Conjectures and Unsolved Problems

In this column *Social Choice and Welfare* will present conjectures and unsolved problems dealing with social choice theory. Suggestions, unsolved problems, conjectures and solutions should be sent to the column editor: Jerry S. Kelly, Department of Economics, Syracuse University, Syracuse, NY 13244-1090, USA

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6. The Ostrogorski Paradox

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There are two candidates, *A* and *B*, and three political issues, *X*, *Y*, *Z*, on which the candidates have taken positions. There are five voters; each will vote for the candidate with whom he agrees on more issues. The position and preference data are given in Table 1.

Table 1

	Issue		
Candidate	*X*	*Y*	*Z*
A	for	for	for
B	against	against	against
Voter #			
1	for	for	against
2	for	against	for
3	against	for	for
4	against	against	against
5	against	against	against

Voters #1, #2, and #3 agree with *A* on two of the three issues and with *B* on only one; there are three votes for *A*. Voters #4 and #5 agree with *B* on all three issues and vote for him. A wins. *But*, issue by issue, *A* has taken the *minority* position. This possibility, of a candidate winning by majority vote, when it holds a minority position on every single issue was first announced in 1976. One version was presented by political scientists Douglas Rae and Hans Daudt [6], who called this the *Ostrogorski paradox* in honor of Moisei Ostrogorski who "devoted his major work [5] to the proposition that all manner of mischief can result when issues are mixed together in a single contest". A related version, with a different interpretation, was presented by British philosopher Gertrude Elizabeth Margaret

Anscombe [1]. In this column, we will explore the Ostrogorski paradox and raise questions about *(i)* the probability of its occurrence and *(ii)* a connection with cycles in the binary defeats-by-majority-vote relation among candidates.

1. Probability of Occurrence

How likely is the Ostrogorski paradox? As usual in these things, real world voting situations almost never permit extraction of enough information to determine whether or not a paradox has occurred (curiously, Anscombe urges study of early 18th century English Parliament voting). So we turn to simulations of the sort that have been used elsewhere [4] to estimate the likelihood of no Condorcet winner. One such approach has been presented by psychologists Bezembinder and van Acker [2]; here I present another.

Assume an odd number, m, of issues and an odd number, n, of voters. Each individual voter's preferences are for $(+1)$ or against (-1) each issue; non-trivial indifference is disallowed. A profile of preferences is thus represented by an n by m matrix of $+1$'s and -1's. There are 2^{mn} such matrices. For each choice of m and n, we treat these 2^{mn} matrices as our population and randomly draw (with replacement) a sample of 100,000. For each matrix in the sample, we determine the majority outcome ($+1$ or -1) for each issue. One candidate, A, is then assumed to adopt as his position the pattern of $+1$'s and -1's that is this set of majority outcomes. The second candidate, B, makes his position the opposite choices. Assume each individual would vote for the candidate with the greater number of position choices agreeing with that individual's preferences. When B defeats A on this majority vote we have an occurrence of the Ostrogorski paradox. The number of such occurrrences divided by 100,000 is our sample estimate of the proportion of the 2^{mn} population members that display this "paradoxical" behavior. Table 2 shows our sample relative frequency results ($n=3$ will be discussed later):

Table 2

$n \backslash m$	3	5	7
5	0.01504	0.00622	0.00230
7	0.02301	0.00894	0.00399
9	0.02708	0.01234	0.00475
11	0.03007	0.01237	0.00495
13	0.03086	0.01284	0.00619
15	0.03197	0.01388	0.00574
17	0.03405	0.01416	0.00633
19	0.03548	0.01411	0.00693
21	0.03554	0.01505	0.00651

There would, of course, be no reason to carry out these simulations if we were able to present analytic results, but no such results are known. While the sample frequency rates are low, and decrease rapidly with an increase in m, the number of issues, they do increase with an increase in n, the number of individuals. This increase with the number of individuals is, in the sample, not quite always monotonic, but nearly so. I conjecture that the underlying population proportions increase monotonically (but not to 1) with n and decrease monotonically to 0 with m.

2. Connection with Voting Cycles

So far we have discussed a majority vote relationship between just two candidates, one that is in agreement on every single issue with a majority of individuals and a second candidate that is in exact opposition. Rae and Daudt observed that if we consider introducing new candidates (who must agree with a majority of voters on some issues but with only a minority on some others), we will observe intransitivities, voter paradoxes, in the binary defeats-by-majority-vote relation. They illustrate this by a profile we present in Table 3.

Table 3

	Preference by issue		
Voter type	1	2	3
1 (20%)	+1	−1	−1
2 (20%)	−1	+1	−1
3 (20%)	−1	−1	+1
4 (40%)	+1	+1	+1

On every issue, position $+1$ wins (60% to 40%). But voters of types 1 to 3 would vote for a candidate, X, choosing positions $(-1, -1, -1)$ over a candidate, Y, choosing $(+1, +1, +1)$. Now introduce a new candidate, Z, choosing $(-1, +1, +1)$. We continue to assume that when individuals face two candidates, they vote for the one that has the greater number of agreements between candidate positions and voter preferences on issues. Then:

X defeats Y (Types 1, 2, 3 voting for X)
Y defeats Z (Types 1 and 4 voting for Y)
Z defeats X (Type 1 votes for X, Type 4 votes for Z, Types 2 and 3 are indifferent between Z and X).

So a cycle has occurred in the defeats-by-majority-vote relation.

But we may still ask how serious these cycles are. For a first observation, suppose that we are not restricted to just two or three candidates (Rae and Daudt, for example, used four). There are $2^3 = 8$ possible candidates if they differ only by issue choices; might one of these be a Condorcet winner even though there is a cycle among other candidates? We will show that, in this case of three issues, this is not possible, that every occurrence of the Ostrogorski paradox implies that among the 8 possible candidates there can be no Condorcet winners. To do this, we will exploit some notation and some inequalities introduced by Shelley [7]. Without loss of generality, we can assume that, in the Ostrogorski paradox occurrence we are

Table 4

	Preference by Issue		
Voter group	1	2	3
A	−1	−1	−1
B	−1	−1	+1
C	−1	+1	−1
D	+1	−1	−1
E	−1	+1	+1
F	+1	−1	+1
G	+1	+1	−1
H	+1	+1	+1

examining, +1 wins on every issue. In Table 4 we show a general profile of preferences. Voter groups A, B, C, and D would vote for a candidate choosing positions $(-1, -1, -1)$ over a candidate choosing $(+1, +1, +1)$. Groups E, F, G, and H would vote for $(+1, +1, +1)$ over $(-1, -1, -1)$. So, if we let each letter refer to the size of its respective group, we have

$$A+B+C+D>E+F+G+H \ . \tag{1}$$

But, since +1 is preferred by a majority on each issue, we also have

$$D+F+G+H>A+B+C+E \ ; \tag{2}$$

$$C+E+G+H>A+B+D+F \ ; \tag{3}$$

$$B+E+F+H>A+C+D+G \ . \tag{4}$$

Shelley observed that these imply also

$$D>E \ ; \quad C>F \ ; \quad B>G \ ; \quad \text{and} \quad H>A$$

(these last show that an occurrence of the Ostrogorski paradox requires more than three voters which is why Table 2 started with the case $n=5$).

We wish to show that inequalities (1)–(4) imply that each of the eight candidates loses to at least one other. (1) tells us

$$(+1, +1, +1) \quad \text{loses to} \quad (-1, -1, -1) \ .$$

It thus will be sufficient to show that $(+1, +1, +1)$ defeats all the other six position lists and that $(-1, -1, -1)$ loses to at least one of them (in fact $(-1, -1, -1)$ loses to all six). (2) tells us

$$(-1, +1, +1) \quad \text{loses to} \quad (+1, +1, +1) \ ;$$

(3) implies

$$(+1, -1, +1) \quad \text{loses to} \quad (+1, +1, +1) \ ;$$

while (4) implies both

$$(+1, +1, -1) \quad \text{loses to} \quad (+1, +1, +1)$$

and

$$(-1, -1, -1) \quad \text{loses to} \quad (-1, -1, +1)\ .$$

Adding (3) and (4) gives

$$C+E+G+H+B+E+F+H>A+B+D+F+A+C+D+G$$

or

$$2(E+H)>2(A+D)$$

or

$$E+H>A+D$$

which implies

$$(+1, -1, -1) \quad \text{loses to} \quad (+1, +1, +1)\ .$$

Finally, adding (2) and (3) gives $G+H>A+B$ which tells us

$$(-1, -1, +1) \quad \text{loses to} \quad (+1, +1, +1)\ .$$

For the case of three issues, then, any occurrence of the Ostrogorski paradox implies that there does not exist a Condorcet winner among all possible candidate positions. I conjecture that this is also true for larger odd numbers of issues.

Secondly, we observe that even more extreme cycling may occur. Still within the three issue case, consider the general profile in Table 4 and suppose that we have, for example,

$$A=1\ , \quad B=5\ , \quad C=8\ , \quad D=7\ , \quad E=2\ , \quad F=1\ , \quad G=2\ , \quad \text{and} \quad H=13\ .$$

Then there exists a *global cycle*, a cycle of the defeats-by-majority-vote relation involving all eight candidates:

$(-1, -1, -1)$ defeats $(+1, +1, +1)$ which defeats
$(+1, +1, -1)$ which defeats $(+1, -1, +1)$ which defeats
$(-1, +1, +1)$ which defeats $(+1, -1, -1)$ which defeats
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$(-1, -1, -1)$.

But, such a global cycle need not always exist when an Ostrogorski paradox occurs. Consider the rather similar example

$$A=1,\ B=5,\ C=8,\ D=7,\ E=2,\ F=4,\ G=2, \quad \text{and} \quad H=13\ .$$

Then it can be shown there is no global cycle. There is, however, one of length six. It can be shown that, with three issues, there will always be a cycle of length six and, usually, a cycle of length eight. It would be interesting to run additional simulation experiments and examine maximal cycle lengths for those profiles exhibiting an Ostrogorski paradox. How does the average maximal cycle length vary as you increase the number of voters, holding the number of issues fixed? As a fraction of the number 2^m of possible candidates, how does average maximal cycle length vary as you increase the number of issues?

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