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The Liberal Paradox with Fuzzy Preferences

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Abstract. The present note reformulates Amartya Sen's (1970) result on the 'impossibility of a Paretian liberal' in a collective choice framework in which both individual and social preferences are allowed to be *fuzzy*: the result of this exercise is not found to be encouraging in terms of escaping Sen's liberal paradox in the exact framework.

1. Introduction

In a recent paper, Barrett, Pattanaik and Salles (henceforth BPS) have pursued the implications for Arrow's General Possibility Theorem (and related results) when preferences – both individual and social – are regarded as being *fuzzy* rather than exact. Using the same notion of fuzziness of (strict) preference that BPS employ, this note explores the consequences for Sen's (1970) well-known result on "the impossibility of a Paretian liberal" when the conventional social choice framework of exact preferences is relaxed to allow for fuzzy preferences. Just as the BPS paper indicates that the Arrow result is robust even when preferences are fuzzy, this note, too, suggests that the liberal paradox remains largely intact in spite of allowing for vagueness in personal and collective preferences.

2. Some Elements of a Fuzzy Preference Framework

Let X, with $3 \leqq |X| < \infty$, be the set of all possible *social states*, or *alternatives*. Following the notation of BPS, define a *fuzzy binary relation* (FBR) on X as a function $g : X \times X \rightarrow [0, 1]$. Note that an *exact binary relation* (EBR) on X would be a

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function $g: X \times X \to \{0, 1\}$. Let T be the set af all FBRs on X, and H_1 the set of all $h \in T$ such that:

(2.1) for all $x \in X$: $h(x, x) = 0$;

(2.2) for all distinct $x, y \in X$: $h(x, y) = 1 \to h(y, x) = 0$; and

(2.3) for all distinct $x^1, x^2, \ldots, x^m \in X$: $[h(x^1, x^2) > h(x^2, x^1)$ & $h(x^2, x^3) > h(x^3, x^2)$ & $\ldots$ & $h(x^{m-1}, x^m) > h(x^m, x^{m-1})] \to \sim [h(x^m, x^1) = 1$ & $h(x^1, x^m) = 0]$.

Let H_2 be the set of all $h \in T$ such that h satisfies (2.1) and (2.2) above, and (2.4) and (2.5) below:

(2.4) for all distinct $x, y, z \in X$: $[h(x, y) > 0$ & $h(y, z) > 0] \to h(x, z) > 0$;

(2.5) for all distinct $x, y, z \in X$: $[h(x, y) = h(y, x) = h(y, z) = h(z, y) = 0] \to [h(x, z) = h(z, x) = 0]$.

Let H_E be the set of all $h \in T$ such that h is *exact*, and define the sets $K_1 := H_1 \cap H_E$ and $K_2 := H_2 \cap H_E$.

Every $h \in H_1$ will be interpreted as a *fuzzy strict preference relation;* and for all $x, y \in X$, $h(x, y)$ will be taken to reflect the 'degree of confidence' with which x is strictly preferred to y. If h were in H_E, *(2.1)* and *(2.2)* would be the definitions of the *irreflexivity* and *asymmetry* properties, respectively, of the strict preference relation. (Note that the asymmetry of the strict preference relation in the fuzzy framework is quite weak: it permits simultaneously both x to be strictly preferred to y and y to be strictly preferred to x with a positive degree of confidence except only when one of the alternatives is preferred to the other with *complete* confidence.) *(2.3)* is clearly a weak *acyclicity* condition, while *(2.4)* is a weak *transitivity* condition, with *(2.4)* implying (without being implied by) *(2.3)*. If indifference over a pair of alternatives is interpreted as the absence of strict preference in either direction, then *(2.5)* is a condition of transitivity on the indifference relation. From the difinitions of K_1 and K_2 it is obvious that K_1 is a set of exact strict preference relations which are acyclic, while K_2 is a set of transitive and exact strict preference relations such that the corresponding indifference relations are transitive. It is immediate that $K_2 \subseteq K_1 \subseteq H_1$, and $K_2 \subseteq H_2 \subseteq H_1$.

Let $N = \{1, \ldots, i, \ldots, n\}$, with $n \geqq 2$, be the set of all individuals constituting society. By a *fuzzy aggregation rule* (FAR) is meant a function $f: \tilde{T}^n \to \hat{T}$, where $\emptyset \neq \tilde{T} \subseteq T$ and $\emptyset \neq \hat{T} \subseteq T$. In what follows $\tilde{T}$ will be identified with K_2 or H_2, while $\hat{T}$ will be identified with H_2 or H_1. A fuzzy aggregation rule f is thus a function which for every n-tuple (h_i) in its domain specifies a unique h in its range; each element in the domain of f will be interpreted as a configuration of *individual* FBRs on X (one FBR for each individual), and each element in the range of f as a *social* FBR on X. Our concern in this note will be with the existence of a fuzzy aggregation rule that satisfies the ethical principles of Paretianism and personal liberty.

3. Rights and Unanimity when Preferences are Fuzzy

Two conditions one can impose on a fuzzy aggregation rule are the following:

(3.1) *Pareto Condition* (PC), which is satisfied if and only if for all $x, y \in X$, and for all $(h_i)_{i \in N}$ in the domain of the FAR:

(3.1.1) $[\forall i \in N: \{h_i(x, y)=1 \;\&\; h_i(y, x)=0\}] \rightarrow [h(x, y)=1 \;\&\; h(y, x)=0]$;

(3.2) Minimal Liberalism (ML), which is satisfied if and only if there exist at least two distinct individuals j and k and two distinct doubletons $\{x, y\}$ and $\{w, z\}$ of alternatives such that for all $(h_i)_{i \in N}$ in the domain of the FAR:

(3.2.1) $[h_j(x, y)=1 \;\&\; h_j(y, x)=0]$ [resp., $\{h_j(y, x)=1 \;\&\; h_j(x, y)=0\}] \rightarrow [h(x, y)=1 \;\&\; h(y, x)=0]$ [resp., $\{h(y, x)=1 \;\&\; h(x, y)=0\}$]; and

(3.2.2) $[h_k(w, z)=1 \;\&\; h_k(z, w)=0]$ [resp., $\{h_k(z, w)=1 \;\&\; h_k(w, z)=0\}] \rightarrow [h(w, z)=1 \;\&\; h(z, w)=0]$ [resp., $\{h(z, w)=1 \;\&\; h(w, z)=0\}$].

Sen (1970) demonstrated that there exists no FAR $f: K_2^n \rightarrow K_1$ which simultaneously satisfies conditions PC and ML. It is of interest to see if the impossibility result persists when the range of the FAR is expanded to allow for vagueness in social preferences. Before coming to that, we take note of two weakened versions of the Minimal Liberalism condition one can impose on an FAR:

(3.3) Weak Minimal Liberalism-1 (WML-1), which is satisfied if and only if there exist at least two distinct individuals j and k and two distinct doubletons $\{x, y\}$ and $\{w, z\}$ of alternatives such that for all $(h_i)_{i \in N}$ in the domain of the FAR:

(3.3.1) $[h_j(x, y)=1 \;\&\; h_j(y, x)=0]$ [resp., $\{h_j(y, x)=1 \;\&\; h_j(x, y)=0\}] \rightarrow [h(x, y) > h(y, x)]$ [resp., $\{h(y, x) > h(x, y)\}$]; and

(3.3.2) $[h_k(w, z)=1 \;\&\; h_k(z, w)=0]$ [resp., $\{h_k(z, w)=1 \;\&\; h_k(w, z)=0\}] \rightarrow [h(w, z) > h(z, w)]$ [resp., $\{h(z, w) > h(w, z)\}$];

(3.4) Weak Minimal Liberalism-2 (WML-2), which is obtained from WML-1 by replacing '$[h(x, y) > h(y, x)]$ [resp., $\{h(y, x) > h(x, y)\}$]' by '$[h(x, y) \geqq h(y, x)]$ [resp., $\{h(y, x) \geqq h(x, y)\}$]' in (3.3.1), and '$[h(w, z) > h(z, w)]$ [resp., $\{h(z, w) > h(w, z)\}$]' by '$[h(w, z) \geqq h(z, w)]$ [resp., $\{h(z, w) \geqq h(w, z)\}$]' in (3.3.2).

It is easy to verify that Minimal Liberalism implies (without being implied by) Weak Minimal Liberalism-1 which, in turn, implies (without being implied by) Weak Minimal Liberalism-2. Condition WML-1 is a weakened version of Sen's ML condition: it envisages that for at least two individuals there exists at least one pair of alternatives each such that the concerned individual's preference over the alternatives assigned to him is "weakly decisive", in the sense that if the individual strictly prefers one of the alternatives (say x) in his assigned pair to the other alternative (say y) with complete confidence, then society should strictly prefer x to y with a greater degree of confidence than it does y to x. Condition WML-2 can be seen as relaxing "weak decisiveness" to "weak semidecisiveness": an individual no longer "dictates" (however weakly) on the pair of alternatives in his protected sphere, he only has a "weak veto" over the pair. Motivationally, WML-2 is a fuzzy counterpart of Karni's (1978) weakening of Sen's liberal axiom in the exact framework.

4. The Liberal Paradox when Preferences are Fuzzy

The following proposition is true.

Proposition 4.1. *There exists no* FAR $f: K_2^n \to H_1$ *which simultaneously satisfies conditions* PC *and* WML-1.

Proof. Two cases must be distinguished: *(i)* the doubletons $\{x, y\}$ and $\{w, z\}$ have exactly one alternative in common, say $y = w$; and *(ii)* $\{x, y\} \cap \{w, z\} = \emptyset$.

Case (i). Let the following be the configuration of preferences of individual j and of all individuals other than j over the triple $\{x, y, z\}$:

j's preference ordering over $\{x, y, z\}$: $h_j(x, y) = 1$, $h_j(y, x) = 0$; $h_j(y, z) = 0$, $h_j(z, y) = 1$; $h_j(x, z) = 0$, $h_j(z, x) = 1$.

Preference ordering over $\{x, y, z\}$ *of all* $i \in N \setminus \{j\}$: $h_i(x, y) = 0$, $h_i(y, x) = 1$; $h_i(y, z) = 1$, $h_i(z, y) = 0$; $h_i(x, z) = 0$, $h_i(z, x) = 1$.
[It is easy to verify that the n-tuple of individual preference orderings (h_i) that we have employed is indeed an element of the set K_2^n.]

Since $h_j(x, y) = 1$ and $h_j(y, x) = 0$, we must have, by WML-1 to person j over the pair (x, y):

(4.1.1) $h(x, y) > h(y, x)$.
Since $h_k(y, z) = 1$ and $h_k(z, y) = 0$, we must have, by WML-1 to person k over the pair (y, z):

(4.1.2) $h(y, z) > h(z, y)$.
Since, by *(4.1.1)* and *(4.1.2)* respectively, $h(x, y) > h(y, x)$ and $h(y, z) > h(z, y)$, acyclicity over the triple $\{x, y, z\}$ – see *(2.3)* – requires that

(4.1.3) $\sim [h(z, x) = 1 \,\&\, h(x, z) = 0]$.
Given that $\forall i \in N: \{h_i(z, x) = 1 \,\&\, h_i(x, z) = 0\}$, by PC over the pair (z, x), we must have:

(4.1.4) $h(z, x) = 1 \,\&\, h(x, z) = 0$.
From *(4.1.3)* and *(4.1.4)* we obtain a contradiction.

Case (ii). Let the following be the configuration of preferences of individual j and of all individuals other than j over the designated set $\{x, y, w, z\}$:

j's preference ordering over $\{x, y, w, z\}$: $h_j(x, y) = 1$, $h_j(y, x) = 0$; $h_j(x, w) = 1$, $h_j(w, x) = 0$; $h_j(x, z) = 0$, $h_j(z, x) = 1$; $h_j(y, w) = 1$, $h_j(w, y) = 0$; $h_j(y, z) = 0$, $h_j(z, y) = 1$; $h_j(w, z) = 0$, $h_j(z, w) = 1$.

Preference ordering over $\{x, y, w, z\}$ *of all* $i \in N \setminus \{j\}$: $h_i(x, y) = 0$, $h_i(y, x) = 1$; $h_i(x, w) = 0$, $h_i(w, x) = 1$; $h_i(x, z) = 0$, $h_i(z, x) = 1$; $h_i(y, w) = 1$, $h_i(w, y) = 0$; $h_i(y, z) = 1$, $h_i(z, y) = 0$; $h_i(w, z) = 1$, $h_i(z, w) = 0$.

[It is, again, easy to varify that the configuration of individual preference orderings (h_i) that we have employed is an element of K_2^n.]
By WML-1 to person j over the pair (x, y),
(4.1.5) $h(x, y) > h(y, x)$.
By PC over the pair (y, w),
(4.1.6) $h(y, w) = 1 > h(w, y) = 0$.
By WML-1 to person k over the pair (w, z),
(4.1.7) $h(w, z) > h(z, w)$.

Given *(4.1.5)*, *(4.1.6)*, and *(4.1.7)*, acyclicity over the set of alternatives $\{x, y, w, z\}$ demands that

(4.1.8) $\sim[h(z,x)=1 \,\&\, h(x,z)=0]$.

But by PC over the pair (z,x), we have

(4.1.9) $h(z,x)=1 \,\&\, h(x,z)=0$.

(4.1.8) and *(4.1.9)* are mutually incompatible, and this completes the proof of the proposition. □

Remark 4.1. Proposition 4.1 asserts that Sen's impossibility result remains unaffected when the range of the aggregation rule is expanded from K_1 to H_1 to admit genuinely fuzzy social preferences; and this, even when Sen's liberal axiom ML is weakened to condition WML-1.

The next two propositions are concerned with the consequences of weakening the liberty axiom even further, from Weak Minimal Liberalism-1 to Weak Minimal Liberalism-2.

Proposition 4.2. *There exists no* FAR $f: K_2^n \rightarrow H_2$ *which simultaneously satisfies conditions* PC *and* WML-2.

Proof. Again two cases must be distinguished, just as in the proof of Proposition 4.1. To avoid tedious repetitiveness – and particularly in view of the fact that it is quite straightforward – case *(ii)* of the proof is omitted.

Case (i). Using exactly the same configuration of individual preferences over the triple $\{x, y, z\}$ as in the proof of Proposition 4.1, we have, by WML-2 to person j over the pair (x, y), and by WML-2 to person k over the pair (y, z):

(4.2.1) $h(x,y) \geqq h(y,x)$; and

(4.2.2) $h(y,z) \geqq h(z,y)$.

By PC over the pair (z,x), we obtain

(4.2.3) $h(z,x)=1 \,\&\, h(x,z)=0$.

Suppose now that $h(x,y)>0$. Then, since $h(z,x)>0$ by *(4.2.3)*, transitivity over the triple $\{z, x, y\}$ dictates that $h(z,y)>0$ and therefore, by *(4.2.2)*, that $h(y,z)>0$. With $h(x,y)>0$ and $h(y,z)>0$ we must, by transitivity over the triple $\{x, y, z\}$, have $h(x,z)>0$ which however contradicts *(4.2.3)*. Thus, our supposition that $h(x,y)>0$ is false, so that, given *(4.2.1)*, we must conclude that

(4.2.4) $h(x,y)=h(y,x)=0$.

Suppose, next, that $h(y,z)>0$. Then, by transitivity over $\{y, z, x\}$, we must have $h(y,x)>0$ which however violates *(4.2.4)*. Therefore, $h(y,z)=0$, and in view of *(4.2.2)*, it must be the case that

(4.2.5 $h(y,z)=h(z,y)=0$.

Given *(4.2.4)* and *(4.2.5)*, transitivity of the indifference relation – see *(2.5)* – must dictate:

(4.2.6) $h(x,z)=h(z,x)=0$.

(4.2.6) is, however, incompatible with *(4.2.3)*. □

With the mild notion of individual liberty embodied in Weak Minimal Liberalism-2, we can secure a possibility result if the range of the aggregation rule is expanded from H_2 to H_1; indeed, the domain of the aggregation rule can also be expanded to admit n-tuples of genuinely fuzzy individual preference orderings. This is demonstrated in Proposition 4.3 below.

Proposition 4.3. *There exists an* FAR $f: H_2^n \to H_1$ *which simultaneously satisfies conditions* PC *and* WML-2.

Proof. Construct the following fuzzy aggregation rule $\tilde{f}$:

(4.3.1) $\forall x, y \in X, \forall (h_i)_{i \in N} \in H_2^n: \tilde{h}(x, y) = \min_i h_i(x, y).$

Note first that $\tilde{h}$ satisfies *(2.1)*, *(2.2)* and *(2.3)*. Since the h_i satisfy *(2.1)* we have: $\forall x \in X, \forall i \in N: h_i(x, x) = 0$, so that, by construction of $\tilde{f}$, $\tilde{h}(x, x) = 0 \, \forall x \in X$, and *(2.1)* is satisfied by $\tilde{h}$. Further, for all distinct $x, y \in X$, $\tilde{h}(x, y) = 1$ can happen only if $\min_i h_i(x, y) = 1$; since the h_i satisfy *(2.2)*, it must be the case that $\forall i \in N: h_i(y, x) = 0$ which ensures, by construction of $\tilde{f}$, that $\tilde{h}(y, x) = 0$, as required for $\tilde{h}$ to satisfy *(2.2)*. To see that $\tilde{h}$ satisfies *(2.3)*, suppose, to the contrary, that there exists an m-tuple of distinct alternatives $\{x^1, x^2, \ldots, x^m\}$ such that $\tilde{h}(x^1, x^2) > \tilde{h}(x^2, x^1) \,\&\, \tilde{h}(x^2, x^3) > \tilde{h}(x^3, x^2) \,\&\, \ldots \,\&\, \tilde{h}(x^{m-1}, x^m) > \tilde{h}(x^m, x^{m-1})$, and $\tilde{h}(x^m, x^1) = 1$. By construction of $\tilde{f}$ we must have: $\forall i \in N: \{h_i(x^1, x^2) > 0 \,\&\, h_i(x^2, x^3) > 0 \,\&\, \ldots \,\&\, h_i(x^{m-1}, x^m) > 0\}$. Since the h_i satisfy *(2.4)*, it must be the case that $\forall i \in N: h_i(x^1, x^m) > 0$, so that $\tilde{h}(x^1, x^m) := \min_i h_i(x^1, x^m) > 0$, which contradicts the supposition that $\tilde{h}(x^m, x^1) = 1$. Hence, $\tilde{h}$ satisfies *(2.3)*. That $\tilde{f}$ satisfies the Pareto Condition is trivial. It only remains to demonstrate that $\tilde{f}$ satisfies Weak Minimal Liberalism-2. To see this, consider any pair of distinct alternatives (x, y) and any individual j, and let it be the case that $h_j(x, y) = 1$ and $h_j(y, x) = 0$. Then, by construction of $\tilde{f}$, $\tilde{h}(y, x) = \min_i h_i(y, x) = 0 \; (= h_j(y, x))$, and since $\tilde{h}(x, y) \geqq 0$ $(= \tilde{h}(y, x))$, WML-2 is satisfied by $\tilde{f}$. This completes the proof of the proposition. □

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