discovery that a natural generalization of consistency in a ccpo is overlap in a csl.

Let $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant \rangle$ be a csl.

(a) x overlaps y: $\Leftrightarrow x \circ y$

$:\Leftrightarrow x \leqslant \hat{y}.$

We write $x \circ Y$ for: for all $y \in Y, x \circ y$.

Note the existence of the dual notion: x underlaps y: $\Leftrightarrow \hat{x} \leqslant y$.

(b) x is *overlap Y-intrinsic*: $\Leftrightarrow x \circ Y$.

(c) $x^{\Delta} := x \sqcap \hat{x}$.
(On $\mathbb{T}$: $T^{\Delta} = T$, $F^{\Delta} = F$, $*^{\Delta} = TF^{\Delta} = *$.)

2.2.6. FACT. Under the conditions of Section 2.2.5:

(a) $x \circ y \Rightarrow y \circ x$,

(b) $z \leqslant x, x \circ y \Rightarrow z \circ y$,

(c) $x \circ x \Leftrightarrow x$ in E-under,

(d) if x, y in E-under, then ($x \circ y \Leftrightarrow \{x, y\}$ is consistent in $\boldsymbol{E}$-under),

(e) $x \circ Y \Leftrightarrow x \circ (\sqcup Y(\boldsymbol{E}))$,

(f) $x \circ$ (y-under) if $x \circ y$, in case $\boldsymbol{E}$ is rich we have: if and only if,

(g) (x in y-under and $x \circ$ (y-under)) $\Leftrightarrow x$ is intrinsic in y-under,

(h) $x \circ \hat{x}$ and $\hat{x}$ is the maximal z such that $x \circ z$,

(i) x^{Δ} is the maximal intrinsic point in x-under.

Proof. (a), (b), (c), (e), (h) are routine, (f) follows from (e), (g) from (d).

(d) Suppose x, y are in E-under. First assume $x \circ y$, i.e., $x \leqslant \hat{y}$. Clearly $x \leqslant x \sqcup y, y \leqslant x \sqcup y$. Moreover $x \leqslant \hat{x}, x \leqslant \hat{y}, y \leqslant \hat{y}, y \leqslant \hat{x}$, hence $x \sqcup y \leqslant \hat{x} \sqcap \hat{y}$. But $\hat{x} \sqcap \hat{y} = \widehat{x \sqcup y}$. Conclude $x \sqcup y$ is in E-under and $\{x, y\}$ is consistent in $\boldsymbol{E}$-under. Secondly if $\{x, y\}$ is consistent in $\boldsymbol{E}$-under, then for some z in E-under $x \leqslant z$, $y \leqslant z$, hence $x \leqslant z \leqslant \hat{z} \leqslant \hat{y}$.

(i) $x \sqcap \hat{x} \leqslant x \sqcup \hat{x} = \widehat{x \sqcap \hat{x}}$, so $x \sqcap \hat{x}$ is in E-under, hence $x \sqcap \hat{x}$ is in x-under. $x \sqcap \hat{x} \leqslant \hat{x}$ so $x \sqcap \hat{x} \circ x$ and $x \sqcap \hat{x} \circ$ (x-under) by f. By g: $x \sqcap \hat{x}$ is intrinsic in x-under. Finally if y is intrinsic in x-under then by $h, y \leqslant \hat{x}$, hence $y \leqslant x \sqcap \hat{x}$. □

2.2.7. FACT. Let $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant \rangle$ be a rcsl, then

(a) For y in E: $y = \sqcap y\text{-over}\,(\boldsymbol{E})$.

(b) Let x be in E-under, y in E-over, $x \leqslant y$. Then there is a u in $E - U$ with $x \leqslant u \leqslant y$.

(c) Define x-up: $= \{u \in E - U \mid u \geqslant x\}$, x-down: $= \{u \in E - U \mid u \leqslant x\}$.
We have:
$x \in E\text{-under} \Rightarrow x = \sqcap x\text{-up}\,(\boldsymbol{E})$,
$x \in E\text{-over} \Rightarrow x = \sqcup x\text{-down}\,(\boldsymbol{E})$.

Proof. (a) $y = \hat{\hat{y}} = (\sqcup \hat{y}\text{-under}\,(\boldsymbol{E}))\hat{\ } = \sqcap y\text{-over}\,(\boldsymbol{E})$.

(b) $x \leqslant x \sqcup \hat{y} \leqslant \hat{x} \sqcap y \leqslant y$ and $\widehat{x \sqcup \hat{y}} = \hat{x} \sqcap y$, so $x \sqcup \hat{y} \in E$-under, hence there is a u in $E - U$ such that $x \sqcup \hat{y} \leqslant u$, and $u = \hat{u} \leqslant \widehat{x \sqcup \hat{y}} = \hat{x} \sqcap y$. Conclude $x \leqslant u \leqslant y$.

(c) We treat, e.g., the x-up case: suppose $x \in E$-under. $x = \sqcap x\text{-over}\,(\boldsymbol{E})$ by (a). Clearly x-up $\subseteq$ x-over. Moreover by (b): for every y in x-over there is a u in $E - U$ with $x \leqslant u \leqslant y$. Hence x-up *minorizes* x-over. Conclude: $\sqcap x\text{-up}\,(\boldsymbol{E}) = \sqcap x\text{-over}\,(\boldsymbol{E})$. □

2.2.8. U-REPRESENTATION THEOREM. Let $\boldsymbol{E}$ be a rcsl then:

(a) $x \in E$-under iff $x = \sqcap S(\boldsymbol{E})$ for some non empty $S \subseteq E - U$.

(b) $x \in E$-over iff $x = \sqcup S(\boldsymbol{E})$ for some non empty $S \subseteq E - U$.

(c) for any x in E:
$x = \sqcup \{\sqcap S(\boldsymbol{E}) \mid S \subseteq E - U, S \neq \emptyset, \sqcap S(\boldsymbol{E}) \leqslant x\}(\boldsymbol{E})$,
$= \sqcap \{\sqcup S(\boldsymbol{E}) \mid S \subseteq E - U, S \neq \emptyset, \sqcup S(\boldsymbol{E}) \geqslant x\}(\boldsymbol{E})$.

Proof. (a) and (b) are directly from 2.2.7 (c), the fact that S is non empty follows from 2.2.2 (d_1). (c) rephrases 2.2.2 (d_2) and 2.2.7 (a) using (a) and (b). □

In the present paper I don't take a stand on what a valuation scheme for four valued logic should be. Two interesting restrictions are selfduality, and overlap preservation. Remember that for f from a csl to a csl: f is selfdual iff $f(\hat{x}) = \widehat{f(x)}$ and that f preserves overlap iff for all x, y in E, $x \circ y \Rightarrow f(x) \circ f(y)$. One may show: f preserves overlap iff $f(\hat{x}) \leqslant \widehat{f(x)}$.

The following result partly describes the world in which the various possible schemes live.

2.2.9. DEFINITION. Let $\boldsymbol{E}_1 = \langle E_1, (\,\hat{}\,), \leqslant\rangle$, $\boldsymbol{E}_2 = \langle E_2, (\,\hat{}\,), \leqslant\rangle$ be csl's. We conveniently confuse $(\,\hat{}\,), \leqslant$ on $\boldsymbol{E}_1$ and $(\,\hat{}\,), \leqslant$ on $\boldsymbol{E}_2$. Define: $\mathrm{Mon}(E_1, E_2) := \{f \mid f \text{ monotonic from } \boldsymbol{E}_1 \text{ to } \boldsymbol{E}_2\}$ and $\mathrm{Mon}(\boldsymbol{E}_1, \boldsymbol{E}_2) := \langle \mathrm{Mon}(E_1, E_2), (\,\check{}\,), \leqslant\rangle$, where $\check{f}(x) := \widehat{f(\hat{x})}$ and $\leqslant$ is the pointwise induced ordering.

2.2.10. THEOREM. Let $\boldsymbol{E}_1 = \langle E_1, (\,\hat{}\,), \leqslant\rangle$, $\boldsymbol{E}_2 = \langle E_2, (\,\hat{}\,), \leqslant\rangle$ be csl's. We have

(a) $\mathrm{Mon}(\boldsymbol{E}_1, \boldsymbol{E}_2)$ is a csl.

(b) f is in $\mathrm{Mon}(E_1, E_2)$-under $\Leftrightarrow$ f is overlap preserving.

(c) $f = \check{f} \Leftrightarrow f$ is selfdual.

(d) Suppose $\boldsymbol{E}_2$ is a rcsl, then $\mathrm{Mon}(\boldsymbol{E}_1, \boldsymbol{E}_2)$ is a rcsl.

Proof (sketch). Only (d) is not immediate.
Clearly $\langle \mathrm{Mon}(E_1, E_2), \leqslant\rangle$ is a complete lattice. We check 2.2.2 (d_1, d_2):

(d_2): Let $e \in E_1$, $x \in E_2$-under. Define $g_{e,x}: E_1 \to E_2$ as follows:

$$g_{e,x}(e') := \begin{cases} x \text{ if } e \leqslant e', \\ * \text{ otherwise.} \end{cases}$$

Clearly $g_{e,x}$ is monotonic and overlap preserving, i.e., $g_{e,x}(\hat{e}') \leqslant \widehat{g_{e,x}(e')}$, or: $g_{e,x} \in \mathrm{Mon}(E_1, E_2)$-under.

Consider any f in $\mathrm{Mon}(E_1, E_2)$, we have, $\boldsymbol{E}_2$ being rich:

$$f = \sqcup\{g_{e_1,x} \mid f(e_1) = e_2, \\ x \in E_2\text{-under}, x \leqslant e_2\}\,(\mathrm{Mon}(\boldsymbol{E}_1, \boldsymbol{E}_2)).$$

Hence: $f = \sqcup f$-under $(\mathrm{Mon}(\boldsymbol{E}_1, \boldsymbol{E}_2))$.

(d_1): It would be sufficient to show that every g in $\mathrm{Mon}(E_1, E_2)$-under that is not selfdual can be extended to a g' in $\mathrm{Mon}(E_1, E_2)$-under with $g < g'$. I give a slightly more elaborate construction that is a bit more informative.

Consider $\boldsymbol{X} := \{X \subseteq E_1 \mid \text{for every } x, y \text{ in } X \textit{ not } \hat{x} \leqslant y\}$. Using Zorn's

Lemma we can find a maximal X_0 in X. One may show: $\{X_0, \hat{X}_0, E_1 - U\}$ is a partition of E_1.

Consider any f in Mon(E_1, E_2)-under, define g: $E_1 \to E_2$ by:

$$g(e) := \begin{cases} f(e) & \text{if } e \in X_0 \\ \widehat{f(\hat{e})} & \text{if } e \in \hat{X}_0 \\ \text{some } u \in E_2 - U,\ u \geqslant f(e) & \text{if } e \in E_1 - U. \end{cases}$$

If $e \in E_1 - U$, then $f(e) = f(\hat{e}) \leqslant \widehat{f(e)}$, hence $f(e) \in E_2$-under; E_2 is rich, so certainly there is a $u \in E_2 - U$ with $u \geqslant f(e)$.

Clearly $g \geqslant f$ and by inspecting cases one shows: g is monotonic and selfdual, i.e., $g \in \text{Mon}(E_1, E_2) - U$. □

We have seen that csl is closed under E^A for set A and Fix(E, f) for f monotonic and selfdual on E. The following result shows that conversely csl can be generated from $\mathbb{T}$ using these operations (up to isomorphism).

2.2.11. FACT. Let $E = \langle E, (\hat{\ }), \leqslant \rangle$ be a csl, then there is an F, monotonic and selfdual on $\mathbb{T}^E$ such that E is isomorphic to Fix$(\mathbb{T}^E, F)$.

Proof (sketch). Define G: $E \to \{T, F, *, TF\}^E$ as follows:

$$G(x)(y) :\simeq \begin{cases} T & \text{if } y \leqslant x, \\ F & \text{if } y \nleqslant \hat{x}. \end{cases}$$

(If none of the conditions obtain the value is $*$, if both obtain TF.)

We have: $(x_1 \leqslant x_2 \Leftrightarrow G(x_1) \leqslant G(x_2))$ and $G(\hat{x}) = \widehat{G(x)}$. Hence G embeds E in $\mathbb{T}^E$. Define F_1, F_2: $\{T, F, *, TF\}^E \to \{T, F, *, TF\}^E$ as follows:

$$F_1(f) := G(\sqcup\{x \in E \mid G(x) \leqslant f\}(E)),$$

$$F_2(f) := G(\sqcap\{x \in E \mid G(x) \geqslant f\}(E)).$$

F_1, F_2 are monotonic from $\mathbb{T}^E$ to $\mathbb{T}^E$ and $F_1 \leqslant F_2$, $F_2(f) = \widehat{F_1(\hat{f})}$, so F_1 is in Mon$(\mathbb{T}^E, \mathbb{T}^E)$-under. As is easily shown E is isomorphic to Fix$(\mathbb{T}^E, F_1)$, Fix$(\mathbb{T}^E, F_2)$. If we extend F_1 carefully to a selfdual F we get the desired result.

Let X_0 be a subset of $\{T, F, *, TF\}^E$ as constructed in the proof of 2.2.10. Define F by:

$$F(e) := \begin{cases} F_1(e) & \text{if } e \in X_0 \text{ or } (e \in \{T, F\}^E \text{ and } F_1(e) \in \{T, F\}^E), \\ F_2(e) & \text{if } e \in \hat{X}_0 \\ \text{some } v \text{ in } \{T, F\}^E,\ v \geqslant F_1(e),\ v \neq e, & \text{if } e \in \{T, F\}^E \text{ and } F_1(e) \notin \{T, F\}^E. \end{cases}$$

We find: $F \in U\text{-Mon}(\mathbb{T}^E, \mathbb{T}^E)$ or F is selfdual and monotonic and: $\boldsymbol{E}$ is isomorphic to $\text{Fix}(\mathbb{T}^E, F)$. ⊏

2.3. *Van Fraassen Valuations*

For this section let $\boldsymbol{E} = \langle E, (\hat{\ }), \leqslant \rangle$ be a fixed rcsl and let $\boldsymbol{H} = \langle H, (\hat{\ }), \leqslant \rangle$ be a fixed csl. $\sqcap$, $\sqcup$ without further comment will be $\sqcap(\boldsymbol{E})$, $\sqcup(\boldsymbol{E})$ or $\sqcap(\boldsymbol{H})$, $\sqcup(\boldsymbol{H})$. 'u', 'u'', 'v' will range over $E-U$; 'S', 'S_1', 'S'' will range over non empty subsets of $E-U$.

Let g: $E-U \to H-U$; g is our analogue of a classical valuation scheme.

The *Van Fraassen Problem* is to extend g to $\tilde{g}$ defined on all of E in a natural way.

Some plausible constraints are:

(a) $\tilde{g}$ must be defined in terms of the structure of $\boldsymbol{E}$, $\boldsymbol{H}$.

(b) $\tilde{g}$ must be monotonic.

(c) $\tilde{g}$ must correspond to the obvious choice for a Van Fraassen Valuation on $\boldsymbol{E}$-under, $\boldsymbol{E}$-over, i.e.:
for x in E-under: $\tilde{g}(x) = \sqcap\{g(u) \mid u \geqslant x\}$,
for x in E-over: $\tilde{g}(x) = \sqcup\{g(u) \mid u \leqslant x\}$.

Let us look at an obvious proposal. Consider any y in E. $y = \sqcup y$-under. Constraint (c) tells us what $\tilde{g}$ is going to be on elements of y-under. So why not take: $\tilde{g}(y) := \sqcup \tilde{g}(y\text{-under})$?

Define: S is closed iff $S = \{u \mid u \geqslant \sqcap S\}$. Note that it follows that S is closed iff $S = \{u \mid u \leqslant \sqcup S\}$. Note also that the U-Representation Theorem 2.2.8 could be rewritten in terms of closed S.

Now we are in a position to rewrite our definition of $\tilde{g}$:

$$\begin{aligned} \tilde{g}(y) &= \sqcup\{\sqcap g(x\text{-up}) \mid x \in y\text{-under}\} \\ &= \sqcup\{\sqcap gS \mid S \text{ closed}, \sqcap S \leqslant y\}. \end{aligned}$$

Can we drop the condition that S is closed? No.

EXAMPLE: Take $E := \mathbb{T}^{\{p_0, p_1\}}$, $H := \mathbb{T}$. $E - U$ will be $\{T, F\}^{\{p_0, p_1\}}$. Let $G := [p_0 \wedge p_1]^c$, i.e., the classical valuation function such that for assignments f in $\{T, F\}^{\{p_0, p_1\}}$, $[p_1 \wedge p_2]f$ is the truthvalue of $(p_1 \wedge p_2)$ at f. If we took $\tilde{G}(g) := \sqcup\{\sqcap gS \mid \sqcap S \leqslant g\}$, we would have $\tilde{G}(^T_T) = TF$; for $\sqcap\{^T_T\} \leqslant {}^T_T$, $\sqcap\{^T_F, {}^F_T\} \leqslant {}^T_T$ and $\sqcap g\{^T_T\} = T, \sqcap g\{^T_F, {}^F_T\} = F$. Note that the example is blocked for closed S. □

Clearly there is a dual to the definition of $\tilde{g}$:

$$\tilde{\tilde{g}}(y) := \sqcap\{\sqcup gS \mid S \text{ closed}, \sqcup S \geqslant y\}.$$

2.3.1. FACT

(a) $\tilde{g}, \tilde{\tilde{g}}$ are monotonic.

(b) $x \in E\text{-under} \Rightarrow \tilde{g}(x) = \tilde{\tilde{g}}(x) = \sqcap g(x\text{-up})$.

(c) $x \in E\text{-over} \Rightarrow \tilde{g}(x) = \tilde{\tilde{g}}(x) = \sqcup g(x\text{-down})$.

(d) $\tilde{g}(u) = \tilde{\tilde{g}}(u) = g(u)$.

(e) $\tilde{g}(x) \leqslant \tilde{\tilde{g}}(x)$.

(f) $\tilde{\tilde{g}}(x) = \widehat{\tilde{g}(\hat{x})}$.

(g) Let $H_1 := \langle\{S \mid S \text{ closed}, \sqcap S \leqslant x\}, \subseteq\rangle$, then for every S, S' minimal in H_1: $S \neq S' \Rightarrow S \cap S' = \emptyset$. Moreover $\tilde{g}(x) = \sqcup\{\sqcap gS \mid S \text{ minimal in } H_1\}$.

(h) Let $H_2 := \langle\{S \mid S \text{ closed}, \sqcup S \geqslant x\}, \subseteq\rangle$, then for every S, S' minimal in H_2: $S \neq S' \Rightarrow S \cap S' = \emptyset$. Moreover $\tilde{\tilde{g}}(x) = \sqcap\{\sqcup gS \mid S \text{ minimal in } H_2\}$.

Proof. (a) and (f) are easy, (c) is dual to (b), (d) follows from (b), (h) is dual to (g).

(b) Suppose $x \in E$-under. x-up is closed, $\sqcap x\text{-up} = x$. Moreover for every closed S with $\sqcap S \leqslant x$, clearly $x\text{-up} \subseteq S$. Hence $\sqcap gS \leqslant \sqcap g(x\text{-up})$. Conclude $\tilde{g}(x) = \sqcap g(x\text{-up})$.

Consider $u \in x$-up. $\{u\}$ is closed, $x \leqslant \sqcup\{u\}$. So $g(x\text{-up}) \subseteq \{\sqcup gS \mid S \text{ closed}, x \leqslant \sqcup S\}$. Moreover if S is closed and $x \leqslant \sqcup S$ then there is a u such that $x \leqslant u \leqslant \sqcup S$. Hence $u \in x$-up, and $u \in S$ (S being closed). It follows that $g(u) \leqslant \sqcup gS$. So $g(x\text{-up})$ minorizes $\{\sqcup gS \mid S \text{ closed}, \sqcup S \geqslant x\}$. Hence $\tilde{\tilde{g}}(x) = \sqcap g(x\text{-up})$.

(e) It is sufficient to show that for S, S' closed with $\sqcap S \leqslant x \leqslant \sqcup S'$: $\sqcap gS \leqslant \sqcup gS'$. From $\sqcap S \leqslant \sqcup S'$ we have for some u: $\sqcap S \leqslant u \leqslant \sqcup S'$, hence by closedness: $u \in S \cap S'$. Conclude $\sqcap gS \leqslant g(u) \leqslant \sqcup gS'$.

(f) Consider the maximal elements m of **x**-under. Clearly m-up is a minimal element of $\boldsymbol{H}_1$. Above every y in **x**-under there is a maximal m in **x**-under, hence below every closed S' with $\sqcap S' \leqslant x$ there is a minimal closed $S \subseteq S'$ with $\sqcap S \leqslant x$. Clearly $\sqcap gS \geqslant \sqcap gS'$. Hence $\tilde{g}(x) = \sqcup\{\sqcap gS \mid S$ minimal in $\boldsymbol{H}_1\}$.

Consider S, S' minimal in $\boldsymbol{H}_1$, $S \neq S'$. If u were in S and in S' then $\sqcap S \leqslant u \geqslant \sqcap S'$. So $\sqcap S, \sqcap S'$ is consistent in $\boldsymbol{E}$-under, and thus in **x**-under, hence $\sqcup\{\sqcap S, \sqcap S'\}$ is in x-under. But $\sqcap S, \sqcap S'$ were maximal and distinct in **x**-under. Contradiction. □

2.3.2. EXAMPLE. We show that *not* generally $\tilde{g} = \tilde{\tilde{g}}$.

(a) Take $\boldsymbol{E} = \mathbb{T}^{\{p_0, p_1\}}$, $\boldsymbol{H} := \mathbb{T}$, $g := [p_0 \leftrightarrow p_1]^c$.

Calculation of $\tilde{g}(\genfrac{}{}{0pt}{}{*}{TF})$

The minimal closed S with $\sqcap S \leqslant \genfrac{}{}{0pt}{}{*}{TF}$ are:

$$S_0 = \{\genfrac{}{}{0pt}{}{T}{T}, \genfrac{}{}{0pt}{}{F}{T}\}, \quad S_1 = \{\genfrac{}{}{0pt}{}{T}{F}, \genfrac{}{}{0pt}{}{F}{F}\}. \quad gS_0 = gS_1 = \{T, F\}.$$

So $\sqcap gS_0 = \sqcap gS_1 = *$. Hence $\tilde{g}(\genfrac{}{}{0pt}{}{*}{TF}) = *$.

Calculation of $\tilde{\tilde{g}}(\genfrac{}{}{0pt}{}{*}{TF})$

The minimal closed S such that $\sqcup S \geqslant (\genfrac{}{}{0pt}{}{*}{TF})$ are:

$$S'_0 = \{\genfrac{}{}{0pt}{}{T}{T}, \genfrac{}{}{0pt}{}{T}{F}\}, \quad S'_1 = \{\genfrac{}{}{0pt}{}{F}{T}, \genfrac{}{}{0pt}{}{F}{F}\}.$$ (Note that S_i, S'_j have at least one element in common as should be.)

$$gS'_0 = gS'_1 = \{T, F\}, \text{ hence } \sqcup gS'_0 = \sqcup gS'_1 = TF.$$

Conclude $\tilde{\tilde{g}}(\genfrac{}{}{0pt}{}{*}{TF}) = TF$.

Remarks

(1) $g := [p_0 \leftrightarrow p_1]^c$ and $g := [p_0 \leftrightarrow \neg p_1]^c$ are the only examples in *two* propositional variables where $\tilde{g}, \tilde{\tilde{g}}$ differ.

(2) The usual Strong Kleene paraphrases for $\leftrightarrow$ differ on $\genfrac{}{}{0pt}{}{*}{TF}$:

$$[(\neg p_0 \vee p_1) \wedge (\neg p_1 \vee p_0)](\genfrac{}{}{0pt}{}{*}{TF}) = T,$$

$$[(p_0 \wedge p_1) \vee (\neg p_0 \wedge \neg p_1)](\genfrac{}{}{0pt}{}{*}{TF}) = F.$$

(b) Let $E := \mathbb{T}^{\{p_0, p_1, p_2\}}$, $H := \mathbb{T}$,

$g := [(p_0 \leftrightarrow p_1) \wedge p_2]^c$. We have:

$$\tilde{g}\begin{pmatrix} * \\ TF \\ TF \end{pmatrix} = F, \quad \tilde{\tilde{g}}\begin{pmatrix} * \\ TF \\ TF \end{pmatrix} = TF.$$

This illustrates that $\tilde{g}$, $\tilde{\tilde{g}}$ may differ *even* if one of them has a definite value.

2.3.3. REMARK. Combining 2.3.1 (a, e, f) we see that $\tilde{g}$, $\tilde{\tilde{g}}$ are in $\mathrm{Mon}(E, H)$ and $\check{\tilde{g}} = \tilde{\tilde{g}}$ and thus $\tilde{g} \leqslant \check{\tilde{g}}$ or $\tilde{g} \in \mathrm{Mon}(E, H)$-under. Example 2.3.2 shows $\tilde{g} \neq \check{\tilde{g}}$ and hence $\tilde{g}$ is not selfdual. Suppose H is rich, then by 2.2.10 there is a selfdual g^* with $\tilde{g} \leqslant g^* \leqslant \tilde{\tilde{g}}$. Note however that such g^* need not be very 'natural'.

3. THE LIAR: PROPOSALS AND COMPARISONS

First I want to give some examples of the kind of csl/rcsl and the kind of transition function I will be interested in. The examples are just a small sample of the possibilities one could think of, but for our purposes they seem to provide sufficiently rich and illustrative material.

3.1. *The First Example*

Let $\mathbb{N}$ be the standard model of the natural numbers with zero, successor, addition, multiplication and exponentiation. (More generally we could consider an acceptable structure in the sense of Moschovakis (see Moschovakis [16]). Let L be the language of $\mathbb{N}$ with one predicate symbol 'True' added. We may assume that the sentences of L are coded in some standard way in the natural numbers. The fact that in $\mathbb{N}$ we can represent a substitution function will allow us to explain quantification into contexts involving truth. We write, e.g., '$\forall x$ True($\phi(x)$)' for '$\forall x$ True(subst(num(x), $\overline{\phi(v_0)}$))'. Here 'num' stands for the (representation of) the function taking a number to the numerical code of its numeral, and '$\overline{\phi(v_0)}$' stands for the numeral of the code of $\phi(v_0)$. For details see, e.g., Smoryński [17].

Let Q be in $\mathbb{T}^N$, where N is the set of natural numbers. $\mathbb{N}(Q)$ is the four valued model we get from $\mathbb{N}$ by adding Q as the interpretation of 'True'.

Let us write '$\phi \simeq_Q \psi$' if the formulas ϕ, ψ have the same truth value in $\mathbb{N}(Q)$ for all assignments, and '$\phi \simeq \psi$' if $\phi \simeq_Q \psi$ for all Q in $\mathbb{T}^N$. Here and elsewhere I suppress the mention of the valuation scheme: the facts stated will hold for any of the schemes we have considered. One can prove for models $\mathbb{N}(Q)$ a fixed point theorem: for every $\psi(x, \vec{y})$ in L there is a $\phi(\vec{y})$ in L such that $\phi(\vec{y}) \simeq \psi(\underline{\phi(\vec{y})}, \vec{y})$. For example, we can produce a sentence L such that $L \simeq \neg\mathrm{True}(\underline{L})$ and a formula $B(x)$ such that $B(x) \simeq ((\forall y < x\ \mathrm{True}(B(y))) \wedge (\exists z\ \neg\mathrm{True}(B(z))))$.

The transition functions will correspond to the chosen valuation schemes. Some examples are:

(a) F_N: $\mathbb{T}^N \to \mathbb{T}^N$ with:

$$F_N(Q)(n) = \begin{cases} \text{the Strong Kleene Valuation of } \phi \text{ in } \mathbb{N}(Q) \text{ in case } n \text{ is the code of sentence } \phi. \\ F \text{ if } n \text{ is not the code of a sentence.} \end{cases}$$

As is easily seen F_N is monotonic from $\mathbb{T}^N$ to $\mathbb{T}^N$. Moreover F_N is selfdual, i.e., $F_N(\hat{Q}) = \hat{F}_N(Q)$.

(b) $\tilde{G}_N$ and $\tilde{\tilde{G}}_N$ from $\mathbb{T}^N$ to $\mathbb{T}^N$ induced by G_N: $\{T, F\}^N \to \{T, F\}^N$, where for $Q \in \{T, F\}^N$:

$$G_N(Q)(n) = \begin{cases} \text{the Classical Valuation of } \phi \text{ in } \mathbb{N}(Q) \text{ in case } n \text{ is the code of a sentence } \phi. \\ F \text{ if } n \text{ is not the code of a sentence.} \end{cases}$$

$\tilde{G}_N$, $\tilde{\tilde{G}}_N$ are monotonic, $\tilde{G}_N$ is overlap preserving, $\tilde{\tilde{G}}_N$ is underlap preserving.

3.2. *The Second Example*

It would be nice to have examples that are purely in propositional logic. But how could that be: truth is essential in the construction of the Liar and truth is a predicate. That may be, but the effect of the Liar can be achieved in several ways. One of them is to employ the idea of *stipulative definitions*.

Those familiar with recursion theory would not balk at the following definition of $+$:

$$x + y = z :\Leftrightarrow (y = 0 \wedge x = z) \vee \exists uv(y = u' \wedge z = v' \wedge x + u = v).$$

I want to exploit the extreme of this form of definition and consider for example:

$$l: \neg l, \qquad a: a, \qquad b: l \vee b.$$

Here, e.g., 'l' is stipulated to have – in some sense – the same meaning as '$\neg l$'. Note that in the stipulation 'l' is introduced as an *atomic sentence*, not as a *name* of a sentence. A series of interconnected stipulations as above I will call a *stipulation list*.

We could work out the idea for a language also involving the usual propositional atoms, but it is simpler and for our purposes sufficient not to do this.

Let us implement the idea. Let A be a set and let $L(A)$ be the minimal class such that $A \subseteq L(A)$, $\perp \in L(A)$ and if ϕ, ψ are in $L(A)$ then $(\phi \wedge \psi)$, $(\phi \vee \psi)$ and $(\neg\phi)$ are in $L(A)$. (Sometimes I write, e.g., '$(\phi \rightarrow \psi)$'; this will just stand for '$(\neg\phi \vee \psi)$'.) A stipulation list S is a function from A to $L(A)$. To represent the idea of stipulation we must specify the appropriate transition functions:

(a) $F_S: \mathbb{T}^A \rightarrow \mathbb{T}^A$ is given by

$$F_S(f)(a) = [S(a)]\,f$$

$$= \text{the Strong Kleene Valuation of } S(a) \text{ at } f.$$

F_S is monotonic and selfdual.

(b) $\tilde{G}_S$, $\tilde{\tilde{G}}_S$ are the Van Fraassen transition functions induced by G_S:

$$\{T, F\}^A \rightarrow \{T, F\}^A \text{ with}$$

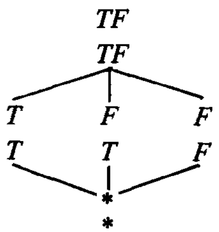
$$G_S(f)(a) = [S(a)]^c f$$

$$= \text{the Classical Valuation of } S(a) \text{ at } f.$$

We write '$\phi \simeq_f \psi$' if ϕ, ψ have the same truthvalue at f; '$\phi \simeq \psi$' if for all f in $\mathbb{T}^A$ $\phi \simeq_f \psi$. If f is a fixed point of F_S, $\tilde{G}_S$ or $\tilde{\tilde{G}}_S$ (depending on the chosen scheme) then we have $a \simeq_f S(a)$ for any $a \in A$.

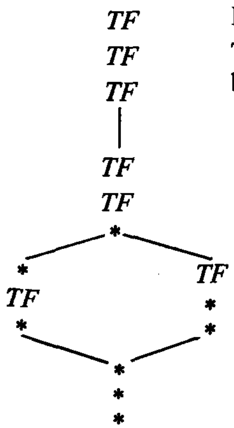
It is more or less routine to check that examples in terms of stipulations can always be translated into examples in terms of truth + selfreference. E.g., examples given for *finite* A can be translated to examples involving $\mathbb{N}(Q)$.

3.2.1. SUBEXAMPLE



Let S be: a: $(a \vee (b \wedge \neg b)) \wedge b$,
b: $(b \wedge (a \vee \neg a)) \vee a$.
Then $\mathrm{Fix}(\mathbb{T}^{\{a,b\}}, F_S)$ is as shown; a *non* distributive lattice.

3.2.2. SUBEXAMPLE



Let S be a: $\neg a$, b: $\neg b$, c: $((a \leftrightarrow b) \leftrightarrow c)$.
Then $\mathrm{Fix}(\mathbb{T}^{\{a,b,c\}}, \tilde{G}_S)$ is as shown; a lattice, but *not* a csl.

□

The next ingredient we need is some insight in iterations of the transition functions.

3.3. *The Basics of Iterations*

Let $E = \langle E, (\hat{\ }), \leqslant \rangle$ be a csl.

3.3.1. DEFINITION. Let $(c_\alpha)_{\alpha<\lambda}$ be an ordinal sequence of elements of E, λ a limit ordinal.

$$\limsup_{\alpha\to\lambda} c_\alpha := \bigsqcap_{\alpha<\lambda} \bigsqcup_{\alpha\leqslant\beta<\lambda} c_\beta$$

$$\liminf c_\alpha := \bigsqcup_{\alpha<\lambda} \bigsqcap_{\alpha\leqslant\beta<\lambda} c_\beta.$$

3.3.2. DEFINITION. Let f be monotonic on E, $x \in E$. Define the *upper fundamental sequence from x based on f* as follows:

$$(x)^0 := x,$$
$$(x)^{\alpha+1} := f((x)^\alpha),$$
$$(x)^\lambda := \limsup_{\alpha\to\lambda} (x)^\alpha.$$

Define the *lower fundamental sequence from x based on f* as follows:

$$(x)_0 := x,$$
$$(x)_{\alpha+1} := f((x)_\alpha),$$
$$(x)_\lambda := \liminf_{\alpha\to\lambda} (x)_\alpha.$$

If we consider different fundamental sequences based on, say, g and h, we write, e.g., $(x)_\alpha(g)$, $(x)_\alpha(h)$.

3.3.3. THEOREM. Under the conditions of 3.3.2

(a) Let κ be $(\max(\text{card}(E), \aleph_0))^+$. For some $\gamma < \kappa$, $f((x)_\gamma) = (x)_\gamma$, similarly for some $\delta < \kappa$, $f((x)^\delta) = (x)^\delta$. We write $(x)_f$ or $(x)_\infty$ for $(x)_\gamma$ and $(x)^f$ or $(x)^\infty$ for $(x)^\delta$. (Actually one may show that if $E = H^A$, H a csl, then we can take: $\kappa = (\max(\text{card}(H), \text{card}(A), \aleph_0))^+$. This last point is essentially due to Vann McGee).

(b) $(x)_\alpha \leqslant (x)^\alpha$.

(c) Suppose that E-under is closed under f, i.e. that if $y \in E$-under then $f(y) \in E$-under. Then $(x)_\alpha$ is in E-under if $x \in E$-under. Similarly in case E-over is closed under f and x is in E-over, then $(x)^\alpha$ is in E-over.

Proof. (a) $(x)_\alpha$ considered as a function from κ to E cannot be injective, hence for some η, $\eta + \theta$ with $\theta \neq 0$, $\eta < \kappa$, $\eta + \theta < \kappa$, we have $(x)_\eta = (x)_{\eta+\theta}$. One easily shows that $(x)_{\eta+\theta\omega} = \sqcap_{\eta<\alpha\leqslant\eta+\theta} (x)_\alpha$. Let $\nu := \eta + \theta\omega$. Clearly $\nu < \kappa$, κ being an infinite successor cardinal. Consider α with $\eta \leqslant \alpha \leqslant \eta + \theta$, we have $(x)_\nu \leqslant (x)_\alpha$, hence $(x)_{\nu+1} = f((x)_\nu) \leqslant f((x)_\alpha) = (x)_{\alpha+1}$. By transfinite induction one shows: for α with $\eta < \alpha \leqslant \eta + \theta$ we have

$(x)_{\nu+1} \leqslant (x)_\alpha$. Hence $(x)_{\nu+1} \leqslant (x)_\nu$. Again by transfinite induction one shows that the sequence will be descending from ν on. Finally by a cardinality argument as above it is proved that for some γ with $\nu \leqslant \gamma < \kappa$, $f((x)_\gamma) = (x)_\gamma$.

The argument for $(x)^\alpha$ is dual.

(b) By transfinite induction.

(c) Using the fact that E-under is a ccpo one shows that E-under is closed under liminfs of sequences in E-under. The rest of the argument is by transfinite induction.

The argument for E-over is dual. □

3.3.4. FACT. Under the conditions of 3.3.2:

(a) $x \leqslant y$ and $f \leqslant g \Rightarrow (x)_f \leqslant (y)_g$ and $(x)^f \leqslant (y)^g$.

(b) $((x)_f)_f = ((x)_f)^f = (x)_f$ and $((x)^f)^f = ((x)^f)_f = (x)^f$.

3.3.5. FACT. Under the conditions of 3.3.2, let $(\ \ \check{})$ be as in 2.2.9.

(a) $(x)_\alpha(\check{f}) = ((\hat{x})^\alpha(f))\hat{}$, $(x)^\alpha(\check{f}) = ((\hat{x})_\alpha(f))\hat{}$.

(b) $(\check{\ })^f = (\ \ )_{\check{f}}$, $(\check{\ })_f = (\ \ )^{\check{f}}$.

(c) If f is overlap preserving then $(\ \ )_f$ is overlap preserving.

Proof. (a) By transfinite induction, using that liminf is dual to limsup.

(b) From (a).

(c) $(\ \ )_f \leqslant (\ \ )_{\check{f}} \leqslant (\ \ )^{\check{f}} = (\check{\ })_f$. □

3.4. *The Liar: Introductory Remarks*

3.3 provides us with a powerful method of access to fixed points, but of course not all fixed points are equally interesting in connection with the Liar. Certainly fixed points that can be reached from interesting starting points are interesting.

Two eminently salient starting points are bottom and top. From bottom we go up to Kripke's minimal fixed point, from top we go down to its dual, the maximal fixed point (both for the Strong Kleene and for the two Van Fraassen valuations). The maximal fixed point may be interesting from the philosophical point of view. Technically speaking it has nothing new to offer over the minimal one. A further idea is to 'load' the starting point with certain properties that are desirable in the fixed point and are preserved in the process. Such is Dowden's plan (see Dowden [4]). He takes 'everything undefined, except the Liar. The Liar both true and false' as starting point. This procedure seems to me rather ad hoc: intuitive distinctions are supposed to roll out of the process, not to be preprogrammed. Moreover there are many 'alphabetic' variants of the Liar and many paradoxical sentences of different design (in fact the set of sentences paradoxical in Kripke's sense over the natural numbers as in 3.1 is complete Π^1_1 – in contrast to the set of ungrounded sentences which is complete Σ^1_1).

Question. What happens if we start from 'Kripke Paradoxical both true and false, rest undefined'?

The idea I will consider in this paper is access of fixed points from starting points that are themselves the end product of certain variants of Gupta/Belnap processes. (I take it that a Herzberger process most naturally starts with 'all false'; I will not consider such processes here.) In outline the idea is this: we start iterating 'simultaneously' on all unique valued points. We iterate over limit ordinals by taking some preferred kind of limit. In the 'end' we take the intersection of the limits of the sequences thus obtained. This gives us an object that summarizes certain stable features common to all these sequences. Then we use this object as starting point for a final iteration to a fixed point.

I will not be exclusively interested in fixed points. After all it could very well be the iterative process itself that best reflects our semantic understanding. Even if this process leads to a fixed point it could be that this fixed point only summarizes certain aspects of the 'solution' proper. Moreover, suppose forceful philosophical arguments back up some iterative process that does not lead to a fixed point. Is the fixed point intuition strong enough to be the sole reason to reject this process? It seems to me that this is not the case, especially when the process would elucidate how the fixed point intuition could arise.

Let me briefly review the basic features of the variants of Gupta/Belnap processes we will be considering.

3.4.1. *Starting Points*

The basic *intention* in Semantics is to get Unique Valued or Classical Interpretations. We start our process by having things as they ought to be at least in this respect. We solve the problem of arbitrariness by 'quantifying out', i.e., by considering all Unique Valued starting points.

3.4.2. *Stages and Transition Functions*

Our stages will be generally four valued. So we will need four valued transition functions. We may opt for, e.g., the Strong Kleene or one of the Van Fraassen transition functions. We may choose to accumulate truth-values in the transition: with every transition function f an accumulating one viz. $f \sqcup id$ is naturally associated.

3.4.3. *Limits*

What is stable is our process should be preserved. In technical language, the limits we choose should be between the liminf and the limsup.

Gupta chooses to adapt the unstable to the starting point, Belnap solves the problem by again quantifying out. Both succeed in having all the stages classical. I want to consider the minimal choice liminf and the maximal one limsup. 'Liminf' corresponds to the idea that at the limit we simply don't get the necessary information to give the unstable sentences a value, hence we leave them undefined. 'Limsup' follows the intuition that – on the contrary – the alternating character of the unstable elements tells us that these are both true and false (as far as T, F alternations are concerned).

3.4.4. *Endevaluation*

As endevaluation we take some appropriate limit of the processes corresponding to the different starting points and then take the intersection of all these limits. (This last step corresponds to the idea that something is stably true if it is stably true for *all* starting points).

3.4.5. *Intuitive Distinctions*

Intuitive distinctions can be made in terms of the processes or in terms of the endevaluations. The four valued endevaluations do rather well: they yield distinctions but do not proliferate them to where we have no intuitions anyway. For example we can get the Liar both true and false ('paradoxical') and the Samesayer ('This sentence is true') undefined ('need not be true, need not be false').

3.5. *Three Possibilities*

Let $E = \langle E, (\hat{\ }), \leqslant \rangle$ be a rcsl.

3.5.1. *Liminf*

Let f be a monotonic transition function such that E-under is closed under f. We consider the processes $((u)_\alpha)_{\alpha \in ON}$ for u in $E-U$.

Clearly E-under is closed under the $(u)_\alpha$, hence the fixed points $(u)_\infty$ are in E-under. Define:

$$j_0 = \sqcap\{(u)_\infty \mid u \in E-U\}, \quad j := (j_0)_\infty .$$

We have: $j_0 \leqslant (u)_\infty$, hence $j = (j_0)_\infty \leqslant ((u)_\infty)_\infty) = (u)_\infty$

Conclude $j \leqslant j_0$. (In fact $j = \sqcap\{(u)_\infty \mid u \in E-U\}$ (Fix(E-under, f))). j_0 – and hence j – is consistent with every fixed point a of f in E-under: pick any fixed point a in E-under; for some u in $E-U$, $u \geqslant a$, so $(u)_\infty \geqslant (a)_\infty = a$. Also $(u)_\infty \geqslant j_0$. Hence j_0 is consistent with every underdefined fixed point a. It follows that j is an intrinsic fixed point, i.e., intrinsic in Fix(E-under, f).

3.5.1.1. EXAMPLE. Let S be $(a{:}\ a)$, $(b{:}\ a \vee \neg a)$, $(c{:}\ \neg c \wedge \neg d)$, $(d{:}\ \neg c \vee \neg d)$, $(e{:}\ e \vee \neg e)$. $E := \mathbf{T}^{\{a,b,c,d,e\}}$, $f := F_S$.

We have: $[b]j_0 = T$, $[b]j = *$. Hence $j \lneq j_0$. $[c]j = [d]j = *$, but $[c]i = F$, $[d]i = T$. (Hence i is the maximal intrinsic fixed point, i.e., the maximal intrinsic point in Fix(E-under, f).) Here $j \lneq i$. (One can give similar examples for the Van Fraassen case.) Finally e is $*$ at the minimal fixed point, but $[e]j = T$. Hence j is not minimal.

In case our stipulation list is finite we can often simplify the description of salient points or give faster procedures to calculate these. For example:

3.5.1.2. LITTLE THEOREM. Let E be finite. Suppose $E-U$ is closed under f. A *finite cycle C* is a set of elements $c_0, \ldots, c_{k-1}$ in $E-U$ such that $c_{i+1} = f(c_i)$ and $c_0 = f(c_{k-1})$. An element of $E-U$ is *finitely cyclic* if it is on some finite cycle.

Call the set of finitely cyclic elements FC. We have: $j = (\sqcap FC)_\infty$.

Proof. Consider c on cycle C. We have: $(c)_\infty = (\sqcap C)_\infty \leqslant \sqcap C \leqslant c$, hence $j_0 \leqslant \sqcap FC$ and $j \leqslant (\sqcap FC)_\infty$. On the other hand for any u in $E-U$: $(u)_\infty = (c)_\infty$ for some c in FC.

Hence: $(\sqcap FC)_\infty \leqslant (c)_\infty = (u)_\infty$.
Conclude $(\sqcap FC)_\infty \leqslant j_0$ and $(\sqcap FC)_\infty = ((\sqcap FC)_\infty)_\infty \leqslant (j_0)_\infty = j$. □

3.5.2. *Limsup*

Let f be monotonic, such that for x in E-under $\cup$ E-over $f(\hat{x}) = \widehat{f(x)}$. It follows that E-under and E-over are closed under f, e.g., if $x \in E$-under then $x \leqslant \hat{x}$ hence $f(x) \leqslant f(\hat{x}) = \widehat{f(x)}$, so $f(x)$ is in E-under.

We consider the processes $((u)^\alpha)_{\alpha \in ON}$ for $u \in E-U$. Clearly E-over is closed under the $(u)^\alpha$, hence the fixed points $(u)^\infty$ are in E-over.

$$w_0 := \sqcap\{(u)^\infty \mid u \in E-U\}, \quad w := (w_0)^\infty.$$

Let us first remark that from $w_0 \leqslant (u)^\infty$, we have $f(w_0) \leqslant f((u)^\infty) = (u)^\infty$. Hence $f(w_0) \leqslant w_0$. It follows that $(w_0)^\alpha$ is descending. By transfinite induction and the fact that on descending sequences liminf and limsup coincide: $(w_0)_\infty = (w_0)^\infty \leqslant w_0$.

Consider a in Fix(E-over, f). Surely for some u in $E-U$, $u \leqslant a$, hence $(u)^\infty \leqslant (a)^\infty = a$. Conclude that $\{(u)^\infty \mid u \in E-U\}$ is a subset of Fix(E-over, f) *and* minorizes this set. Hence $w_0 = \sqcap$Fix(E-over, f).

It follows that w_0 is the maximal overlap Fix(E-under, f)-intrinsic point, by 2.2.5(f, h) and the fact that (Fix(E-under, f))$\hat{}$ = Fix(E-over, f) (f being selfdual on (E-under $\cup$ E-over)). As is easily seen w is the maximal fixed point $\leqslant w_0$. In other words w is the maximal overlap Fix(E-under, f)-intrinsic fixed point.

Consider any of the examples 3.1, 3.2 with any of the treated Strong Kleene/Van Fraassen transition functions. We have for ϕ either in N or in A:

$w_0(\phi) = TF$	iff $a(\phi) = TF$ for all overdefined fixed points a, iff $b(\phi) = *$ for all underdefined fixed points b, iff ϕ is (code of a) Kripke paradoxical (sentence) for the given valuation scheme.
$w_0(\phi) = *$	iff ϕ is T in some Kripke (i.e., underdefined) fixed point, F in some other.
$w_0(\phi) = T$	iff ϕ is T in some Kripke fixed point, F in no other
$w_0(\phi) = F$	iff ϕ is F in some Kripke fixed point, T in no other.

Warning. w_0 is not generally a fixed point, so we must keep in mind that the *declared* truths *of* w_0 not the truths at $\mathbf{N}(w_0)$ reflect Kripke's distinctions (in case we are considering 3.1). Similarly in case of stipulations, the atoms whose meaning is stipulated not the stipulated meanings reflect these distinctions.

3.5.2.1. EXAMPLE. Let S be l: $\neg l$, s: s, a: $l \wedge (s \vee \neg s)$, and define $f := F_S$, $E := \mathrm{T}^{\{l,s,a\}}$. Then $[a]\,w_0 = TF$, $[a]\,w = [l \vee (s \vee \neg s)]\,w_0 = F$. So w does not reflect Kripke paradoxality but rather carries its own notion of paradoxality.

3.5.3. *Accumulate at Every Step*

Let f be monotonic and suppose E-over is closed under f. We consider the processes $((u)^{\alpha}(f \sqcup id))_{\alpha \in ON}$ for u in $E-U$. As is easily seen $(u)^{\alpha}(f \sqcup id)$ is ascending, hence each $(u)^{\alpha}(f \sqcup id)$ is in E-over. Moreover $(u)^{\alpha}(f \sqcup id) = (u)_{\alpha}(f \sqcup id)$, because on ascending sequences limsup and liminf coincide. Clearly $f \sqcup id$ is monotonic. Define:

$$t_0 := \sqcap\{(u)^{f \sqcup id} \mid u \in E-U\}.$$

We have:

(a) Let a be in Fix(E-over, f). For some u in $E-U$, $u \leqslant a$. $(f \sqcup id)(a) = f(a) \sqcup a = a$, so $(u)^{f \sqcup id} \leqslant (a)^{f \sqcup id} = a$. Hence $\{(u)^{f \sqcup id} \mid u \in E-U\}$ *minorizes* Fix(E-over, f).

(b) Let $b = (u)^{f \sqcup id}$. We have $b = (f \sqcup id)(b) = f(b) \sqcup b$, so $f(b) \leqslant b$, hence $(b)^f \leqslant b$. Moreover $(b)^f$ in E-over. Conclude: Fix(E-over, f) *minorizes* $\{(u)^{f \sqcup id} \mid u \in E-U\}$.

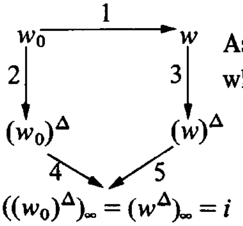
(c) Combining a, b: $t_0 = \sqcap$Fix(E-over, f).

For further considerations we may clearly return to 2.5.2 (under its conditions). Note however that the processes of 2.5.3 reflect another semantic 'plan' than those of 2.5.2. So if we adhere to the idea that not just the endresult has explanatory value we should not consider 2.5.2 and 2.5.3 as just variants.

3.6. *Accessability and Comparison, a Small Bestiary of Fixed Points and Assorted Structures*

3.6.1. *Projecting w_0 and w Down*

Different reasons can be given for considering $(w_0)^\Delta$ and $(w)^\Delta$ (where w_0, w are as in 3.5.2). First it is plausible to treat overdefined and underdefined to be on a par as far as *judgement* is concerned. $(\ )^\Delta$ obliterates the distinction between over- and underdefined. Secondly we may be interested in methods of *access* to Kripkean fixed points, as we will see $(w_0)^\Delta$, $(w)^\Delta$ are interesting starting points. Thirdly we may want to *compare* w_0, w with various Kripkean structures. This can be done in several ways; one of these is downwards projection. Here Kripkean structures play as it were a homegame and w_0, w play on the road.



Assume f is monotonic, overlap preserving and selfdual when restricted to (E-under $\cup$ E-over).

(1) As we have seen $w_0 \geqslant w$ (3.5.2).

(2) $(w_0)^\Delta \leqslant w_0$. As we noted in 3.2.5 w_0 is maximal such that $w_0 \circ \mathrm{Fix}(E\text{-under}, f)$. For i, the maximal intrinsic fixed point, we have: $i \circ \mathrm{Fix}(E\text{-under}, f)$, hence $i \leqslant w_0$. Also trivially $\hat{w}_0 = \sqcup \mathrm{Fix}(E\text{-under}, f) \geqslant i$. Conclude $(w_0)^\Delta \geqslant i$. Moreover $(w_0)^\Delta \leqslant w_0$, hence $(w_0)^\Delta$ is consistent with all underdefined fixed points.

(3) $w = (w_0)^\infty \leqslant w_0$, hence $w \circ \mathrm{Fix}(E\text{-under}, f)$. Moreover $w = (w_0)^\infty \geqslant (i)^\infty = i$. Further $\hat{w}_0 \geqslant i$, so $\hat{w} = \widehat{(w_0)^\infty} \geqslant (\hat{w}_0)_\infty \geqslant (i)_\infty = i$. (The first $\geqslant$ is because f is overlap preserving, in slightly different notation: $(\hat{w}_0)_f \leqslant (\hat{w}_0)\check{}_f = \widehat{(w_0)}^f$ by 3.3.5 (b).) Conclude $(w)^\Delta \geqslant i$. Also $(w)^\Delta \leqslant w_0$ hence $(w)^\Delta$ is consistent with all underdefined fixed points.

(4, 5) Suppose z is underdefined and consistent with every underdefined fixed point. Then $(z)_\infty$ is intrinsic in $\mathrm{Fix}(E\text{-under}, f)$, for if a is an underdefined fixed point, and $a \leqslant x$, $z \leqslant x$ for x underdefined, then $a = (a)_\infty \leqslant (x)_\infty$ and $(z)_\infty \leqslant (x)_\infty$ and $(x)_\infty$ is an underdefined fixed point. Conclude: $((w_0)^\Delta)_\infty$ and $((w)^\Delta)_\infty$ are intrinsic in $\mathrm{Fix}(E\text{-under}, f)$. Moreover as $(w_0)^\Delta$ and $(w)^\Delta$ are $\geqslant i$, $((w_0)^\Delta)_\infty$ and $((w)^\Delta)_\infty$ are $\geqslant i$, hence $((w_0)^\Delta)_\infty = ((w)^\Delta)_\infty = i$.

3.6.1.1. EXAMPLE. Let $\boldsymbol{S}$ be $l\colon \neg l$, $s\colon s$, $a\colon l \wedge (s \vee \neg s)$, $b\colon s \vee \neg s$. $f := F_S$, $\boldsymbol{E} := \mathbf{T}^{\{l,s,a,b\}}$. Then:

$$w_0 = \begin{matrix} TF \\ * \\ TF \\ T \end{matrix}, \quad w = \begin{matrix} TF \\ * \\ F \\ * \end{matrix}, \quad (w_0)^{\Delta} = \begin{matrix} * \\ * \\ * \\ T \end{matrix}, \quad (w)^{\Delta} = \begin{matrix} * \\ * \\ F \\ * \end{matrix}, \quad i = \begin{matrix} * \\ * \\ * \\ * \end{matrix}.$$

So in this example we see that $w_0 > w$; $(w_0)^{\Delta}$ and $(w)^{\Delta}$ are incomparable ($\nleqslant$ and $\ngeqslant$) and both are $> i$.

3.6.1.2. REMARK. The fact that $((w_0)^{\Delta})_{\infty} = i$ gives us in a sense iterative access to i. First we iterate from the u in $E-U$ to the $(u)^{\infty}$, then we take the intersection (a 'monotonic' step), then project down and iterate to i. Our process contains one 'non-monotonic' step: $(\ \ )^{\Delta}$.

I want to point out that there is no iterative process that gives access to i, such that:

(1) the starting points are valuation independent,

(2) all steps are monotonic,

(3) all stages are underdefined.

If there were such a process then surely for a given $\boldsymbol{S}$ the Van Fraassen i (say i_{VF}) would extend the Strong Kleene i (say i_{SK}) for $\tilde{G}_S(x) \geqslant F_S(x)$ if x underdefined. Consider the following $\boldsymbol{S}$:

$$l\colon \neg l,\ a\colon l \vee \neg l,\ b\colon (l \vee \neg l) \wedge b,$$

then:

$$i_{\mathrm{VF}} = \begin{matrix} * \\ T \\ * \end{matrix} \quad \text{and} \quad i_{\mathrm{SK}} = \begin{matrix} * \\ * \\ F \end{matrix},$$

so i_{VF} and i_{SK} are incomparable. One can show that they *must* be consistent.

3.6.2. *Structures from Processes in E–U*

To save our bestiary from overpopulation I will not go into the comparison of Gupta-structures with Belnap-structures, nor will I provide Van Fraassen examples. All this I leave to the reader's diligence.

Consider f monotonic, overlap preserving and selfdual on (E-over $\cup$ E-under). As is easily seen: $E-U$ is closed under f.

$([u]^\alpha)_{\alpha \in ON}$ is an *acceptable E–U process* if

(a) $[u]^{\alpha+1} = f([u]^\alpha)$,

(b) $[u]^\lambda \geqslant \liminf_{\alpha\to\lambda} [u]^\alpha$ and $[u]^\lambda \in E-U$. (Note that automatically $[u]^\lambda \leqslant \limsup_{\alpha\to\lambda} [u]^\lambda$, so the constraints of 3.4.3 are satisfied.)

A class P of acceptable $E-U$ processes is *full* if for each u in $E-U$ there is a process starting on u in P.

The Belnap class B is just the set of all acceptable $E-U$ processes. The Gupta class G consists of all processes satisfying the following constraint: let $x: = \liminf [u]^\alpha$, then $[u]^\lambda \geqslant x \sqcup (\hat{x} \sqcap u)$ and $[u]^\lambda \in E-U$.

x is clearly in E-under, hence $x \leqslant \hat{x}$. Moreover $x \leqslant x \sqcup u$, $\hat{x} \sqcap u \leqslant \hat{x}$, $\hat{x} \sqcap u \leqslant u \leqslant x \sqcup u$ so $x \sqcup (\hat{x} \sqcap u) \leqslant \hat{x} \sqcap (x \sqcup u) = (x \sqcup (\hat{x} \sqcap u))\hat{}$, hence $x \sqcup (\hat{x} \sqcap u) \in E$-under. E is rich, so certainly some v in $E-U$ with $v \geqslant x \sqcup (\hat{x} \sqcap u)$ can be found. Conclude: G is full. In case E is *modular* we have: $x \sqcup (\hat{x} \sqcap u) = \hat{x} \sqcap (x \sqcup u)$, so $x \sqcup (\hat{x} \sqcap u) \in E-U$; in other words the Gupta rule determines a unique limit. The examples we consider are in fact modular: the $\mathbb{T}^A$ are even distributive.

Consider any full P. Define:

$$j_0^P := \sqcap \{\liminf \pi \mid \pi \in P\},$$
$$j^P := (j_0^P)_\infty .$$

For $P = G/B$ in the case of Example 3.1 we have ϕ is *declared* true at j_0^P iff ϕ is stably true in all $\pi \in P$.

We have for $\pi \in P$: $f(\liminf \pi) \leqslant \liminf \pi$. The proof is by noting that there is a κ such that all $[u]^\alpha$ for $\alpha \geqslant \kappa$ occur cofinally in π. Moreover $\liminf \pi = \sqcap \{[u]^\alpha \mid \alpha \geqslant \kappa\}$. It follows that $f(\liminf \pi) \leqslant f([u]^\alpha) = [u]^{\alpha+1}$ for $\alpha \geqslant \kappa$. By transfinite induction one shows that for all $\alpha > \kappa\, [u]^\alpha \geqslant f(\liminf \pi)$. $[u]^\kappa$ is itself cofinal, hence for all $\alpha \geqslant \kappa$ $[u]^\alpha \geqslant f(\liminf \pi)$. Conclude $\liminf \pi \geqslant f(\liminf \pi)$.

It follows that $f(j_0^P) \leqslant j_0^P$ and hence $j^P \leqslant j_0^P$.

One definition we didn't consider is:

$$j_1^P := \sqcap \{(\liminf \pi)_\infty \mid \pi \in P\}, \text{ but as is easily seen } (j_1^P)_\infty = j^P.$$

By an easy induction $(u)_\alpha \leqslant [u]^\alpha$ for any $\pi = ([u]^\alpha)_{\alpha \in ON}$ in P, hence $j_0 \leqslant j_0^P$ and $j \leqslant j^P$.

3.6.2.1. *Little Theorem.* If E is finite and $P = G/B$ then $j^P = j = (\sqcap FC)_\infty$.

Proof. First consider any finite cycle $C = \{c_0, \ldots, c_{k-1}\}$. We claim $[c_0]^\alpha := c_{\alpha \bmod k}$ is a G – and hence B – process. Clearly it is sufficient to show that $[c_0]^\lambda = c_0$ is an admissible limit in G. Assuming this for all $\beta < \lambda$ we have $\liminf_{\alpha \to \lambda} [c_0]^\alpha = \sqcap C$ and $\sqcap C \sqcup (\sqcup C \sqcap c_0) = c_0$, so c_0 is the unique correct choice. Conclude $\sqcap\{\liminf \pi \mid \pi \in P\} \leqslant \sqcap\{\sqcap C \mid C \text{ a finite cycle}\} = \sqcap FC$. Hence $j^P \leqslant (\sqcap FC)_\infty$. Secondly in 3.5.1.2 we have seen $j = (\sqcap FC)_\infty$. Moreover $j \leqslant j^P$. It follows $j = j^P$. □

Is j^P ever *not* equal to j? Let $E := \mathbf{T}^N$ and $f := F_N$ as in 3.1. Both Anil Gupta and Vann McGee produced sentences ψ in L paradoxical for Gupta processes but stably true for *some* Belnap process. Consider: $\chi \tilde{\leftrightarrow} \text{True}(\chi) \wedge \psi$. χ will be eventually false in all Gupta processes and hence will be false in j_0^G. On the other hand χ may be true in a Belnap process where ψ is stably true and χ may also be false. Hence χ is undefined in j_0^B and hence in j^B. As is easily seen $j^G \geqslant j^B \geqslant j$, so $j^G \neq j$.

Open problem. Is j^B always j?

Consider any full P. Define:

$$w_0^P := \sqcap\{\limsup \pi \mid \pi \text{ in } P\},$$

$$w^P := (w_0^P)^\infty,$$

$$i_-^P := ((w_0^P)^\Delta)_\infty,$$

$$i_+^P := ((w^P)^\Delta)_\infty.$$

One other possibility reduces immediately to what we did before: $\sqcap\{(\limsup \pi)^\infty \mid \pi \in P\} = w_0$ as is easily shown.

w_0^P summarizes some G/B distinctions: for $P = G/B$ we have (in case of Example 3.1):

> ϕ is *declared* T at w_0^P iff ϕ is stably true in some π in P, stably false in no π in P.
>
> ϕ is *declared* TF at w_0^P iff ϕ is G/B paradoxical.
>
> ϕ is *declared* $*$ at w_0^P iff ϕ is stably true in some π in P, stably false in some other π' in P.

Consider any overdefined fixed point a. For some u in $E-U$, $u \leqslant a$.

Hence for $\pi = ([u]^\alpha)_{\alpha \in ON}$ in P: $[u]^\alpha \leqslant a$ (by transfinite induction) and hence $\limsup_{\alpha \to \infty} [u]^\alpha \leqslant a$. It follows that $w_0^P \leqslant w_0$ and hence $w^P \leqslant w \leqslant w_0$. w_0 overlaps all underdefined fixed points, hence so do w_0^P, w^P, $(w_0^P)^\Delta$, $(w^P)^\Delta$, i_-^P, i_+^P. As in the case of $((w_0)^\Delta)_\infty$ we may conclude that i_-^P and i_+^P are intrinsic in Fix(E-under, f).

Moreover:

$$i^P = ((w_0^P)^\Delta)_\infty = (w_0^P \sqcap \hat{w}_0^P)_\infty \leqslant (w_0^P)_\infty \sqcap (\hat{w}_0^P)_\infty \leqslant (w_0^P)^\infty \sqcap \widehat{(w_0^P)}^\infty = w^P \sqcap \hat{w}^P = (w^P)^\Delta.$$

($(\hat{w}_0^P)_\infty \leqslant \widehat{(w_0^P)}^\infty$ because f is overlap preserving). Conclude $i_-^P \leqslant ((w^P)^\Delta)_\infty = i_+^P$.

Finally, as is easily seen, $j^P \leqslant i_-^P$.

3.6.2.2. EXAMPLES

(a) Let S be a: $\neg b$, b: $\neg a$, c: $c \wedge (a \leftrightarrow b)$.
$E := \mathbb{T}^{\{a,b,c\}}$, $f := F_S$, $P :\in \{G, B\}$.
Then

$$w_0 = w = (w_0)^\Delta = (w)^\Delta = i = \begin{matrix} * \\ * \\ F \end{matrix},$$

and

$$w_0^P = w^P = (w_0^P)^\Delta = (w^P)^\Delta = i_+^P = i_-^P = \begin{matrix} * \\ * \\ * \end{matrix}.$$

So $w_0 \neq w_0^P$ etecetera.

(b) Let S be: a: $\neg a \vee \neg b$, b: $\neg a \wedge \neg b$.
$E := \mathbb{T}^{\{a,b\}}$, $f := F_S$, $P :\in \{G, B\}$, then:

$$j^P = \begin{matrix} * \\ * \end{matrix}, \quad i_-^P = \begin{matrix} T \\ F \end{matrix}. \text{ So } j^P \neq i_-^P.$$

(c) Let S be: l: $\neg l$, s: s, a: $(s \vee \neg s) \wedge \neg a$, b: $(l \vee \neg l) \wedge b$.
$E := \mathbb{T}^{\{l,s,a,b\}}$, $f := F_S$, $P :\in \{G, B\}$, then:

$$w_0^P = \begin{matrix} TF \\ * \\ TF \\ * \end{matrix}, \quad w^P = \begin{matrix} TF \\ * \\ * \\ F \end{matrix}, \quad i_-^P = (w_0^P)^\Delta = \begin{matrix} * \\ * \\ * \\ * \end{matrix}, \quad i_+^P = (w^P)^\Delta = \begin{matrix} * \\ * \\ * \\ F \end{matrix}$$

Hence w_0^P, w^P are incomparable (this is in contrast to w_0, w) and $i_-^P \neq i_+^P$.

3.6.2.3. REMARK. We have: $j \leqslant j^P \leqslant i_-^P \leqslant i_+^P \leqslant i$. Each of these fixed points represents a different notion of intrinsicity and the sentences separating them illustrate genuinely different selfreferential phenomena. In a sense a with a: $a \vee \neg a$ is the 'most intrinsic' of all examples considered; c of Example 3.6.2.2 (a) the least.

I finish with one of the Little Theorems.

3.6.2.4. LITTLE THEOREM. Let P be G or B. Suppose E is finite. Then $w_0^P = \sqcap\{\sqcup C \mid C \text{ a finite cycle}\}$.

Proof. Consider a finite cycle $C = \{c_0, \ldots, c_{k-1}\}$, $[c_0]^\alpha := c_{\alpha \bmod k}$ is in G/B and $\limsup_{\alpha \to \infty} c_{\alpha \bmod k} = \sqcup C$. Hence $\{\sqcup C \mid C \text{ is a finite cycle}\}$ is a subset of $\{\limsup \pi \mid \pi \in P\}$. On the other hand consider any π in P. There are only finitely many finite cycles, so some finite cycle C' must occur cofinally in π. Hence $\sqcup C' \leqslant \limsup \pi$. Conclude that $\{\sqcup C \mid C \text{ is a finite cycle}\}$ minorizes $\{\limsup \pi \mid \pi \in P\}$. It follows that $\sqcap\{\limsup \pi \mid \pi \in P\} = \sqcap\{\sqcup C \mid C \text{ is a finite cycle}\}$. □

NOTE

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This paper is a successor of an earlier unpublished draft "Sketchy Notes on Four Valued Logic and a 'Solution' of the Liar". Shortly after writing Sketchy Notes I received Peter Woodruff's "Paradox, Truth and Logic. Part I: Paradox and Truth". In the process of transmutation of "Sketchy Notes" into the present paper I have not hesitated to use certain ideas of Peter Woodruff – especially his notion of overlap – to improve both contents and presentation.

Section 3.3 on Iterations is an adaptation of Herzberger's earlier work on iterations. Most of the ideas of that section have to be credited to him.

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