Soc Choice Welfare (1984) 1:113–122

Social Choice and Welfare

An Axiomatic Characterization of the Lexicographic Maximin Extension of an Ordering Over a Set to the Power Set

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Received August 1, 1983 / Accepted November 7, 1983

Abstract. The lexicographic maximin extension of an ordering is an important and widely used tool in social choice theory. We provide an axiomatization of it by means of five axioms. When the basic ordering is linear the following four (independent) axioms are sufficient: (1) Gärdenfors principle; (2) Neutrality; (3) Strong Fishburn monotonicity; and (4) Extension. Our result may also have applications in the theory of individual choice under uncertainty.

1. Introduction

Let R be an ordering defined over a finite set X. Let a binary relation $\succeq$ over $\pi(X) \equiv 2^X - \{\phi\}$ be called an *extension* of R to $\pi(X)$ if and only if for all $x, y \in X$, $x R y$ iff $\{x\} \succeq \{y\}$. Are there extensions of R to $\pi(X)$, satisfying certain reasonable axioms? This is the problem considered in a number of recent papers (see [3, 5, 8, 9, 14, and 15]; see also earlier contributions [2, 4, 6, 7, 10, 16, and 18] in this general area). The main purpose of this paper is to provide necessary and sufficient conditions for an ordering $\succeq$ over $\pi(X)$ to be the "lexicographic maximin extension" (see Definition 3.1) of R. In Sect. 2 we first examine the intuitive significance of the general problem of inducing an extension of R to $\pi(X)$, and then comment on the interest of the lexicographic maximin extension. In Sect. 3 we introduce some basic notation and the notion of the lexicographic maximin extension. In Sect. 4 we characterize the lexicographic maximin extension of R. In Sect. 5 we consider this characterization problem again for the special case where R is assumed to be a linear ordering. We conclude in Sect. 6 with a characterization of the lexicographic maximax extension which is, intuitively, the 'dual' of the lexicographic maximin extension.

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2. Intuitive Motivation

One of the earliest motivations for extending an ordering R over X to $\pi(X)$ is to be found in the literature on manipulability in social choice theory (see [2, 4, 6, 7, 10, and 16]). In a voting game where the outcome function is represented by a social choice correspondence rather than a social choice function (so that given the individual voting strategies we have a set of possibly many outcomes rather than a single outcome), the game cannot be defined properly unless the honest preference ordering R_i (over the set X of possible outcomes) of each voter i is extended to derive an ordering $\succeq_i$ over alternative sets of outcomes. Of course, if the social choice correspondence is supplemented by a deterministic tie-breaking mechanism which is assumed to be known to the voters and which figures explicitly in the model, then the problem of inducing $\succeq_i$ from R_i becomes trivial. (A *tie-breaking mechanism* can be defined to be a function p which for every $A \in \pi(X)$ and for every $x \in A$, specifies a non-negative real number $p(x, A)$ such that $\sum_{y \in B} p(y, B) = 1$ for all $B \in \pi(X)$; a tie-breaking mechanism is *deterministic* iff for every $A \in \pi(X)$ and every $x \in A$, $p(x, A) = 0$ or $p(x, A) = 1$.) Therefore, one can distinguish at least three types of situations where the 'extension problem' will be non-trivial in voting theory.

1) The first case arises when p is assumed to be known to the voters but is non-deterministic. Then the problem is one of extending R_i to derive $\succeq_i$ when the voters are facing risk.

2) Even when the voter is assumed to know the tie-breaking mechanism which may or may not be deterministic, the model-builder may, for reasons of generality, like to consider a class of tie-breaking mechanisms rather than a single tie-breaking mechanism. In that case, the model-builder may specify a few properties common to all extensions of an ordering generated by the different tie-breaking mechanisms belonging to this class, and then analyse the game in terms of all possible extensions which satisfy these specified properties.

3) Lastly, consider the case when the tie-breaking mechanism p is not known to the voter, and the voter does not have subjective probabilities for the elements belonging to different possible sets of outcomes. In this case the problem is one of extending R_i to arrive at $\succeq_i$ when a voter is facing a situation of uncertainty rather than risk (to use the terminology of Luce and Raiffa [12]). It may, however, be remarked that this case may be rare in so far as social choice correspondences are often explicitly supplemented by tie-breaking mechanisms.

Apart from voting theory, the extension problem seems to be important in the wider context of choice under uncertainty. Consider an agent faced with the problem of choosing an action from a finite set of actions, $Y = \{y_1, \ldots, y_m\}$, under uncertainty. The set of all possible outcomes is X and R is the agent's preference ordering over X. For each $y_i \in Y$ we have a non-empty set of possible outcomes $F(y_i) \subseteq X$; however, the agent does not have any subjective or objective probability distribution over the outcomes in $F(y_i)$. For this agent the problem of ranking the actions under uncertainty can be interpreted as a problem of inducing an extension over $\{F(y_1), \ldots, F(y_m)\}$ given the ordering R over X. Since this way of posing the problem of an agent's choice under uncertainty is somewhat different from certain earlier formulations (see for example, Milnor [13], Luce and Raiffa [12], and Arrow and Hurwicz [1]), it may be worthwhile spelling out the difference.

In these earlier formulations, a finite set $S = \{s_1, \dots, s_n\}$ of possible states of the world is explicitly introduced into the model. For all $y_i \in Y$ and all $s_j \in S$, there is a specific outcome z_{ij} belonging to X. Thus, we have an $m \times n$ matrix $[z_{ij}]$ of outcomes. The problem of ranking actions y_i under uncertainty is then identified as the problem of ranking the rows r_i in the matrix $[z_{ij}]$ given the ordering R over the set X of distinct outcomes figuring in the matrix $[z_{ij}]$. This formulation of the problem of choice under uncertainty (for convenience, we call it Type *I* formulation) is not identical with our earlier formulation (we call it Type *II* formulation) in terms of $\succeq$ defined over $\{F(y_1), \dots, F(y_m)\}$, i.e. the class of sets of outcomes corresponding to different actions, though the two formulations are closely related. To illustrate the difference, consider the following example with $Y = \{y_1, y_2, y_3\}$, $S = \{s_1, s_2, s_3\}$ and the outcomes being represented by ordinal utility numbers.

	s_1	s_2	s_3
y_1	0	0	2
y_2	2	0	2
y_3	3	1	2

Under Type *I* formulation, y_1, y_2 and y_3 will be ranked by ranking the rows (0, 0, 2), (2, 0, 2) and (3, 1, 2) while under Type *II* formulation, the ranking of y_1, y_2, and y_3 is intuitively identified with the ranking of the sets $\{0, 2\}$ and $\{3, 1, 2\}$. It is clear that Type *II* formulation involves some loss of information as compared to Type *I*. A striking example of such loss of information arises when the information about weak dominance between actions gets lost in the process of transforming the rows of outcomes into the corresponding sets of outcomes. For example, y_2 weakly dominates y_1 (i.e. y_2 yields at least as good an outcome as y_1 for every state of the world and a strictly better outcome than y_1 under some state of the world). However, there is no way of retaining this information in Type *II* formulation where each of y_1 and y_2 is seen to yield the same set $\{0, 2\}$ of outcomes. To put the point slightly differently, in Type *I* formulation one views each action as a function from S to the set of outcomes and ranks these functions on the basis of their complete description and R. In Type *II* formulation one ranks the functions solely on the basis of their ranges (given the ordering R); this inevitably leads to some loss of information in Type *II* formulation. (To some extent this loss of information, especially the loss of information regarding weak dominance, can be prevented by labelling the outcome for any y_i and s_j as 0_{ij}, and by treating the outcomes 0_{ij} as all different. However, this raises other conceptual problems regarding the interpretation of certain axioms used in this paper, and this needs further investigation.)

Despite what we have said above, there are many circumstances where one may like to have a Type *II* formulation (involving extension of R to $\pi(X)$) of the problem of choice under uncertainty. First, when the number of possible states of the world is large, an agent of bounded rationality may be incapable of undertaking (or unwilling to undertake) the complex calculations which consideration of the entire rows in the outcome matrix will involve. For example, consider a two-stage election process where in the first stage a relatively small committee, say C, chooses a panel of alternatives out of a given set and in the second stage a larger electorate E chooses a single alternative out of this panel (the pre-Eurovision song contests in Israel seem to be of this type). Consider any $i \in C$ and assume that i has very imperfect information about the preferences of members of E. In this case, in principle, i can list all the different possible contingencies (i.e. different voting patterns) that can arise in the second stage and construct a matrix which will show the final outcomes corresponding to each panel of alternatives chosen in the first stage and each contingency (voting pattern) in the second. However, if E is even moderately large, then this may prove to be such a demanding task that i may very well decide to view the information in a compressed from by just considering the sets of possible outcomes corresponding to different panels of alternatives chosen in the first stage. Secondly, several writers (see Arrow and Hurwicz [1] and Luce and Raiffa [12]) have commented on the arbitrariness of partitioning all possible contingencies in a particular way so as to have a specific set of states of the world. For example, if the amount of rain in a particular season can vary from 5 to 40 inches, any partitioning of these infinite number of contingencies for a farmer into, say, adequate rain, excessive rain and inadequate rain, would seem somewhat arbitrary. In view of this arbitrariness it would seem desirable to construct a model of choice under uncertainty where the problem is posed not in terms of the outcomes of each action corresponding to different states of the world but in terms of the sets of outcomes associated with each action. Lastly, there are certain ethical frameworks such as that of Rawls [17] which involve the conceptual experiment of putting an individual under uncertainty (e.g. the Rawlsian 'veil of ignorance') where there does not seem to be any obvious non-trivial formulation of the notion of states of the world as distinguished from the possible outcomes corresponding to an action. In such situations, the agent's choice problem under uncertainty can again be formulated in our Type *II* fashion.

So far we have considered interpretations of the extension problem in terms of voting theory and choice under uncertainty. One can think of a third interpretation distinct from these. Here every $A \in \pi(X)$ is interpreted not as a set of mutually exclusive outcomes only one of which will materialize finally, but as a set of objects which the agent can have simultaneously. This is, for example, the case with the problem of plausible reasoning considered by Rescher [18] and Packard [15] (see also Heiner and Packard [8] for another interpretation along this line). In Rescher [18] and Packard [15], X is interpreted as a set of hypotheses; R is interpreted as a ranking of these hypotheses in terms of plausibility; and $\succeq$ is an induced plausibility ranking of sets of consistent hypotheses (note that here $\succeq$ is not defined over all possible non-empty subsets of X since inconsistent sets of hypotheses are excluded; also the empty set is not excluded in Packard [15]. However, these are technical details which we have ignored in this discussion of intuitive interpretation.)

It is clear that the problem of extending R over X to generate $\succeq$ over $\pi(X)$ admits a wide range of applications. In this paper we have chosen to study one particular type of such extension, where $\succeq$ is a lexicographic maximin extension of R. We provide an axiomatic characterization of the lexicographic maximin extension. Such characterization seems to be of interest in view of the fact that the maximin extension in various forms has figured extensively in the theory of justice (see Rawls [17]), voting theory (see Pattanaik [16]), the theory of games, and the theory of choice under uncertainty (see Milnor [13]).

3. The Lexicographic Maximin Extension of R

As indicated earlier, R is a given ordering (i.e., it is complete and transitive) over a finite set X; and a binary relation $\succsim$ over $\pi(X) \equiv 2^X - \{\phi\}$ is called an extension of R to $\pi(X)$ iff for all $x, y \in X$, $[x R y$ iff $\{x\} \succsim \{y\}]$. Let P and I respectively be the asymmetric and symmetric factors of R; and let $\succ$ and $\sim$ respectively be the asymmetric and symmetric factors of $\succsim$. For all $A, B \in \pi(X)$, $[A\mathbf{R}B$ iff xRy for all $x \in A$ and all $y \in B]$; $[A\mathbf{P}B$ iff xPy for all $x \in A$ and all $y \in B]$; and lastly, $[A\mathbf{I}B$ iff xIy for all $x \in A$ and all $y \in B]$. We write $x\mathbf{R}A$, $A\mathbf{R}x$ etc. instead of $\{x\}\mathbf{R}A$, $A\mathbf{R}\{x\}$ etc.

Let $|X| = n$. Let Re and Re^+ respectively indicate the set of real numbers and the set of positive real numbers. Let u: $X \to \mathrm{Re}^+$ satisfying $u(x) \geq u(y)$ iff xRy. If $A = \{a_1, a_2, \ldots, a_s\}$ such that $u(a_1) \leq u(a_2) \leq \ldots \leq u(a_s)$, then we denote

$$v_*(A) = (u(a_1), \ldots, u(a_s), \overbrace{0, \ldots, 0}^{n-s}).$$

Definition 3.1: A binary relation $\succsim$ over $\pi(X)$ is called the lexicographic maximin extension of R to $\pi(X)$ iff for all $A, B \in \pi(X)$

(3.1) if $A\mathbf{I}B$, then $A \sim B$; and

(3.2) if not $A\mathbf{I}B$, then $[A \succsim B$ iff $v_*(A) \geq_L v_*(B)]$ where $\geq_L$ is the lexicographic ordering on $\mathrm{Re}^{(n)}$.

The lexicographic maximin extension of R to $\pi(X)$ will be indicated by $\succsim_*$.

Remark 3.1. If R is a linear (i.e., antisymmetric) ordering then for all $A, B \in \pi(X)$, $A \succsim_* B$ iff $v_*(A) \geq_L v_*(B)$.

Remark 3.2. It can be easily checked that $\succsim_*$ is an ordering.

4. Characterization of the Lexicographic Maximin Extension

In this section we provide a characterization of $\succsim_*$.

For all $A, B \in \pi(X)$, let $A \bigcap\!\!\!\!\mathrm{I}\; B = \phi$ iff for all $x \in A$ and all $y \in B$, not xIy.

Definition 4.1. Let $\succsim$ be a binary relation over $\pi(X)$. $\succsim$ satisfies

(4.1.1) *Gardenfors Principle* (GP) iff for all $A \in \pi(X)$ and for all $x \in X$,

(i) $x\mathbf{I}A$ implies $\{x\} \cup A \sim A$;
(ii) ($x \notin A$ and $x\mathbf{R}A$ and not $A\mathbf{R}x$) implies $\{x\} \cup A \succ A$; and
(iii) ($x \notin A$ and $A\mathbf{R}x$ and not $x\mathbf{R}A$) implies $A \succ \{x\} \cup A$;

(4.1.2) *Extension* (E) iff for all $A, B \in \pi(X)$, and for all $x \in X - (A \cup B)$, $[(A \cup B)\mathbf{R}x$ and $A \succ B]$ implies $\{x\} \cup A \succ \{x\} \cup B$;

(4.1.3) *Strong Fishburn Monotonicity* (SFM) iff for all $A, B \in \pi(X)$ such that $A \bigcap\!\!\!\!\mathrm{I}\; B = \phi$, and for all $x \in X - (A \cup B)$, $[\{x\} \cup A \succ \{x\} \cup B$ iff $A \succ B]$;

(4.1.4) *Neutrality* (N) iff for all $A, B \in \pi(X)$, all one-to-one f: $A \to X$, and all one-to-one g: $B \to X$, if [for all $x \in A$ and all $y \in B$, (xPy iff $f(x)P\, g(y)$) and (xIy iff $f(x)I\, g(y)$)] then $[A \succsim B$ iff $f(A) \succsim g(B)]$; and

(4.1.5) *Union* (U) iff for all $a \in X$, and all $B, C \in \pi(X)$, $[\{a\} \succ B$ and $\{a\} \succ C]$ implies $\{a\} \succ B \cup C$.

GP in a somewhat different form originated in Gardenfors [6]; the present definition is a slightly modified version of the definition given in Kannai and Peleg [9]. GP requires three things. First, if everything in A is indifferent to everything in B, A must be indifferent to B. Secondly, if x, not belonging to A, is at least as good as everything in A and is strictly preferred to something in A, then $\{x\} \cup A$ must be better than A. Lastly, if everything in A is at least as good as x (not belonging to A) and something in A is strictly better than x, then A is better than $A \cup \{x\}$. E requires that if A is better than B, then adding to A and also to B, an element which is outside both A and B and which is not better than any element in A or in B, does not change the ranking of the sets. A weaker version of SFM is to be found in Fishburn [5]. SFM requires that if nothing in A is indifferent to anything in B, and if x is outside both A and B, then the ranking of $\{x\} \cup A$ and $\{x\} \cup B$ will be exactly analogous to the ranking of A and B. N is reminiscent of the corresponding condition in social choice theory. U requires that if $\{a\}$ is better than B and also better than C, then $\{a\}$ is better than $B \cup C$.

Theorem 4.1. *Assume that there exist $\hat{x}, \hat{y}, \hat{z}, \hat{w} \in X$ such that $\hat{x} \, P \, \hat{y} \, P \, \hat{z} \, P \, \hat{w}$, and let $\succeq$ be an ordering over $\pi(X)$. Then $\succeq \, = \, \succeq_*$ iff $\succeq$ satisfies* GP, E, SFM, N *and* U.

Proof: The necessity part of the theorem is straightforward. So we prove only the sufficiency part which says that if $\succeq$ satisfies GP, E, SFM, N, and U, then $\succeq \, = \, \succeq_*$.

Let $\succeq$ satisfy GP, E, SFM, N and U. Let $\succ_*$ and $\sim_*$ respectively be the asymmetric factor and the symmetric factor of $\succeq_*$. For all $A \in \pi(X)$, let $\min(A)$ be the set of R-least elements in A and let $\max(A)$ be the set of R-greatest elements in A. We first prove

(4.1) For all $A, B \in \pi(X)$ and for all f: $A \to B$, if f is one-to-one onto B such that for all $x \in A$, $x I f(x)$, then $A \sim B$. Consider $A, B \in \pi(X)$ and f: $A \to B$ where f is one-to-one and onto B and $x I f(x)$ for all $x \in A$. Since $\succeq$ is complete, without loss of generality assume $A \succeq B$. By N, $f(A) \succeq f^{-1}(B)$, i.e., $B \succeq A$. Hence $A \sim B$ which proves (4.1).

Now consider any $A, B \in \pi(X)$ such that $A \sim_* B$. Given $A \sim_* B$, there are two possibilities: (i) $A \, \mathbf{I} \, B$ or (ii) there exists f: $A \to B$ such that f is one-to-one onto B and for all $x \in A$, $x I f(x)$. If (i) holds, then by GP, $A \sim B$. If (ii) holds, then by (4.1), $A \sim B$. Therefore

(4.2) For all $A, B \in \pi(X)$, if $A \sim_* B$, then $A \sim B$.

Next we show

(4.3) For all $x, y, z \in X$, if $x P y P z$, then $\{y\} \succ \{x, z\}$. By assumption there exist $\hat{x}, \hat{y}, \hat{z}, \hat{w} \in X$ such that $\hat{x} \, P \, \hat{y} \, P \, \hat{z} \, P \, \hat{w}$. By GP, $\{\hat{z}\} \succ \{\hat{z}, \hat{w}\}$ and $\{\hat{z}, \hat{w}\} \succ \{\hat{w}\}$. Hence by transitivity of $\succeq$, $\{\hat{z}\} \succ \{\hat{w}\}$. Then by SFM, $\{\hat{x}, \hat{z}\} \succ \{\hat{x}, \hat{w}\}$. Now by GP, $\{\hat{x}\} \succ \{\hat{x}, \hat{y}\}$. Hence by E, $\{\hat{x}, \hat{w}\} \succ \{\hat{x}, \hat{y}, \hat{w}\}$. Given $\{\hat{x}, \hat{z}\} \succ \{\hat{x}, \hat{w}\}$, it follows that $\{\hat{x}, \hat{z}\} \succ \{\hat{x}, \hat{y}, \hat{w}\}$. Hence by SFM, $\{\hat{z}\} \succ \{\hat{y}, \hat{w}\}$. Noting $\hat{y} \, P \, \hat{z} \, P \, \hat{w}$, (4.3) now follows by N.

We now prove

(4.4) For all $A, B \in \pi(X)$, if $\min(A) \, \mathbf{P} \, \min(B)$, then $A \succ B$. Suppose $\min(A) \, \mathbf{P}$ $\min(B)$. Let $\tilde{x} \in \min(A)$ and $\tilde{y} \in \min(B)$. Then $\tilde{x} \, P \, \tilde{y}$. If $B = \{\tilde{y}\}$, then by GP, $\{\tilde{x}\} \succ \{\tilde{x}, \tilde{y}\}$ and $\{\tilde{x}, \tilde{y}\} \succ \{\tilde{y}\} = B$. By repeated application of GP and transitivity of $\succeq$, $A \succeq \{\tilde{x}\}$ and hence it follows (by transitivity of $\succeq$) that $A \succ B$ in the case where $B = \{\tilde{y}\}$. Now suppose $B \neq \{\tilde{y}\}$. Then let $\tilde{z} \in B - \{\tilde{y}\}$. There are three subcases here: (i) $\tilde{x} \, P \, \tilde{z}$ or (ii) $\tilde{x} \, I \, \tilde{z}$ or (iii) $\tilde{z} \, P \, \tilde{x}$. If $\tilde{x} \, P \, \tilde{z}$, then by GP $\{\tilde{x}\} \succ \{\tilde{x}, \tilde{z}\}$, $\{\tilde{x}, \tilde{z}\} \succ \{\tilde{x}, \tilde{z}, \tilde{y}\}$, and

$\{\tilde{x},\tilde{z},\tilde{y}\} \succ \{\tilde{z},\tilde{y}\}$; and hence $\{\tilde{x}\} \succ \{\tilde{z},\tilde{y}\}$ by transitivity of $\succeq$. If $\tilde{x}\,I\,\tilde{z}$, then by (4.1), $\{\tilde{x},\tilde{y}\} \sim \{\tilde{z},\tilde{y}\}$. Since $\{\tilde{x}\} \succ \{\tilde{x},\tilde{y}\}$ by GP, it follows that $\{\tilde{x}\} \succ \{\tilde{z},\tilde{y}\}$ in the case where $\tilde{x}\,I\,\tilde{z}$. Finally, if $\tilde{z}\,P\,\tilde{x}$, then $\{\tilde{x}\} \succ \{\tilde{z},\tilde{y}\}$ by (4.3). Thus in all cases $\{\tilde{x}\} \succ \{\tilde{z},\tilde{y}\}$ for all $\tilde{z} \in B - \{\tilde{y}\}$. Hence by U, $\{\tilde{x}\} \succ B$. Since $A \succeq \{\tilde{x}\}$ (by repeated application of GP and transitivity of $\succeq$), it follows that $A \succ B$. This completes the proof of (4.4).

Next, we prove

(4.5) For all $A, B \in \pi(X)$, $A \succ_* B$ implies $A \succ B$.

(4.5) is proved by first proving

(4.6) For all $A, B \in \pi(X)$, if $|A| = 1$ and $A \succ_* B$ then $A \succ B$, and then proving

(4.7) For all $k (1 \le k < |X|)$, if for all $A', B' \in \pi(X)$, $[(|A'| \le k$ and $A' \succ_* B')$ implies $(A' \succ B')]$, then for all $A, B \in \pi(X)$, $[(|A| \le k+1$ and $A \succ_* B)$ implies $(A \succ B)]$.

Consider (4.6). If $|A| = 1$, then given $A \succ_* B$, it is clear that $\min(A)\,\mathbf{P}\,\min(B)$ and hence by (4.4), $A \succ B$. This proves (4.6). Now consider (4.7). Suppose the hypothesis of (4.7) holds. Consider A such that $|A| = k+1$. Let $A \succ_* B$. If $\min(A)\,\mathbf{P}\,\min(B)$, then $A \succ B$ follows by (4.4). Suppose $\min(A)\,\mathbf{I}\,\min(B)$. Since $A \succ_* B$, it is clear that $A - \min(A) \neq \phi$. There are two possibilities: *(I)* $\min(A) \cap \min(B) \neq \phi$ or *(II)* $\min(A) \cap \min(B) = \phi$. Suppose *(I)* holds so that $\min(A) \cap \min(B) \neq \phi$. Let $a \in \min(A) \cap \min(B)$. If $B - \{a\} = \phi$, then given $A - \min(A) \neq \phi$, repeated application of GP and transitivity of $\succeq$ gives us $A \succ B$. If $B - \{a\} \neq \phi$, then indicating $A - \{a\}$ by A' and $B - \{a\}$ by B', we have two possibilities: either $[A' \succ_* B']$ or $[A'\,\mathbf{I}\,B'$ and $|A'| > |B'|]$. If $A' \succ_* B'$, then by the hypothesis of (4.7), $A' \succ B'$, and hence by E, $A \succ B$. Suppose $[A'\,\mathbf{I}\,B'$ and $|A'| > |B'|]$. Let $A'' \subset A'$ be such that $|A''| = |B'|$. Then by (4.1), $(A'' \cup \{a\}) \sim (B' \cup \{a\}) = B$. Hence by GP $A' \cup \{a\} \succ B$, i.e., $A \succ B$. Thus if *(I)* holds, $A \succ B$. Suppose *(II)* holds so that $\min(A) \cap \min(B) = \phi$. Let $a^* \in \min(A)$ and $b^* \in \min(B)$. If $B - \{b^*\} = \phi$, then given $A - \min(A) \neq \phi$, GP and transitivity of $\succeq$ ensure $A \succ B$. If $B - \{b^*\} \neq \phi$, then indicating $A - \{a^*\}$ by A^* and $B - \{b^*\}$ by B^*, we have two possibilities: either $[A^* \succ_* B^*]$ or $[A^*\,\mathbf{I}\,B^*$ and $|A^*| > |B^*|]$. Suppose $A^* \succ_* B^*$. Then by the hypothesis of (4.7), $A^* \succ B^*$. Then by E, $A \succ B^* \cup \{a^*\}$. By (4.1) $(B^* \cup \{a^*\}) \sim B$. Hence by transitivity of $\succeq$, $A \succ B$. If $[A^*\,\mathbf{I}\,B^*$ and $|A^*| > |B^*|]$, then the proof consists of a slight modification of the proof of the corresponding subcase considered earlier when we assumed $\min(A) \cap \min(B) \neq \phi$, $a \in \min(A) \cap \min(B)$ and $B - \{a\} \neq \phi$. Thus in all cases, when *(II)* holds, $A \succ B$. This completes the proof of (4.7).

Since $\succeq_*$ is complete, (4.2) and (4.5) together imply that $\succeq\, = \,\succeq_*$.

Remark 4.1. Under the assumption of Theorem 4.1, the four axioms – GP, E, SFM, and N, are independent in the sense that any one of them can be violated while the other three and U are satisfied. This is shown by the following example. The independence of U is still an unresolved problem.

Example 4.1: (i) Let $X = \{x, y, z, w\}$ and $x\,P\,y\,P\,z\,P\,w$. Let the ordering $\succeq$, be such that for all $A, B \in \pi(X)$, $A \sim B$. Then $\succeq$ violates GP but trivially satisfies E, SFM, N and U.

(ii) The lexicographic maximax extension (see Definition 6.1 below) violates E but satisfies GP, SFM, N and U.

(iii) Let $X = \{x, y, z, w\}$ and $x\,P\,y\,P\,z\,P\,w$. Let the ordering $\succeq$ be as follows: $\{x\} \succ \{x,y\} \succ \{x,z\} \succ \{x,w\} \succ \{x,y,z\} \succ \{x,y,w\} \succ \{y\} \succ \{y,z\} \succ \{x,z,w\} \succ \{x,y,z,w\} \succ$

$\{y,w\} \succ \{y,z,w\} \succ \{z\} \succ \{z,w\} \succ \{w\}$. $\succeq$ violates SFM since $\{y,w\} \succ \{z\}$ and $\{x,z\} \succ \{x,y,w\}$. However, it can be checked that $\succeq$ satisfies GP, E, N and U.

(iv) Let $X = \{x,y,z,w\}$ and $x\,P\,y\,P\,z\,P\,w$. Let the ordering $\succeq$ be as follows: $\{x\} \succ \{x,y\} \succ \{y\} \succ \{x,z\} \succ \{x,y,z\} \succ \{x,w\} \succ \{x,w,y\} \succ \{y,z\} \succ \{z\} \succ \{y,w\} \succ \{x,z,w\} \succ \{x,y,z,w\} \succ \{y,z,w\} \succ \{z,w\} \succ \{w\}$. $\succeq$ violates N since $\{y\} \succ \{x,z\}$ and $\{z\} \prec \{x,w\}$. However, $\succ$ satisfies GP, E, SFM and U.

5. Characterization of the Lexicographic Maximin Extension when *R* is a Linear Ordering

In the special case where R is a linear ordering, the characterization of $\succeq_*$ becomes simpler and intuitively more transparent. In this section we discuss this special case.

First, we note that when R is a linear ordering, GP, E and SFM take somewhat simpler forms.

Proposition 5.1. If R is a linear ordering, then a reflexive binary relation $\succeq$ over $\pi(X)$ satisfies

(5.1.1) GP iff for all $A \in \pi(X)$ and all $x \in X$, [$x\,\mathbf{P}\,A$ implies $A \cup \{x\} \succ A$] and [$A\,\mathbf{P}\,x$ implies $A \succ A \cup \{x\}$];

(5.1.2) E iff for all $A, B \in \pi(X)$ and for all $x \in X$ such that $(A \cup B)\,\mathbf{P}\,x$, $A \succ B$ implies $(A \cup \{x\} \succ B \cup \{x\})$;

(5.1.3) SFM iff for all $A, B \in \pi(X)$ and all $x \in X$ such that $x \notin A \cup B$ and $A \cap B = \phi$, [$\{x\} \cup A \succ \{x\} \cup B$ iff $A \succ B$].

The proof of Proposition 5.1 is obvious and we omit it.

Proposition 5.2. If $|X| \geq 4$, R is a linear ordering and $\succeq$ is an ordering over $\pi(X)$, then GP, E, SFM and N together imply U.

Proof: Let $\succeq$ satisfy GP, E, SFM and N. First, we claim

(5.1) for all $x,y,z \in X$, if $x\,P\,y\,P\,z$, then $\{y\} \succ \{x,z\}$. The proof of (5.1) is exactly similar to the proof of (4.3) and is therefore omitted. (Note that the proof of (4.3) involved only GP, E, SFM, and N and did not involve U.)

Next, we show that

(5.2) For all $A \in \pi(X)$, if $|A| \geq 3$, then $(\min(A) \cup \max(A)) \succ A$. Let $A = \{a_1, \ldots, a_s\}$ where $a_i\,P\,a_{i-1}$, $i = 2, \ldots, s$. By GP, $\{a_s\} \succ \{a_s, a_{s-1}\}$ & $\{a_s, a_{s-1}\} \succ \{a_s, a_{s-1}, a_{s-2}\}$ & … & $\{a_s, \ldots, a_3\} \succ \{a_s, \ldots, a_2\}$. Hence by transitivity of $\succeq$, $\{a_s\} \succ \{a_s, \ldots, a_2\}$. Hence by E, $\{a_1, a_s\} \succ A$. This proves (5.2).

Now consider $x \in X$, and $B, C \in \pi(X)$ such that $\{x\} \succ B$ and $\{x\} \succ C$. By GP, and transitivity of $\succeq$, it is clear that $\{x\}\,\mathbf{P}\,\min(B)$ and $\{x\}\,\mathbf{P}\,\min(C)$. Hence $\{x\}\,\mathbf{P}\,\min(B \cup C)$. Let $\max(B \cup C) = \{y\}$ and $\min(B \cup C) = \{z\}$. Then $x\,P\,z$. If $x\,R\,y$ then by GP, it is clear that $\{x\} \succ B \cup C$. Suppose $y\,P\,x$. Then $y\,P\,x\,P\,z$. Then by (5.1) $\{x\} \succ \{y,z\}$, and noting (5.2), $\{y,z\} \succeq B \cup C$. Hence by transitivity of $\succeq$, $\{x\} \succ B \cup C$. This completes the proof of the proposition.

Theorem 5.1. *Let* $|X| \geq 4$ *and let R be a linear ordering. Let* $\succeq$ *be an ordering over* $\pi(X)$. *Then* $\succeq = \succeq_*$ *iff* $\succeq$ *satisfies* GP, E, SFM *and* N.

Proof: Theorem 5.1 follows immediately from Theorem 4.1 and Proposition 5.2.

Remark 5.1. It is possible to generalize Theorem 5.1 somewhat, by requiring $\succeq$ to be only reflexive and transitive (but not necessarily connected) over $\pi(X)$ instead of requiring $\succeq$ to be an ordering.

Remark 5.2. Under the assumption of Theorem 5.1, GP, E, SFM and N are independent. This is shown by the orderings $\succeq$ constructed in cases (i), (ii), (iii) and (iv) of Example 4.1 (note that in each of these four cases the ordering R was assumed to be linear).

6. Characterization of the Lexicographic Maximax Extension

Though our primary interest in this paper is to provide axiomatic characterization of the lexicographic maximin extension, our investigation also yields, as a by-product, a characterization of the lexicographic maximax extension of R.

Let $u^* \in \mathrm{Re}^+$ be such that $u^* > u(x)$ for all $x \in X$. Then for all $A = \{a_1, \ldots, a_s\}$ such that $u(a_s) \geq u(a_{s-1}) \geq \ldots \geq u(a_1)$, we denote

$$v^*(A) = (u(a_s), \ldots, u(a_1), \overbrace{u^*, \ldots, u^*}^{n-s})$$

Definition 6.1. A binary relation $\succeq$ over $\pi(X)$ is called the *lexicographic maximax extension* of R to $\pi(X)$ iff for all $A, B \in \pi(X)$,

(6.1) If $A\mathbf{I}B$, then $A \sim B$; and

(6.2) If not $A\mathbf{I}B$, then $[A \succeq B$ iff $v^*(A) \geq_L v^*(B)]$ where $\geq_L$ is the lexicographic ordering on $\mathrm{Re}^{(n)}$.

The lexicographic maximax extension of R to $\pi(X)$ will be indicated by $\succeq^*$.

Definition 6.2. Let $\succeq$ be a binary relation over $\pi(X)$. $\succeq$ satisfies

(6.2.1) (E′) iff for all $A, B \in \pi(X)$ and for all $x \in X - (A \cup B)$, $[x\mathbf{R}(A \cup B)$ and $A \succ B]$ implies $\{x\} \cup A \succ \{x\} \cup B$; and

(6.2.2) (U′) iff for all $a \in X$, and all $B, C \in \pi(X)$, $[B \succ \{a\}$ and $C \succ \{a\}]$ implies $[B \cup C \succ \{a\}]$.

E′ and U′ are, intuitively, the duals of E and U respectively.

We now state Theorem 6.1 which provides characterization of the lexicographic maximax extension for the general case where R is an ordering (not necessarily linear), and Theorem 6.2 which provides a characterization of the lexicographic maximax extension for the special case where R is a linear ordering. The proofs of these theorems are omitted since the proof of Theorem 6.1 is exactly analogous to the proof of Theorem 4.1 and the proof of Theorem 6.2 is exactly analogous to the proof of Theorem 5.1.

Theorem 6.1. *Assume that there exist $\hat{x}, \hat{y}, \hat{z}, \hat{w} \in X$ such that $\hat{x}\, P\, \hat{y}\, P\, \hat{z}\, P\, \hat{w}$, and let $\succeq$ be an ordering over $\pi(X)$. Then $\succeq\, = \,\succeq^*$ iff $\succeq$ satisfies* GP, E′, SFM, N *and* U′.

Theorem 6.2. *Let $|X| \geq 4$ and let R be a linear ordering. Let $\succeq$ be an ordering over $\pi(X)$. Then $\succeq\, = \,\succeq^*$ iff $\succeq$ satisfies* GP, E′, SFM *and* N.

References

1. Arrow KJ, Hurwicz L (1972) An optimality criterion for decision-making under ignorance. In: Carter CF, Ford JL (eds) Uncertainty and expectations in economics. Augustus M. Kelley, Fairfield, NJ
2. Barbera S (1977) The manipulation of social choice mechanisms that do not leave too much to chance. Econometrica 45 : 1573–1588
3. Barbera S, Pattanaik PK Extending an order on a set to the power set: Some remarks on Kannai and Peleg's approach. (mimeograph, 1982); J Econ Theory (unknown)
4. Fishburn PC (1972) Even-chance lotteries in social choice theory. Theory Decision 3 : 18–40
5. Fishburn PC (1981) Comments on the Kannai-Peleg impossibility theorem for extending orders. (mimeograph, 1981); J Econ Theory (unknown)
6. Gardenfors P (1976) Manipulation of social choice functions. J Econ Theory 13 : 217–228
7. Gardenfors P (1979) On definitions of manipulation of social choice functions. In: Laffont J-J (ed) Aggregation and revelation of preferences. North-Holland, Amsterdam
8. Heiner R, Packard DJ (1982) A uniqueness result for extending orders; with application to collective choice as inconsistency resolution. (mimeograph, 1982); J Econ Theory (unknown)
9. Kannai Y, Peleg B A note on the extension of an order on a set to the power set, (mimeograph, 1981); J Econ Theory (unknown)
10. Kelly J (1977) Strategy-proofness and social choice functions without single-valuedness. Econometrica 45 : 439–446
11. Krantz D, Luce RD, Suppes P, Tversky A (1971) Foundations of measurement. Wiley, New York
12. Luce RD, Raiffa H (1957) Games and decisions. Wiley, New York
13. Milnor J (1954) Games against nature. In: Thrall RM, Coombs CH, Davis RL (eds) Decision processes. Wiley, New York
14. Packard DJ (1979) Preference relations. J Math Psychol 19 : 295–306
15. Packard DJ (1981) Plausibility orderings and social choice. Synthese 49 : 415–418
16. Pattanaik PK (1978) Strategy and group choice. North-Holland, Amsterdam
17. Rawls J (1971) Theory of justice. Harvard University Press, Cambridge, MA
18. Rescher N (1976) Plausible reasoning. Van Gorcum, Assen/Amsterdam