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Ethical Indices of Income Mobility *

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Abstract. Ethical indices of income mobility measure the change in welfare resulting from mobility. The concept of mobility we explore consists of a welfare comparison between the actual time path of the income distribution with a hypothetical time path obtained by supposing that starting from the actual first-period distribution, the remaining income receipts exhibit complete immobility.

1. Introduction

Indices of inequality are typically summary statistics of the dispersion of incomes at a particular point in time. Even if such indices are computed for a number of successive periods, by their very nature they will ignore many features of the time path of incomes which are of interest. As time progresses we observe changes in relative incomes as well as changes in the absolute income differences found in any given time period. Indices of income mobility are meant to measure the magnitude of these changes. Indices of relative mobility measure changes in relative incomes while indices of absolute mobility measure changes in income differences.

In this article we are concerned with ethical indices of relative income mobility. Ethical indices are derived from explicit social welfare functions and are measures of the change in welfare resulting from mobility. Such measures contrast with descriptive indices of mobility which endeavor to measure some objective aspect of mobility. Needless to say, our ethical indices are not meant to supplant descriptive indices; rather they are designed with a different purpose in mind. The concept of mobility we explore here consists of a comparison between the time path of incomes received over a number of periods with a hypothetical time path of incomes obtained by supposing that starting from the actual first-period distribution, the remaining income receipts

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exhibit complete immobility. Our indices are indices of relative income mobility if this benchmark maintains relative incomes through time.

By using a welfare function to perform this comparison, our mobility indices allow us to determine whether the observed changes are socially desirable. The idea that mobility can be either socially desirable or socially undesirable can be illustrated by considering a simple example. In this example, there are two people and two time periods. In the first period incomes are (10, 10), while in the second period they are (19, 1). In this example, the mobility in the income distribution introduces inequality into an initial egalitarian situation and is therefore judged to be socially undesirable; our indices of mobility will assign this change a negative value. In contrast, the reverse change from (19, 1) to (10, 10) is socially desirable, and our indices will indicate this fact by assigning such a change a positive index value. For many descriptive purposes only the magnitude of mobility, and not its direction, is important. For such indices, the mobility associated with the move from (10, 10) to (19, 1) is regarded to be equal to the mobility associated with the move in the reverse direction.

The comparison of the actual time path of the income distribution with a hypothetical immobile benchmark is central to our approach. A natural way to perform this comparison is to introduce an intertemporal social welfare function (defined on time paths of income distributions) and to use this function to directly compare the actual time path of incomes with the benchmark. For this approach to yield interesting conclusions, it is necessary to put some structure on the form of this intertemporal welfare function. We suppose that the welfare function is only sensitive to the total income received by each individual over all of the time periods under consideration. This assumption is introduced to provide a concrete illustration of how to operationalize our concept of mobility. In our concluding remarks we discuss problems with this particular approach and consider an alternative way of measuring mobility consistent with our general ethical perspective.

Recently a substantial number of articles have appeared which concern themselves with various aspects of mobility. Most of these articles assume that the changes over time in the variables of interest are generated by transition matrices, i.e. matrices with entries which show the fraction of the population which move from one category to another in one time period. In the context of income distributions, these categories could be ranks in the income distribution or the actual incomes received. In the latter case it is typically assumed that only a finite set of income levels are attainable and that this set is fixed over time. Shorrocks [16], in a quite abstract setting meant to apply to changes in such diverse phenomena as social class and place of residence, studies mobility indices defined directly on transition matrices. Kanbur and Stiglitz [9] and Markandya [13, 14], among others, evaluate mobility in terms of the time stream of income distributions that would result from the operation of a transition matrix, rather than directly in terms of the transition matrix itself. In either approach, the properties of the transition matrix play a central role in the analysis. In contrast, we do not assume that the time path of incomes is generated by a (fixed) transition matrix.

Of more relevance to the present inquiry are the contributions of Shorrocks [17], Markandya [12], and King [10]. While these authors are concerned with different concepts of mobility from the one considered here, an analysis of their mobility indices will hopefully lead to a deeper understanding of our approach. Accordingly, we discuss their mobility indices in Sect. 4.

2. Ethical Indices of Inequality

In this section we review the theory of ethical indices of relative inequality. For more extensive discussions the reader is referred to Atkinson [1], Blackorby and Donaldson [2, 3], Kolm [11], and Sen [15].

Let $y = (y_1, \ldots, y_n)$ be the vector of incomes received by the n individuals (or households) in society. The mean of y is denoted by $\lambda(y)$. Alternative income distributions are ranked by a social welfare function $W: D \to R^1$ where $D = R^n_{++}$ (the strictly positive orthant).[1] We assume that the social welfare function W is continuous, increasing along the ray of equality, and has every social indifference curve intersect this ray. Welfare functions satisfying these three properties are called *regular*.[2] For some of our results we also assume that W is strictly S-concave.[3] Strict S-concavity of W requires $\hat{y}$ to be socially preferred to y if $\hat{y}$ is Lorenz-superior to y and both distributions have the same mean.

Given W, the equally distributed equivalent income y_e is defined to be that level of income which if received by all individuals generates the same level of social welfare as the actual distribution y. Hence, it is implicitly defined by

$$W(y_e \cdot 1^n) = W(y),$$

where 1^n is an n-vector of ones. If W is regular, y_e exists and is unique for all $y \in D$, so it can be written explicitly as

$$y_e = \Xi(y).$$

The function Ξ is simply an ordinal transform of W; hence, it is also regular.

An inequality index is a function $I: D \to R^1$. Atkinson [1], Kolm [11], and Sen [15] propose the use of

$$I_R^{\Xi}(y) = 1 - \frac{\Xi(y)}{\lambda(y)} \tag{1}$$

as a measure of inequality. Associated with the inequality index I_R^{Ξ} is the equality index

$$E_R^{\Xi}(y) = 1 - I_R^{\Xi}(y) = \frac{\Xi(y)}{\lambda(y)}. \tag{2}$$

$I_R^{\Xi}(y)$ is the fraction of total income which could be disposed of without any welfare loss if society distributed incomes equally. When efficiency considerations are absent,

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1 Blackorby and Donaldson [3] suggest that the terminology "social evaluation function" would be more appropriate for W as it is defined directly on income distributions rather than distributions of utilities. Zero incomes are excluded from consideration to simplify the discussion. The modifications to our analysis which are required if $D = R^n_+ \backslash \{0\}$ are noted at the appropriate points in our argument.

2 If W is assumed to be continuous and strictly monotonic, then it is regular.

3 A function $F: X \to R^1$ where $X \subset R^p$ is S-concave if $F(Bx) \geqq F(x)$ for all $x \in X$ and all bistochastic matrices B of order p. B is a bistochastic matrix of order p if it is a nonnegative $p \times p$ matrix whose row and column sums are all one. The function F is strictly S-concave if the weak inequality is replaced by a strict inequality whenever Bx is not a permutation of x or x itself. S-convex and strictly S-convex functions reverse the inequalities. S-concave and S-convex functions are symmetric. The modifications to our results when strict S-concavity is weakened to S-concavity are left to the reader.

i. e. mean income is fixed, an increase in inequality is equivalent to a reduction in social welfare. For a regular strictly S-concave social welfare function W, I_R^{Ξ} is continuous, strictly S-convex, and bounded by zero and one. The lower bound is obtained whenever incomes are equally distributed. Because the AKS index combines the terms $\Xi(y)$ and $\lambda(y)$ in a ratio form, it is interpreted to be an index of *relative inequality*, which accounts for our use of the subscript R in (1). However, in general, I_R^{Ξ} need not be a *relative index*. An index is relative if it is homogeneous of degree zero in its arguments. Thus, an inequality index I is a relative index if $I(y) = I(\alpha y)$ for all scalars $\alpha > 0$. Blackorby and Donaldson [2] show that I_R^{Ξ} is a relative index if and only if W is homothetic; homotheticity of W is equivalent to Ξ being positively linear homogeneous.[4]

Given a functional form for W or Ξ, (1) shows how to construct the corresponding inequality index I_R^{Ξ}. Equally important is the ability to recover Ξ (and, hence, the ordinal properties of W) from knowledge of the functional form for an index of relative inequality I_R. It is clear from (1) that this recovery is possible, so one can always determine the social welfare function underlying an Atkinson-Kolm-Sen inequality index.[5]

3. Ethical Indices of Relative Mobility

In the previous section only one time period was considered. To study income mobility, income distributions from a sequence of time periods are required. The time interval $[t_0, t_m)$ is partitioned into m equal subperiods $[t_{k-1}, t_k)$, $k = 1, \ldots, m$ where m is a fixed exogenous integer. We refer to $[t_{k-1}, t_k)$ as the k-th period. For period k, let y_i^k be person i's income. The *income distribution* in period k is the (row) vector $y^k = (y_1^k, \ldots, y_n^k)$. Sequences of income distributions, denoted $Y = (y^1, \ldots, y^m)$, are called *income structures*. Income structures are elements of D^m. Person i's *income stream* is $y_i = (y_i^1, \ldots, y_i^m)$. Over the whole time interval $[t_0, t_m)$ person i receives income $y_i^a = \sum_{k=1}^{m} y_i^k$. The corresponding income distribution $y^a = (y_1^a, \ldots, y_n^a)$ is called the *aggregate distribution*.

The mobility concept we wish to explore is the one embodied in a welfare comparison of the actual income structure Y with a hypothetical benchmark structure Y^b. The benchmark Y^b is chosen to be completely immobile and to have the same first-period distribution as the actual structure Y. A comparison of Y and Y^b yields a mobility index precisely because the benchmark is a sequence of incomes which could have resulted in the absence of mobility given the first period distribution y^1. The use of a social welfare function to form the comparison provides the ethical interpretation of our indices. The construction of the reference income structure depends, of course, on how we define complete immobility. We consider a benchmark exhibiting complete relative immobility, and thus obtain indices of relative mobility.

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[4] A function $F\colon X \to R^1$ where $X \subset R^p$ is homothetic if $F(x) = F^*(\bar{F}(x))$ for all $x \in X$ where F^* is increasing in its argument and $\bar{F}$ is positively linear homogeneous. The function $\bar{F}$ is positively linear homogeneous if $\bar{F}(\alpha x) = \alpha \bar{F}(x)$ for all $x \in X$ and all scalars $\alpha > 0$ such that $\alpha x \in X$.

[5] It should be stressed that while social welfare functions are ordinal in this framework, inequality indices are not.

3.1. A Class of Ethical Indices of Relative Mobility

Definition: Y exhibits *complete relative immobility* if and only if for all i, $y_i^k/\lambda(y^k)$ is the same for each period k.

With complete relative immobility, income shares are maintained through time. For an index of relative mobility, the benchmark structure Y^b corresponding to Y is chosen to be completely relatively immobile.[6] It is denoted Y_R^b and is equal to the completely relatively immobile structure with first-period distribution equal to the actual first-period distribution and mean income in each period equal to actual mean income. Formally,

$$Y_R^b = \left[y^1, y^1 \cdot \frac{\lambda(y^2)}{\lambda(y^1)}, \ldots, y^1 \cdot \frac{\lambda(y^m)}{\lambda(y^1)} \right]. \tag{3}$$

Consequently, the aggregate distribution for this benchmark structure, denoted y_R^b, gives each individual the same share of actual total income as they receive in period one.

A mobility index assigns a numerical value to each income structure Y in D^m; i.e. it is a function $M: D^m \to R^1$. Our ethical approach to measuring mobility utilizes an intertemporal social welfare function $\mathscr{W}: D^m \to R^1$; $\mathscr{W}(Y)$ is the social welfare level associated with the income structure Y. Our mobility indices are obtained by comparing this level of social welfare $\mathscr{W}(Y)$ with the level of social welfare $\mathscr{W}(Y_R^b)$ obtained with the benchmark structure Y_R^b.

To provide a concrete illustration of this approach, we make the assumption that the only features of the income structures Y and Y_R^b relevant for our welfare comparison are their aggregate distributions y^a and y_R^b. This assumption, while important for the specific results developed here, is not an essential feature of our mobility concept. Formally, we assume that there exists a regular *aggregate* social welfare function $W: D \to R^1$ such that

$$W(y^a) = \mathscr{W}(Y) \tag{4}$$

for all $Y \in D^m$. Since W is regular, it can be expressed in its normalized form Ξ, as described in Sect. 2.

Given Ξ we propose the use of an index of relative mobility in the class $\mathscr{M}_R^\Xi$ where $M \in \mathscr{M}_R^\Xi$ if and only if M can be written in the form

$$M(Y) = \phi\left(\frac{\Xi(y^a)}{\Xi(y_R^b)}\right), \tag{5}$$

where $\phi: R_{++}^1 \to R^1$ is a continuous increasing function with $\phi(1) = 0$. Indices in this class are ordinally equivalent to each other and to the ratio of the equally distributed equivalent income of the actual aggregate distribution to the equally distributed equivalent income of the aggregate distribution in the hypothetical immobile benchmark structure Y_R^b. The normalization employed ensures that an immobile income structure is assigned a mobility value of zero.

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[6] For an index of absolute income mobility, the benchmark is chosen to be completely absolutely immobile, i.e. income differences are maintained through time.

Among the members of the class $\mathcal{M}_R^{\Xi}$ one index stands out because of its simple interpretation. This index is obtained by setting $\phi(s) = s - 1$ in (5). Formally, this is the index

$$M_R^{\Xi}(Y) = \frac{\Xi(y^a)}{\Xi(y_R^b)} - 1. \tag{6}$$

This index measures mobility as the percentage change in the equally distributed equivalent income of the actual aggregate distribution compared with what it would be with the immobile benchmark structure Y_R^b.[7] Using (2), M_R^{Ξ} can be rewritten in terms of equality indices,

$$M_R^{\Xi}(Y) = \frac{E_R^{\Xi}(y^a)}{E_R^{\Xi}(y_R^b)} - 1 \tag{7}$$

where use is made of the fact that Y and Y_R^b have the same means. In terms of inequality indices, from (1) this index is

$$M_R^{\Xi}(Y) = \frac{I_R^{\Xi}(y_R^b) - I_R^{\Xi}(y^a)}{1 - I_R^{\Xi}(y_R^b)}.$$

Figure 1 illustrates the construction of M_R^{Ξ}. In this diagram there are two individuals and two subperiods. Since $\Xi(y^a) < \Xi(y_R^b)$, the situation illustrated is one where $M_R^{\Xi}(Y)$ is negative; this mobility is socially undesirable.

In constructing our index M_R^{Ξ}, or any of the other indices in $\mathcal{M}_R^{\Xi}$, the welfare (or inequality) comparisons only involve the income distributions y^a and y_R^b, distributions defined on a common time interval. Indeed, the social welfare function W in (4) is only constructed for income distributions on the complete interval $[t_0, t_m)$. If one assumes that we have a social welfare function defined on one-period incomes as well and that this function is the *same* as the m-period welfare function W, for *homothetic* welfare functions we obtain

$$M_R^{\Xi}(Y) = \frac{\lambda(y^1)}{\lambda(y^a)} \cdot \frac{\Xi(y^a)}{\Xi(y^1)} - 1$$

in place of (6) and

$$M_R^{\Xi}(Y) = \frac{E_R^{\Xi}(y^a)}{E_R^{\Xi}(y^1)} - 1$$

in place of (7). The latter expression has a natural interpretation; it is the percentage change in equality of the aggregate distribution compared with the first-period distribution. Thus, *with these additional assumptions* our mobility index M_R^{Ξ} can be viewed as a measure of the change in inequality over the complete time interval compared to the inequality in the first period. As there does not appear to be a convincing ethical justification for employing the same welfare function (or, equivalently, inequality index) over both the whole interval and over each of the subperiods,[8] our mobility index is not, in general, an index of the change in inequality.

[7] Analogously, the AKS inequality index I_R^{Ξ} can be thought of as a member of a general class of inequality indices obtained by taking continuous increasing monotone transforms of I_R^{Ξ}. The appeal of the AKS index in this class lies in its simple interpretation.

[8] Shorrocks [17, p. 391] presents an intriguing justification for using a common functional form.

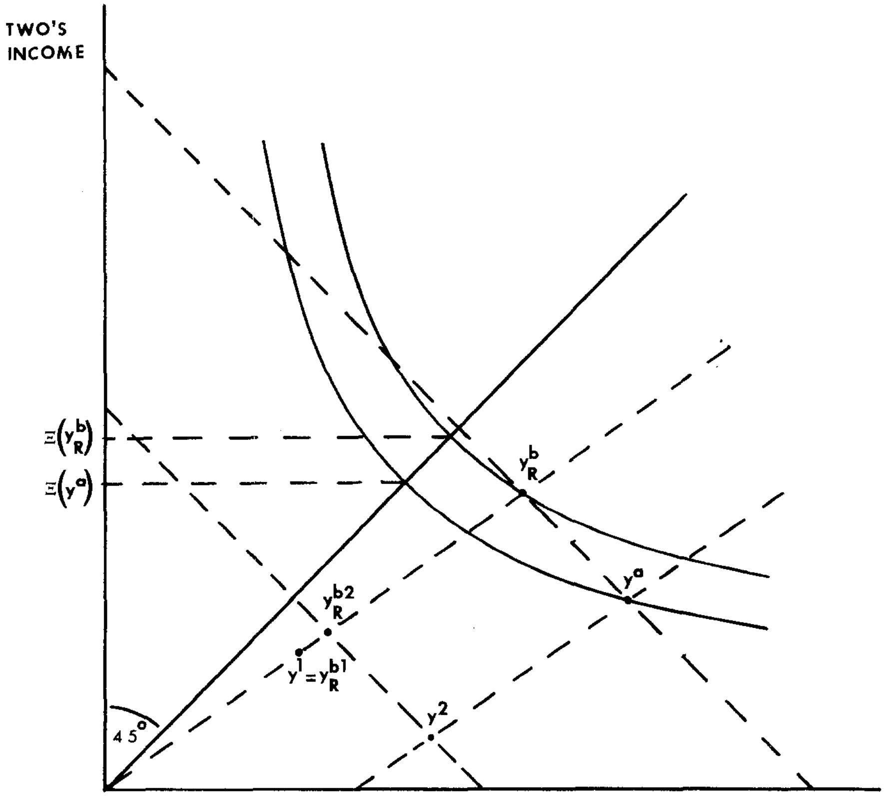

Fig. 1

3.2. Properties of the Class $\mathscr{M}_R^{\Xi}$

We now investigate the properties indices in the class $\mathscr{M}_R^{\Xi}$ inherit from an aggregate social welfare function W. To facilitate comparisons with other indices of mobility, the formal statements of the properties are in terms of an arbitrary mobility index $M\colon D^m \to R^1$ and an arbitrary immobile structure Y^b. By an arbitrary immobile structure Y^b we mean any income structure which embodies some notion of complete immobility and which has the same mean and first-period distribution as the actual income structure Y. To simplify the exposition, we only verify that M_R^{Ξ} satisfies the properties introduced in this subsection; it is an easy exercise to check that all members of $\mathscr{M}_R^{\Xi}$ satisfy them as well.

The index M_R^{Ξ} satisfies the first three properties we consider provided that the social welfare function W is regular.[9]

The idea that an ethical index of mobility is obtained from a welfare comparison of the actual income structure with a reference immobile structure is formalized as Property 1.

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9 Note that regularity of W ensures that M_R^{Ξ}, and hence any member of $\mathscr{M}_R^{\Xi}$, is well-defined for all $Y \in D^m$.

Property 1. *For all* $Y \in D^m$, $M(Y) = f(\mathscr{W}(Y), \mathscr{W}(Y^b))$ *for some function* f, *with* f *increasing in its first argument and decreasing in its second argument.*

If the intertemporal social welfare function $\mathscr{W}$ satisfies (4) for some regular aggregate social welfare function W, Property 1 can be equivalently expressed as Property 1′.

Property 1′. *For all* $Y \in D^m$, $M(Y) = g(\Xi(y^a), \Xi(y^b))$ *for some function* g, *with* g *increasing in its first argument and decreasing in its second argument.*

The index M_R^{Ξ} with the benchmark Y_R^b clearly satisfies Property 1′.

The monotonicity properties of g have natural ethical implications. Assuming that $\mathscr{W}$ satisfies (4) and that the benchmark is Y_R^b, any mobility index satisfying Property 1′ has the following properties as well. First, in a comparison of the income structures Y and $\hat{Y}$, if $y^1 = \hat{y}^1$ and $\lambda(y^a) = \lambda(\hat{y}^a)$, then $M(Y) \geqq M(\hat{Y})$ if and only if $\Xi(y^a) \geqq \Xi(\hat{y}^a)$. This conclusion follows from the observation that $y_R^b = \hat{y}_R^b$ when $y^1 = \hat{y}^1$ and $\lambda(y^a) = \lambda(\hat{Y}^a)$.[10] Thus we have an equivalence between increases in mobility and increases in social welfare whenever two income streams have the same total income (no incentive effects) and the same first-period distributions. Second, for two income structures Y and $\hat{Y}$ with $y^a = \hat{y}^a$, $M(Y) \geqq M(\hat{Y})$ if and only if $\Xi(y_R^b) \leqq \Xi(\hat{y}_R^b)$. When two income structures have identical aggregate distributions, mobility is inversely related to the social desirability of the reference aggregate distribution.

Because data is often measured imperfectly, it is desirable to have an index which is a continuous function of its arguments.

Property 2. *M is continuous in Y.*

M_R^{Ξ} satisfies this property since Ξ is continuous and y^a and y_R^b both vary continuously with Y.

If the actual income structure equals the benchmark structure, there is no mobility. As a normalization rule, we assign immobile income structures an index value of zero.

Property 3. *For all* $Y \in D^m$, $M(Y) = 0$ *if* $Y = Y^b$.

If $Y = Y_R^b$, then $y^a = y_R^b$ as well, so M_R^{Ξ} satisfies this normalization rule.

As noted in the introduction, in our approach mobility can be either socially desirable or socially undesirable. For a mobility index satisfying Properties 1 and 3, socially desirable mobility is associated with income structures having positive index values while socially undesirable mobility is associated with income structures having negative index values.

If an addition to being regular, the social welfare function W is also strictly S-concave, then M_R^{Ξ} satisfies two additional properties.

The first of these properties requires a mobility index to be anonymous. In other words, permutations of individuals in the income structure Y leave the mobility index unchanged.

[10] The benchmark structures Y_R^b and $\hat{Y}_R^b$ could differ.

Property 4. *M is symmetric in the income streams* $(y_1, \ldots, y_n)$.

Permutations of income streams yield permutations of y^a and y^b_R. As strict S-concavity of W implies Ξ is symmetric, M^{Ξ}_R satisfies Property 4.

We now consider the effects on mobility of a transfer of income in a single subperiod from one individual to another. The transfer is assumed to be from an individual who has at least as much income every period as the recipient, with the transfer assumed to be sufficiently small that these rankings do not change as a result of the transfer. If such a transfer occurs in the first period, both the benchmark and the actual income structures are affected, and, in general, the effect on mobility is ambiguous. However, if the transfer occurs in any other period but the first, there is no change in the benchmark distribution. Using the Lorenz criterion, there is a clear-cut reduction in inequality during the period in which the transfer occurs and no change in inequality in any other period. Consequently, it seems reasonable to describe the post-transfer income structure as exhibiting more mobility than the pre-transfer income structure. This property is stated formally as Property 5.

Property 5. *For all* $Y, \hat{Y} \in D^m$, *if* $y^k = \hat{y}^k$ *for all* $k \neq k^*$ *with* $k^* > 1$, $y^{k^*}_i = \hat{y}^{k^*}_i$ *for all* $i \neq j, j'$, $y^k_j \geqq y^k_{j'}$ *for all* $k \neq k^*$, $y^{k^*}_j > y^{k^*}_{j'}$, $\hat{y}^{k^*}_j > \hat{y}^{k^*}_{j'}$, $\hat{y}^{k^*}_j = y^{k^*}_j - \delta$, *and* $\hat{y}^{k^*}_{j'} = y^{k^*}_{j'} + \delta$ *with* $\delta > 0$, *then* $M(\hat{Y}) > M(Y)$.

We now verify that M^{Ξ}_R satisfies Property 5 when W is regular and strictly S-concave. The restriction that the transfer does not occur in period one implies that Y and $\hat{Y}$ have the same benchmark structures, so $y^b_R = \hat{y}^b_R$. Since $\lambda(y^a) = \lambda(\hat{y}^a)$, $\hat{y}^a$ is obtained from y^a by a transfer of δ units of income from j to j'. The satisfaction of Property 5 by M^{Ξ}_R now follows immediately from the Dasgupta et al. [8] equivalence theorem.

An index M is an index of relative income mobility if it is constructed using a completely relatively immobile benchmark. This fact does not mean that M is a relative index. However, if one is concerned with mobility in income shares it seems appropriate to require that the mobility index be a relative index, i.e. an index which is invariant to proportional scalings of the income structure.

Property 6. *M is a relative index.*

If W is both regular and homothetic, M^{Ξ}_R satisfies Property 6. Blackorby and Donaldson [2] show that homotheticity of W is equivalent to Ξ being positively linear homogeneous. If $\hat{Y} = \alpha Y$ for some scalar $\alpha > 0$, then $\hat{y}^a = \alpha y^a$ and $\hat{y}^b_R = \alpha y^b_R$. Hence if Ξ is positively linear homogeneous, from (5) we have $M^{\Xi}_R(\hat{Y}) = M^{\Xi}_R(Y)$, which establishes that M^{Ξ}_R is a relative index.

Given a regular homothetic aggregate social welfare function W, $\mathscr{M}^{\Xi}_R$ contains *all* of the mobility indices which satisfy Properties 1′, 2, 3, and 6. If the social welfare function W is also assumed to be strictly S-concave, the corresponding class of mobility indices $\mathscr{M}^{\Xi}_R$ exhaust the set of indices which satisfy all of the properties introduced in this subsection.

Proposition 1. *If the social welfare function $W\colon D \to R^1$ is regular and homothetic and if the benchmark income structure Y^b is Y^b_R, then for all $Y \in D^m$ a mobility index $M\colon D^m \to R^1$ is in $\mathcal{M}^{\Xi}_R$ if and only if M satisfies Properties 1′, 2, 3, and 6. If in addition W is strictly S-concave, then M also satisfies Properties 4 and 5.*

Proof: In view of the preceding remarks, it is sufficient to show that Properties 1′, 2, 3, and 6 imply M has the form (5). By Property 1′,

$$M(Y) = g(\Xi(y^a), \Xi(y^b_R)).$$

If (ξ_1, ξ_2) is in the domain of g, there must exist an income structure Y such that $\Xi(y^a) = \xi_1$ and $\Xi(y^b_R) = \xi_2$. (Given that W is regular and the benchmark is Y^b_R, the domain of g is R^2_{++}.) If $\hat{Y} = \alpha Y$ for some scalar $\alpha > 0$, then $\hat{y}^a = \alpha y^a$ and $\hat{y}^b_R = \alpha y^b_R$. Since Ξ is positively linear homogeneous, $\Xi(\hat{y}^a) = \alpha\Xi(y^a)$ and $\Xi(\hat{y}^b_R) = \alpha\Xi(y^b_R)$. By Property 6, $M(\hat{Y}) = M(Y)$ so $g(\alpha\xi_1, \alpha\xi_2) = g(\xi_1, \xi_2)$ for all $\alpha > 0$ and all (ξ_1, ξ_2) in the domain of g. Thus, g is homogeneous of degree zero, so M can be rewritten as

$$M(Y) = \phi\left(\frac{\Xi(y^a)}{\Xi(y^b_R)}\right).$$

Since $\Xi(y) > 0$ for all $y \in D$, $\phi(s)$ only needs to be defined for $s > 0$. Property 2 implies ϕ is continuous since Ξ is continuous and y^a and y^b_R are continuous in Y. Property 1′ implies ϕ is increasing, while Property 3 implies $\phi(1) = 0$. □

Suppose M is a mobility index in the class $\mathcal{M}^{\Xi}_R$. We now illustrate its performance in two examples.

In the first example, $Y(\varepsilon) = ((1-\varepsilon, 1+\varepsilon), (1+\varepsilon, 1-\varepsilon))$ where $0 \leqq \varepsilon < 1$. The common aggregate distribution is $y^a = (2,2)$, while the reference aggregate distributions are $y^b_R(\varepsilon) = (2 - 2\varepsilon, 2 + 2\varepsilon)$. In an intuitively obvious sense, the extent of mobility depends on the value of ε – the larger is ε, the greater is the mobility. Since the aggregate social welfare function W is strictly S-concave, $\Xi(y^b_R(\varepsilon))$ is decreasing in ε. Hence, by Property 1′, $M(Y(\varepsilon))$ is increasing in ε, as desired.

In the second example, $Y = ((10,10), (18,2))$ and $\hat{Y} = ((18,2), (10,10))$. These structures differ only in the sequence of the one-period distributions. For Y the income mobility introduces inequality into a completely egalitarian distribution while for $\hat{Y}$ the inequality in the distribution is narrowed by mobility. The second kind of mobility is socially desirable while the first is not. In this example, $y^a = \hat{y}^a = (28,12)$ and the reference aggregate distributions are $y^b_R = (20,20)$ and $\hat{y}^b_R = (36,4)$. By the strict S-concavity of Ξ, $\Xi(20,20) > \Xi(28,12) > \Xi(36,4)$. Thus the argument in the function ϕ found in (5) is less than one for Y and greater than one for $\hat{Y}$, implying $M(Y) < 0$ and $M(\hat{Y}) > 0$.

3.3. A Characterization of M^{Ξ}_R

In the preceding discussion we have assumed that a functional form W for the social welfare function defined on the aggregate income distribution has been prespecified and used this functional form to construct the mobility index M^{Ξ}_R (or some index in the class $\mathcal{M}^{\Xi}_R$). In Sect. 2 we remarked that not only is it possible to determine a functional form for an inequality index given the functional form of a welfare function, it is also possible to start with a functional form for an inequality index and determine the social welfare function which would generate the given inequality index. In this subsection we study the analogous problem for indices of relative mobility. Specifically, we wish to determine a set of necessary and sufficient conditions on the form of a mobility index to guarantee that there exists a social welfare function defined on the aggregate income distribution which would generate the mobility index by our formula (6). In other words, we wish to determine conditions on a mobility index which would enable us to interpret it as an ethical index of relative mobility in the class $\mathcal{M}_R^{\Xi}$. If the mobility index can be given such an interpretation, it is also desirable to determine the functional form of the underlying welfare function from knowledge of the functional form of the mobility index. While it is always possible to recover a welfare function from an inequality index, it is often the case that the terms in the formula for the index must be manipulated before it is in the form (1). The Gini index provides such an example (Blackorby and Donaldson [2]). With mobility indices, it is not a priori obvious that the formula for the mobility index can be rearranged into the form (6), as the unknown welfare function appears twice and in a ratio form. We show that it is possible to determine the functional form of the underlying welfare function by constructing an algorithm designed for this purpose.

Our previous discussion has shown that a mobility index derived from a regular social welfare function defined on aggregate income distributions using (6) must satisfy Properties 1′, 2, and 3. However, to use Property 1′ as one of our necessary and sufficient conditions would be inappropriate, as this axiom explicitly assumes that a welfare function exists; a satisfactory set of conditions on M would not involve reference to a social welfare function. We now introduce two new properties which share this characteristic. While Properties 1–6 are natural restrictions on an ethical index of relative mobility, the two new properties described below do not have the same a priori appeal. However, if a regular aggregate social welfare function is used to derive the index M_R^{Ξ}, this index necessarily satisfies these additional properties and, consequently, these properties must be imposed on an index M if it is to have the ethical interpretation described above.

Our index M_R^{Ξ} is the percentage change in the equally distributed equivalent income of the actual aggregate distribution compared with what it would be with the immobile benchmark structure. Since Ξ is assumed to be regular, $\Xi(y)$ is positive for all $y \in D$. As a consequence, the largest percentage decrease in the equally distributed equivalent income of the actual aggregate distribution compared with what it would be with the immobile benchmark structure is bounded by 100%. (The index M_R^{Ξ} has no upper bound.) This restriction on the value of a mobility index is stated as Property 7.

Property 7. *For all $Y \in D^m$, $M(Y) > -1$.*

The last property we consider requires the introduction of some additional notation. Define

$$\mathscr{Y} := \{Y \in D^m \mid y_i^1 = y_{i'}^1,\ i, i' = 1, \ldots, n\}. \tag{8}$$

The set $\mathscr{Y}$ consists of all income structures which have completely equal first-period distributions. For all $y \in D$, define

$$\psi(y) := \{Y \in D^m \mid Y \in \mathscr{Y} \text{ and } y^a = y\}. \tag{9}$$

The set $\psi(y)$ consists of all income structures with equal first-period incomes which have aggregate distributions equal to y. Similarly, for all $Y \in D^m$, define

$$\Psi(Y) := \{\bar{Y} \in D^m \mid \bar{Y} \in \mathscr{Y} \text{ and } \bar{y}^a = y^a\}. \tag{10}$$

The set $\Psi(Y)$ consists of all income structures with equal first-period incomes which have the same aggregate distribution as the structure Y. For future reference, we demonstrate that $\psi(y)$ and $\Psi(Y)$ are both nonempty.

Lemma 1. (a) *For all* $y \in D$, $\psi(y) \neq \emptyset$. (b) *For all* $Y \in D^m$, $\Psi(Y) \neq \emptyset$.

Proof: (a) The proof is by construction. For $y \in D$, let $\gamma = \min\{y_i\}$, which is a positive number. Set $y^1 = (\gamma \cdot 1^n)/m$. For the remaining periods, let $y_i^k = \left(\frac{1}{m-1}\right)(y_i - y_i^1)$. For the income structure Y obtained by this algorithm we have $Y \in \mathscr{Y}$ and $y^a = y$.

(b) Part (b) follows immediately from the definition (10) and the proof of (a).[11] □

One of the major goals in this section is to provide an algorithm which enables us to generate an aggregate social welfare function which rationalizes a given mobility index. Lemma 1 and its proof provide a key element in this endeavour. Lemma 1 (b) says that for any income structure Y, there exists an income structure Y' with the same aggregate distribution as Y but which has a first-period distribution exhibiting complete equality.

Our mobility indices are normalized so that immobile income structures are assigned a zero value. It is convenient to consider an alternative normalization obtained by setting

$$\bar{M}(Y) = M(Y) + 1 \tag{11}$$

for all $Y \in D^m$. If M satisfies our normalization rule, Property 3, then $\bar{M}(Y) = 1$ for any completely relatively immobile income structure.

Because we are restricting attention to mobility indices which compare the actual aggregate distribution with the aggregate benchmark distribution, all elements of $\Psi(Y)$ which have the same mean first-period income are assigned the same index value. For an income structure $Y \in \mathscr{Y}$, i.e. an income structure with equal first-period incomes, the corresponding benchmark Y_R^b is completely relatively immobile, so $\bar{M}(Y) = \bar{M}(Y)/\bar{M}(Y_R^b)$ as $\bar{M}(Y_R^b) = 1$. Furthermore, for such Y, $Y \in \Psi(Y)$, so $\bar{M}(\hat{Y}) = \bar{M}(Y)$ for all $\hat{Y} \in \Psi(Y)$ which have $\lambda(\hat{y}^1) = \lambda(y^1)$. Analogously, $\bar{M}(\tilde{Y}) = \bar{M}(Y_R^b)$ for all $\tilde{Y} \in \Psi(Y_R^b)$ which have $\lambda(\tilde{y}^1) = \lambda(y^1)$. Consequently, for all $Y \in \mathscr{Y}$,

$$\bar{M}(Y) = \frac{\bar{M}(\hat{Y})}{\bar{M}(\tilde{Y})} \tag{12}$$

if $\hat{Y} \in \Psi(Y)$, $\tilde{Y} \in \Psi(Y_R^b)$, and $\lambda(\hat{y}^1) = \lambda(\tilde{y}^1) = \lambda(y^1)$. Equation (12) can be viewed as a decomposition principle. Property 8 requires that a mobility index satisfies (12) for all $Y \in D^m$, not just for $Y \in \mathscr{Y}$, provided $\hat{y}^1$ and $\tilde{y}^1$ have equal means.

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[11] If the domain $D = R^n_{++}$ is replaced by $R^n_+ \backslash \{0\}$ and if $y_i = 0$ for some i, then the algorithm yields the first-period distribution $0 \cdot 1^n$, a distribution which is not in the domain. Consequently, with $D = R^n_+ \backslash \{0\}$, $\Psi(y) = \emptyset$ for such y. However, this boundary problem is unimportant for Proposition 2.

Property 8. *For all $Y \in D^m$, if $\hat{Y} \in \Psi(Y)$, if $\tilde{Y} \in \Psi(Y^b)$, and if $\hat{y}^1 = \tilde{y}^1$, then $\bar{M}$ satisfies* (12).

Property 8 may be of considerable practical benefit in constructing mobility indices. The income structures $\hat{Y}$ and $\tilde{Y}$ have equal first-period incomes, making it easier to judge the mobility in the income structure than would be the case for an income structure in $D^M \backslash \mathscr{Y}$. Property 8 says that the evaluation of the mobility exhibited by a structure Y can always be decomposed into evaluations of simpler structures which initially have equal incomes.

In Proposition 2, we demonstrate that our index M_R^{Ξ} satisfies Property 8 if it is derived from a regular aggregate social welfare function. First, however, we demonstrate that Property 8 implies: (i) the normalization rule, Property 3, is satisfied and (ii) for all $Y \in D^m$,

$$M(Y) = h(y^a, y^b) \tag{13}$$

for some function h.[12]

Lemma 2. *Property 8 implies Property 3.*

Proof: Since $\lambda(y^a) = \lambda(y^b)$ for all $y \in D^m$, the algorithm used to establish Lemma 1 implies that two structures $\hat{Y}$ and $\tilde{Y}$ can be found to satisfy the antecedent in Property 8 for *any* $Y \in D^m$. If $Y = Y^b$, $\Psi(Y) = \Psi(Y^b)$ and $\hat{Y}$ can be chosen to equal $\tilde{Y}$. Doing this implies $\bar{M}(Y) = 1$, i.e. $M(Y) = 0$. □

Lemma 3. *Property 8 implies* (13).

Proof: Consider Y^* and Y^{**} with $y^{*a} = y^{**a}$ and $y^{*b} = y^{**b}$. By construction, $\Psi(Y^*) = \Psi(Y^{**})$ and $\Psi(Y^{*b}) = \Psi(Y^{**b})$. Using the observation in the first line of the proof of Lemma 2, we can verify the hypothesis of Property 8 for both Y^* and Y^{**} with $\hat{Y}^* = \hat{Y}^{**}$ and $\tilde{Y}^* = \tilde{Y}^{**}$. The conclusion now follows from (11) and (12). □

We are now in a position to state our main theorem concerning the ethical interpretation of a mobility index.

Proposition 2. *There exists a regular aggregate social welfare function W: $D \to R^1$ which generates the mobility index M: $D^m \to R^1$ by means of* (6) *if and only if M satisfies Properties 2, 7, and 8 with $Y^b = Y_R^b$. If M satisfies Properties 2, 7, and 8 and $Y^b = Y_R^b$, then knowledge of the functional form of M is sufficient to determine the functional form of W up to an increasing monotone transformation.*

Because the proof of Proposition 2 is long and complicated, it is presented in the appendix. However, the basic idea of the algorithm used to construct a social welfare function W (more precisely, an equally distributed equivalent income function Ξ) from the functional form of a mobility index M is quite simple, and can be illustrated with a numerical example.

Suppose there are two individuals, two subperiods, the functional form of M is known, and we wish to determine $\Xi(y)$ for $y = (14, 4)$. From Lemma 1, it is known that

[12] Recall that a defining characteristic of an immobile benchmark Y^b is that $\lambda(y^a) = \lambda(y^b)$.

there exists an income structure Y with equal first-period incomes which has y as its aggregate income. Applying the algorithm given in the proof of Lemma 1 to construct such a structure, we obtain $Y = ((2,2),(12,2))$. Since the functional form of M is given, we know the value of $M((2,2),(12,2))$, say 3. The benchmark aggregate distribution corresponding to Y is $y_R^b = (9,9)$. This benchmark aggregate distribution must have equal components since the structure Y has equal first-period incomes. Consequently, the value of $\Xi(y_R^b)$ is known since $\Xi(\alpha \cdot 1^n) = \alpha$ for all $\alpha > 0$. In this example, $\Xi(y_R^b) = \Xi(9,9) = 9$. It is this fact which allows us to untangle the ratio of terms involving the equally distributed income functions in (6). Since $y = y^a$, substituting the mobility index value of 3 for the L.H.S. of (6) and substituting 9 for $\Xi(y_R^b)$, we obtain the unknown value for $\Xi(14,4)$, namely $\Xi(14,4) = (3+1)\cdot 9 = 36$.

In Proposition 2, the aggregate social welfare function W is only required to be regular. The implications of requiring this function to also be homothetic and/or strictly S-concave are stated in Corollary 1. The proof of Corollary 1 is in the appendix.

Corollary 1. (a) *There exists a regular homothetic aggregate social welfare function* $W\colon D \to R^1$ *which generates the mobility index* $M\colon D^m \to R^1$ *by means of* (6) *if and only if* M *satisfies Properties 2, 6, 7, and 8 with* $Y^b = Y_R^b$. (b) *There exists a regular strictly* S*-concave aggregate social welfare function* $W\colon D \to R^1$ *which generates the mobility index* $M\colon D^m \to R^1$ *by means of* (6) *if and only if* M *satisfies Properties 2, 4, 5, 7, and 8 with* $Y^b = Y_R^b$.

Proposition 2 and Corollary 1 are extremely powerful results. Proposition 2 shows that any mobility index M which satisfies Properties 2, 7, and 8 can be thought of as being an index generated by a regular aggregate social welfare function by means of (6). This conclusion is true regardless of whether the mobility index is in fact so constructed. By writing the mobility index explicitly in the form shown in (6), Proposition 2 and Lemmas 2 and 3 tell us that the mobility index will *automatically* possess Property 3 and satisfy Eq. (13). Furthermore, Corollary 1 informs us that the implicit welfare function is homothetic if M is a relative index and is strictly S-concave if M satisfies Properties 4 and 5.

4. Alternative Concepts of Relative Mobility

In this section we contrast our approach to measuring relative income mobility with the alternative mobility concepts considered by Shorrocks [17], Markandya [12], and King [10]. Because each of these authors is interested in a different aspect of mobility from the one considered in this article, not all of the properties of mobility indices considered in the previous section are appropriate for their measures. However, by contrasting these alternative approaches to our own, it is hoped that the implications of adopting our point of view will be more fully appreciated.

4.1. Shorrocks' Concept of Mobility

Shorrocks [17] is primarily interested in constructing a descriptive index of income mobility, although he does suggest a social welfare interpretation for the measure he proposes. In his study "mobility is measured by the extent to which the income distribution is equalized as the accounting period is extended." (Shorrocks [17], p. 378). Shorrocks works with relative inequality indices and to simplify the exposition it is assumed that they satisfy all of the properties listed for I_R^{Ξ} in Sect. 2. Shorrocks' results on descriptive mobility indices hold for a somewhat more general class of inequality indices, but this generality is not an issue for the points we wish to discuss.

Given a particular relative inequality index I, Shorrocks wishes to compare the inequality $I(y^a)$ when the accounting period is the complete interval $[t_0, t_m)$ with the m one-period inequality values $I(y^k)$, $k = 1, \ldots, m$. Thus in contrast to our approach, he assumes that the functional form of an inequality index is the same for each of the subperiods and for the whole time interval. To make this comparison operational, the one-period measures are combined into a single measure by taking a weighted sum, where the k-th weight w^k is the fraction of total income received by society in period k, i.e. $w^k = \lambda(y^k)/\lambda(y^a)$. Formally, Shorrocks' mobility index is the function $M_S\colon D^m \to R^1$ defined by

$$M_S(Y) = 1 - \frac{I(y^a)}{\sum_k w^k\, I(y^k)}. \tag{14}$$

Shorrocks demonstrates that M_S lies in the interval [0, 1], attaining its minimum value when relative incomes remain constant over time and attaining its maximum value when the aggregate incomes are completely equalized.

Even when used as a descriptive index, Shorrocks' measure can yield unituitive conclusions. As an illustration, let us reconsider the first example discussed in Sect. 3.2. In that example we considered the income structure $Y(\varepsilon) = ((1-\varepsilon, 1+\varepsilon), (1+\varepsilon, 1-\varepsilon))$, concluding that mobility increases with ε, a property exhibited by out indices. The aggregate distribution for $Y(\varepsilon)$ is $y^a = (2, 2)$, so $M_S = 1$ (complete mobility) regardless of the value of ε. This feature of M_S arises due to the fact that $I(y^a)$ may be close to zero either because annual inequalities are very low or because annual inequalities are high but the relative income variations cancel out over time. One would surely only wish to call an income structure mobile in the latter case.

If the inequality index in (14) is obtained from a social welfare function as in (1), Shorrocks shows that (14) may be rewritten as

$$M_S(Y) = 1 - \frac{\lambda(y^a) - \Xi(y^a)}{\sum_k \{\lambda(y^k) - \Xi(y^k)\}}. \tag{15}$$

The numerator of the right-hand-term in (15) is interpreted to be the "true welfare loss" per capita while the denominator is the "apparent welfare loss" per capita. As the numerator never exceeds the denominator, Shorrocks concludes that mobility is *always desirable*. Furthermore, he argues that if two income structures have the same total income and have vectors of one-period incomes which are permutations of each other, then social welfare is greater for the structure exhibiting the largest mobility.

In our approach mobility could be either socially desirable or socially harmful, depending on how the actual aggregate distribution is ranked in comparison with the immobile benchmark distribution. Thus, in the example with $Y = ((10, 10), (18, 2))$ and $\hat{Y} = ((18, 2), (10, 10))$ we found $M_R^{\Xi}(Y) < 0$ and $M_R^{\Xi}(\hat{Y}) > 0$. For Shorrocks' index,

$M_S(Y) = M_S(\hat{Y})$ since these structures differ only in the sequence of one-period distributions. We believe that the actual time-sequence of incomes is important in constructing an ethical index of mobility.[13]

4.2. *Markandya's Concept of Mobility*

Markandya's [13] analysis is restricted to situations involving two periods. He views mobility "as something of interest in itself and distinct from changes in equality." (Markandya [13], pp. 76–77). He regards the movement from y^1 to y^2 in the income structure $Y = (y^1, y^2)$ as involving, in general, a change in total income, a change in inequality, and mobility in the income distribution. He eliminates the first source of change by scaling each period's income distribution so that their means are equal. Call this income structure $Y^* = (y^{*1}, y^{*2})$. To measure "mobility as distinct from changes in inequality", Markandya introduces a hypothetical income structure $Y^{\mathrm{M}} = (y^{\mathrm{M1}}, y^{\mathrm{M2}})$ which involves no change in inequality between the two time periods. Specifically, for a relative inequality index I defined on one-period incomes, he chooses $y^{\mathrm{M1}} = y^{*1}$ and y^{M2} to be the element closest to y^{*2} among all the distributions y which have $I(y) = I(y^{*1})$ and $\lambda(y) = \lambda(y^{*1})$. Markandya uses the square of the coefficient of variation as the inequality index and measures mobility by the (normalized) distance between y^{M1} and y^{M2},

$$M_M(Y) = \frac{\left[\sum_{i=1}^{n} (y_i^{\mathrm{M2}} - y_i^{\mathrm{M1}})^2\right]^{\frac{1}{2}}}{2\left[\sum_{i=1}^{n} (y_i^{\mathrm{M1}} - 1)^2\right]^{\frac{1}{2}}}. \tag{16}$$

The denominator in (16) ensures that the index value lies in the interval [0, 1], taking the lower bound if and only if $y^{\mathrm{M1}} = y^{\mathrm{M2}}$ and the upper bound when y^{M2} is the permutation of y^{M1} obtained by reversing the rank order in the initial distribution.[14]

While Markandya often refers to welfare functions, it is not clear if he views his mobility index as having ethical significance. Because M_M measures the distance between two distributions, and is thus nonnegative, it is not possible to distinguish socially desirable and socially undesirable mobility in this framework. Furthermore, one can easily verify that $M_M(Y) = M_M(Y')$ if Y and Y' differ only in the order of the two time periods.[15]

From our perspective, however, the most interesting feature of Markandya's measure is its use of a hypothetical income structure, albeit one quite different from our own. While our counterfactual benchmark is a structure exhibiting no mobility, Markandya's structure exhibits no change in inequality between periods.

[13] Markandya [13] provides further discussion of Shorrocks' index.

[14] When there are only two people mobility is zero if rank orders do not change and one if they do.

[15] If some other inequality index is used in place of the square of the coefficient of variation, this property of Markandya's index need not hold if $I(y^{*1}) \neq I(y^{*2})$. It continues to hold if $I(y^{*1}) = I(y^{*2})$.

4.3. King's Concept of Mobility

King [10] is essentially interested in obtaining a normative mobility index which measures changes in the rank orders of the income distribution. His analysis is conducted in a two-period framework. From the actual income structure $Y = (y^1, y^2)$ King, in effect, constructs a hypothetical benchmark structure $Y^* = (y^{*1}, y^{*2})$ where $y^{*1} = y^1$ and y_i^{*2} is the element in $\{y_1^2, \ldots, y_n^2\}$ which person i would obtain if his rank order in the income distribution did not change (from y^1). This hypothetical structure is used to define the scaled order statistic

$$s_i = \frac{|y_i^{*2} - y_i^2|}{\lambda(y^2)} \qquad i = 1, \ldots, n. \tag{17}$$

This statistic is nonnegative and is zero if and only if the rank order of y^1 and y^2 are identical.

King's social welfare function is defined on the final distribution y^2 and the vector of scaled order statistics $s = (s_1, \ldots, s_n)$,

$$F(y^2, s). \tag{18}$$

King assumes that F is increasing in all of its arguments. Using (18) he determines a "zero mobility equivalent proportion of income ϱ", defined implicitly in

$$F(\varrho y^2, 0) = F(y^2, s). \tag{19}$$

The increasingness of F implies $\varrho \geqq 1$. King's index of mobility is

$$M_K(Y) = 1 - \frac{1}{\varrho}. \tag{20}$$

This index is interpreted to be the "proportion of total income which, from a position of zero mobility, we would be prepared to sacrifice in order to achieve the degree of mobility we observe..." (King [10], p. 109).

According to this concept of mobility, if there is no difference between the rank orders in y^1 and y^2, there is no mobility (since s_i is then zero). In our approach there can be mobility whenever relative incomes change, even if there are no rank order changes.

Again, from our perspective, the most interesting feature of King's measure is its use of a hypothetical benchmark structure. While we use a benchmark which exhibits no mobility in relative incomes, King uses a benchmark which exhibits no change in rank orders.[16]

5. Concluding Remarks

The innovative feature of this article is the suggestion embodied in Property 1 that an ethical index of income mobility is obtained from a welfare comparison of the actual income structure Y with an immobile benchmark structure Y^b. However, for the specific indices considered here, this comparison has been operationalized by making the assumptions that the benchmark is a completely relatively immobile income

[16] A generalization of King's index is developed in Chakravarty [6].

structure and that the intertemporal welfare function used to make the comparison is sensitive only to the aggregate income distribution. Alternatives to both of these assumptions are possible within our general framework. For example, if one is interested in indices of absolute income mobility, it is more appropriate to use a benchmark which preserves income differences through time.

For two-period problems, the assumption that the intertemporal welfare function is sensitive to only the aggregate income distribution does not appear to be an unreasonable restriction as income distributions in both periods are reflected in the construction of our indices, the first-period distribution through its effect on the aggregate benchmark distribution y^b_R and the second-period distribution through its effect on the actual aggregate distribution y^a. With more than two periods, our indices are sensitive to only y^1 and $\sum_{k=2}^{m} y^k$ rather than the whole structure Y. Consequently, our indices have properties which to many would seem inappropriate. For example, consider the following two person, three period income structures. In the first structure $Y = ((2,2),(2,2),(2,2))$, a structure exhibiting no mobility. In the second, $Y = ((2,2),(3,1),(1,3))$, a structure which also exhibits no mobility for any mobility index derived from an intertemporal welfare function sensitive only to aggregate distributions. Examples such as this, we believe, provide an important test of our intuitions concerning the assumption that $\mathscr{W}$ satisfies (4), rather than a critique of our general concept of mobility.

However, in light of such examples, it seems worth exploring other ways of operationalizing our welfare comparison. One such way is suggested by the growing literature on measuring lifetime incomes and lifetime inequality, a literature which includes contributions by Cowell [7] and Blewett [5]. In this work each individual has an intertemporal utility function $U^i: R^m \to R^1$ which is used to evaluate his or her income stream y_i. An individual may care about the time path of their income receipts because of capital market imperfections or a number of the other reasons studied by Cowell. With perfect capital markets these utility functions will depend nontrivially only on the present-value of the income stream. Cowell has suggested a useful way of employing the intertemporal utility functions to construct a summary statistic y^r_i of the income stream y_i, which following Blewett we call representative lifetime income. Representative lifetime income is defined to be the level of income which if received in each subperiod would yield an income stream judged equivalent to the actual income stream.[17] Formally, it is implicitly defined by

$$U^i(y^r_i \cdot 1^m) = U^i(y_i).$$

Thus, the construction of y^r_i is similar in spirit to the construction of the equally distributed equivalent income y_e. Mild restrictions on U^i ensure y^r exists, in which case y^r is an exact index of individual welfare for the same reason that y_e is an exact index of W.

In terms of ethical mobility indices, one could, as we have done, use welfare functions $\Xi: D \to R^1$ but with vectors of representative lifetime incomes as arguments rather than vectors of aggregate incomes. Letting $y^r = (y^r_1, \ldots, y^r_n)$ and y^{br} be the vector of representative lifetime incomes for the benchmark structure, Property 1′ could be replaced by Property 1″.

[17] A somewhat similar idea is briefly considered in Shorrocks [17].

Property 1″. *For all $Y \in D^m$, $M(Y) = d(\Xi(y^r), \Xi(y^{br}))$ for some function d, with d increasing in its first argument and decreasing in its second argument.*

No doubt such modifications would considerably alter the specific results obtained here, but this alternative approach does appear structured enough to yield results of interest. We leave the derivation of these results as an open research programme.

The underlying idea in the mobility concept we have explored is that of treating a mobility index as a comparison between an actual income distribution and a hypothetical income distribution which exhibits no mobility. A similar idea is considered by Blackorby and Donaldson [4] in their study of tax and benefit progressivity. To construct their index of relative tax progressivity, Blackorby and Donaldson compare the actual after-tax income distribution with the distribution which would have resulted if taxes had been raised proportionally. Reinterpreting y^a as after-tax income, and y^b_R as the hypothetical distribution arising from proportional taxation, our index M^{Ξ}_R becomes Blackorby and Donaldson's index of relative tax progressivity. Thus, our indices are formally equivalent.[18]

Mobility is a many-faceted phenomenon. Each of the recent studies by Chakravarty [6], Kanbur and Stiglitz [9], King [10], Markandya [12, 13, 14], and Shorrocks [16, 17] consider different aspects of this notion. By exploring yet another concept of mobility in this paper, we hope we have further demonstrated that mobility indices have an important role to play in social welfare evaluations.

Appendix

Proof of Proposition 2: (a) Necessity. Our informal discussion has already established that M must satisfy Properties 2 and 7 if W is regular. To show that Property 8 is also satisfied suppose $\hat{Y} \in \Psi(Y)$, $\tilde{Y} \in \Psi(Y^b_R)$, and $\hat{y}^1 = \tilde{y}^1$. We have

$$\bar{M}(Y) = \frac{\Xi(y^a)}{\Xi(y^b_R)}$$

$$= \frac{\Xi(\hat{y}^a)}{\Xi(\tilde{y}^a)} \qquad \text{(since } \hat{y}^a = y^a \text{ and } \tilde{y}^a = y^b_R\text{)}$$

$$= \frac{\dfrac{\Xi(\hat{y}^a)}{\lambda(\hat{y}^b_R)}}{\dfrac{\Xi(\tilde{y}^a)}{\lambda(\tilde{y}^b_R)}} \qquad \text{(since } \lambda(\hat{y}^b_R) = \lambda(\tilde{y}^b_R)\text{)}$$

$$= \frac{\dfrac{\Xi(\hat{y}^a)}{\Xi(\hat{y}^b_R)}}{\dfrac{\Xi(\tilde{y}^a)}{\Xi(\tilde{y}^b_R)}} \qquad \text{(since } \hat{y}^b_R \text{ and } \tilde{y}^b_R \text{ are on the ray of equality)}$$

$$= \frac{\bar{M}(\hat{Y})}{\bar{M}(\tilde{Y})}.$$

[18] We are indebted to David Donaldson for drawing this equivalence to our attention. Blackorby and Donaldson [14] do not develop the analogues to our Propositions 1 and 2 for their indices of relative tax progressivity.

(b) Sufficiency. To establish the sufficiency part of the theorem we actually construct the social welfare function, thus also proving the last part of the theorem.

For $y \in D$, let $Y(y)$ denote the income structure generated by the algorithm introduced in the proof of Lemma 1. Define $\Omega: D \to R^1$ by

$$\Omega(y) = [M(Y(y)) + 1] \cdot \lambda(y). \tag{A.1}$$

We demonstrate that Ω is a regular equally distributed equivalent income function which generates M by means of (6).

First we show that Ω is a regular equally distributed equivalent income function. By Property 2, M is continuous. As $Y(\cdot)$ and $\lambda(\cdot)$ are continuous as well, Ω is continuous. If y has equal components, $Y(y)$ also has equal components and the benchmark corresponding to $Y(y)$ is simply $Y(y)$. By Property 8, $M(Y(y)) = 0$. Using (A.1), $\Omega(y) = \lambda(y)$. Thus Ω is increasing along the ray of equality and has the appropriate normalization for an equally distributed equivalent income function. For an arbitrary $y \in D$, $\lambda(y) > 0$ and, by Property 7, $M(Y(y)) > -1$. Thus from (A.1), $\Omega(y) > 0$ and the social indifference curve through y intersects the ray of equality at the point $\lambda(y) \cdot 1^n$.

We now show that using Ω as an equally distributed equivalent income function, $M(Y) = M_R^{\Omega}(Y)$ for all $Y \in D^m$; i.e. M is generated from Ω by means of (6). For an arbitrary $Y \in D^m$, determine the corresponding y^a and y_R^b. Let $Y^* = Y(y^a)$ and $Y^{**} = Y(y^b)$. From (A.1),

$$\frac{\Omega(y^a)}{\Omega(y_R^b)} = \frac{\bar{M}(Y^*)}{\bar{M}(Y^{**})} \cdot \frac{\lambda(y^a)}{\lambda(y_R^b)}.$$

By the definition of the benchmark, $\lambda(y^a) = \lambda(y_R^b)$, so

$$\frac{\Omega(y^a)}{\Omega(y_R^b)} = \frac{\bar{M}(Y^*)}{\bar{M}(Y^{**})}. \tag{A.2}$$

By construction, $Y^* \in \Psi(Y)$, $Y^{**} \in \Psi(Y_R^b)$, and $y^{*1} = y^{**1}$. Hence, by Property 8,

$$\bar{M}(Y) = \frac{\bar{M}(Y^*)}{\bar{M}(Y^{**})}. \tag{A.3}$$

Substituting (A.3) into (A.2),

$$\Omega(y^a) = [M(Y) + 1] \cdot \Omega(y_R^b),$$

which is (6) with Ω substituted for Ξ and M substituted for M_R^{Ξ}.[19] □

Proof of Corollary 1: The remarks in Sect. 3.2 establish the necessity part of the proof, so we restrict attention to sufficiency.

(a) Suppose $\hat{y} = \alpha y$ for some scalar $a > 0$. Construct the structures Y and $\hat{Y}$ according to the algorithm used to prove Lemma 1. Thus $\hat{Y} = \alpha Y$ and $\hat{y}_R^b = \alpha y_R^b$ where $y_R^b = \lambda(Y_R^b) \cdot 1^n$. Hence, $\Xi(\hat{y}_R^b) = \alpha \Xi(y_R^b)$. Combined with the assumption that M is a relative index, we obtain from (6) that Ξ is positively linear homogeneous and, by Blackorby and Donaldson's [2] theorem, W is homothetic.

[19] When D is replaced by $R_{++}^n \backslash \{0\}$, the algorithm is used to determine Ω on R_{++}^n and continuity is used to determine the values on the boundary of the domain.

(b) Suppose $\hat{y}_i = y_i$ for all $i \neq j, j'$, $y_j > y_{j'}$, $\hat{y}_j > \hat{y}_{j'}$, $\hat{y}_j = y_j - \delta$, and $\hat{y}_{j'} = y_{j'} + \delta$. We wish to construct structures from y and $\hat{y}$ which have $y^1 = \hat{y}^1$. Note that the algorithm used in the proof of Lemma 1 will not achieve this objective if j' is the unique recipient of the minimal income in y. To overcome this difficulty we apply the algorithm in the proof of Lemma 1 to obtain the structure Y from y and define the structure $\hat{Y}$ by setting $\hat{y}^1 = y^1$ and letting $\hat{y}^k = (\hat{y} - \hat{y}^1)/(m-1)$ for all $k \geqq 2$. Note that $\hat{y}_i - \hat{y}_i^1 > 0$ for all i. For all $k \geqq 2$, $\hat{y}_i^k = y_i^k$ for $i \neq j, j'$, $y_j^k > y_{j'}^k$, $\hat{y}_j^k > \hat{y}_{j'}^k$, $\hat{y}_j^k = y_j^k - \left(\dfrac{\delta}{m-1}\right)$, and $\hat{y}_{j'}^k = y_{j'}^k + \left(\dfrac{\delta}{m-1}\right)$. By an application of Property 5 to each $k \geqq 2$ separately, we obtain from (6) that $\Xi(\hat{y}) > \Xi(y)$ as $y^a = y$ and $\hat{y}^a = \hat{y}$. Property 4 implies Ξ is symmetric, so by Dasgupta, Sen, and Starrett's [8] theorem, Ξ (and, hence, W) is strictly S-concave. □

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