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Asymptotic Behaviour of Positive Solutions of Elliptic Equations with Critical and Supercritical Growth I. The Radial Case

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1. Introduction and main results

In this paper we consider the singular limit in a family of non-linear elliptic equations with strong growth. The general problem is the following. Consider for a bounded domain Ω in $\mathbf{R}^N$, where $N > 2$, with smooth boundary $\partial\Omega$, the problem

$$(\mathrm{P}) \qquad \begin{aligned} -\Delta u &= f(u) && \text{in } \Omega \\ u &> 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and suppose $f(s)$ is a function whose growth as $s \to \infty$ is such that (P) has no solution. We then consider, what we call the "approach problem",

$$(\mathrm{P}_\varepsilon) \qquad \begin{aligned} -\Delta u &= f_\varepsilon(u) && \text{in } \Omega \\ u &> 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

in which the family of functions f_ε is so chosen that for $\varepsilon > 0$ and small, (P_ε) has a solution u_ε and

$$f_\varepsilon(s) \to f(s) \quad \text{as } \varepsilon \to 0,$$

uniformly on compact sets. The natural question to ask now is what happens to u_ε as $\varepsilon \to 0$.

As a first example we consider the function

$$f(s) = s^p, \quad p > 1 \tag{1.1}$$

and we set

$$p_N = \frac{N+2}{N-2}.$$

As we know, if $p < p_N$ (*p subcritical*) then (P) has a solution [R] but if $p \geqq p_N$ (*p critical* or *supercritical*) and Ω is star shaped, then it has none [P].

For $p = p_N$ this problem was studied in [AP2], [BP] and [H] by means of the family of functions

$$f_\varepsilon(s) = s^{p_N - \varepsilon}, \quad \varepsilon > 0,$$

first when Ω is the unit ball B_1 in $\mathbf{R}^N$ and subsequently in non-radial star-shaped domains, where in addition it was assumed that

$$\frac{\|\nabla u_\varepsilon\|_{L^2}^2}{\|u_\varepsilon\|_{L^{p_N+1-\varepsilon}}^2} \to S_N \quad \text{as } \varepsilon \to 0. \tag{1.2}$$

Here S_N is the best Sobolev constant for the norm in H^1, given by

$$S_N = \pi N(N-2)\left(\frac{\Gamma(N/2)}{\Gamma(N)}\right)^{2/N}.$$

It was shown that the solution $u_\varepsilon(x)$ concentrates at a single point x_0 as $\varepsilon \to 0$ and that

$$\varepsilon \|u_\varepsilon\|_{L^\infty}^2 \to 2c_N^2\sigma_N^2\left(\frac{N(N-2)}{S_N}\right)^{N/2} |g(x_0, x_0)| \quad \text{as } \varepsilon \to 0. \tag{1.3}$$

Here c_N is a normalizing constant and σ_N is the area of the unit sphere in $\mathbf{R}^N$:

$$c_N = \{N(N-2)\}^{(N-2)/4} \quad \text{and} \quad \sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$

The function $g(x, y)$ is the regular part of the Green's function $G(x, y)$ which solves

$$-\Delta G = \delta_y \quad \text{in } \Omega \tag{1.4}$$

$$G = 0 \quad \text{on } \partial\Omega \tag{1.5}$$

and is given by the relation

$$G(x, y) = \frac{1}{(N-2)\,\sigma_N\,|x-y|^{N-2}} + g(x, y), \tag{1.6}$$

and x_0 is a critical point of the function $\phi(y) = g(y, y)$.

About the shape of the solution u_ε it was shown that, away from the point of concentration x_0

$$\varepsilon^{-1/2}\,u_\varepsilon(x) \to c_N\left(\frac{N(N-2)}{S_N}\right)^{N/4} \frac{N-2}{\sqrt{2\,|g(x_0, x_0)|}}\,G(x, x_0) \quad \text{as } \varepsilon \to 0 \tag{1.7}$$

and near the point of concentration:

$$u_\varepsilon(x) \sim \gamma_\varepsilon V(\gamma_\varepsilon^{(p-1)/2}(x - x_0)) \quad \text{as } \varepsilon \to 0, \tag{1.8}$$

where $\gamma_\varepsilon = \|u_\varepsilon\|_{L^\infty}$ and $V(y)$ satisfies

$$-\Delta V = V^{p_N} \quad \text{in } \mathbf{R}^N$$

$$V(0) = 1, \quad 0 < V \leqq 1 \quad \text{in } \mathbf{R}^N,$$

that is

$$V(y) = \left(1 + \frac{|y|^2}{N(N-2)}\right)^{-(N-2)/2}. \tag{1.9}$$

As another example we mention the function

$$f(s) = \lambda s + s^{p_N}.$$

It was shown in [BN] that for this function Problem (P) has a variational solution which satisfies (1.2) when $\lambda \in (\lambda^*, \mu_1)$, where μ_1 is the principal eigenvalue of the Laplacian and $0 \leqq \lambda^* < \mu_1$ ($\lambda^* > 0$ if $N = 3$ and $\lambda^* = 0$ if $N \geqq 4$). If we choose, as functions f_ε the family

$$f_\varepsilon(s) = (\lambda^* + \varepsilon)\, s + s^{p_N}$$

the asymptotic behaviour of u_ε was investigated in [R] for $N \geqq 5$ and in [BP] for $N = 3$.

Finally we mention the example in $N = 3$

$$f_\varepsilon(s) = \lambda^* s + s^{5-\varepsilon}$$

which was studied in [Bu] and [BP] when $\Omega = B_1$ and so $\lambda^* = \pi^2/4$ [BN]l

In all the examples investigated so far, the function g had no more than critica. growth. It is the object of this paper to study in particular the approach to problems involving *supercritical* growth, and compare the resulting asymptotics to the approach to problems with *critical* growth.

We consider again the function f given by (1.1) and we choose as approximating functions

$$f_\varepsilon(s) = s^p - \varepsilon s^q, \qquad s > 0, \tag{1.10}$$

in which

$$q > p \geqq p_N, \qquad \varepsilon > 0.$$

In this paper we shall consider this problem taking for Ω the unit ball B_1. By [GNN] this implies that the solution u_ε has radial symmetry, which allows us to use the techniques for ordinary differential equations. In a forthcoming paper we shall discuss the same problem for general star-shaped domains under the assumption (1.2) if $p = p_N$ and a comparable assumption if $p > p_N$.

Thus in this paper we shall study the problem

(I)

$$-\Delta u = u^p - \varepsilon u^q \quad \text{in } B_1, \tag{1.11}$$

$$u > 0 \quad \text{in } B_1, \tag{1.12}$$

$$u = 0 \quad \text{on } \partial B_1. \tag{1.13}$$

By [GNN], u_ε is decreasing with respect to $r = |x|$ and so $\|u_\varepsilon\|_{L^\infty} = u_\varepsilon(0)$. For convenience we shall sometimes write

$$\gamma_\varepsilon = u_\varepsilon(0) = \|u_\varepsilon\|_{L^\infty}.$$

As in [AP2] and [BP] we find that for any solution u_ε of (I)

$$\gamma_\varepsilon \to \infty \qquad \text{as } \varepsilon \to 0.$$

The existence of a solution u_ε of Problem (I) is ensured for small values of ε by the following theorem.

Theorem A. *For $\varepsilon > 0$ and sufficiently small, Problem* (I) *has at least two solutions. For one solution we have*

$$\lim_{\varepsilon \to 0} \varepsilon \gamma_\varepsilon^{q-p} = 1 \tag{1.14}$$

and for another we have

$$\lim_{\varepsilon \to 0} \varepsilon \gamma_\varepsilon^{q-p} = c^*, \tag{1.15}$$

where c^* is a number which is uniquely determined by p, q and N, and

$$\frac{c(p, N)}{c(q, N)} \leqq c^* < 1 \tag{1.16}$$

in which

$$c(s, N) = \frac{(N-2)\,s - (N+2)}{2(s+1)}. \tag{1.17}$$

Observe that $c(p, N) > 0$ if $p > p_N$ and that $c(p_N, N) = 0$.

In what follows we shall refer to those solutions of Problem (I) for which (1.14) holds as *large solutions* and to those for which (1.15) holds as *small solutions.*

To formulate our results and explain the origin of the number c^*, we need to introduce the notion of a *ground state solution* (or a fast decay solution) of the equation

$$-\Delta V = V^p - cV^q, \quad V > 0 \quad \text{in } \mathbf{R}^N \tag{1.18}$$

which has the properties

$$V(0) = 1 \quad \text{and} \quad V(y) = O(|y|^{-(N-2)}) \quad \text{as } |y| \to \infty. \tag{1.19}$$

There is precisely one value of c for which (1.18)–(1.19) has a radial solution V, which is necessarily unique [KMPT]. This is the value $c^* = c(p, q, N)$ referred to in Theorem A.

The following proposition provides a relation between c^* and V.

Proposition B. *Suppose that $q > p \geqq p_N$. Then*

$$c^* c(q, N) \int_{\mathbf{R}^N} V^{q+1} = c(p, N) \int_{\mathbf{R}^N} V^{p+1}. \tag{1.20}$$

Suppose u_ε is a small solution of Problem (I) so that (1.15) is satisfied. If $p > p_N$, then $c(p, N) > 0$ and therefore we can conclude from (1.16) that $c^* > 0$ and from (1.15) that

$$\|u_\varepsilon\|_{L^\infty} \asymp c^* \varepsilon^{-1/(q-p)} \quad \text{as } \varepsilon \to 0. \tag{1.21}$$

On the other hand, if $p = p_N$, then $c(p, N) = 0$ and we conclude from Proposition B that $c^* = 0$. In this case we find that

$$\|u_\varepsilon\|_{L^\infty} \asymp A(q, N)\, \varepsilon^{-1/(q-p+2)} \quad \text{as } \varepsilon \to 0, \tag{1.22}$$

where

$$A(q, N) = \left\{ \frac{N^2 c(q, N)}{\{N(N-2)\}^{N/2}} B\left(\frac{N}{2}, q \frac{N-2}{2} - 1 \right) \right\}^{-1/(q-p+2)} \tag{1.23}$$

and $B(a, b)$ denotes the beta function [AS], defined by

$$B(a, b) = \int_0^\infty t^{a-1}(1+t)^{-a-b}\, dt.$$

Here we write $f(x) \asymp g(x)$ as $x \to 0$, when $g(x)$ is positive near $x = 0$ and $f(x)/g(x) \to 1$ as $x \to 0$.

As in previous studies of the limiting behaviour of solutions of elliptic equations near criticality [AP2, BP, H, Re] we find that the function $u_\varepsilon(x)$, when suitably scaled, converges to the Green's function $G_0(x) = G(x, 0)$ defined by (1.4)–(1.5). Here we prove the following limit theorem.

Theorem C. *Let u_ε be a small solution of Problem* (I) *so that* (1.15) *is satisfied. Then*

$$\varepsilon^{-\theta} u_\varepsilon(x) \to M G_0(x) \quad \text{as } \varepsilon \to 0, \tag{1.24}$$

where θ and M are positive constants. If $p > p_N$ then

$$\theta = \frac{(N-2)p - N}{2(q-p)}, \qquad M = (c^*)^{-\theta} \left(\int_{\mathbf{R}^N} V^p - c^* \int_{\mathbf{R}^N} V^q \right).$$

If $p = p_N$ then

$$\theta = \frac{1}{q-p+2}, \qquad M = \frac{\{N(N-2)\}^{N/2} \sigma_N}{N A(q, N)},$$

where $A(q, N)$ is given by (1.23).

Remark. We shall see in [MP] that Problem (I) has a variational structure and that what we call a small solution is in fact a variational solution of Problem (I).

Remark. It is easy to see that the asymptotic behaviour for a large solution u_ε is given by

$$\gamma_\varepsilon^{-1} u_\varepsilon \to 1 \quad \text{as } \varepsilon \to 0 \text{ when } x \in B_1.$$

Remark. A similar analysis can be given for radial solutions of Problem (I) with a prescribed number of zeros.

The organisation of the paper is the following. In Section 2 we establish the existence of large and small solutions and prove Theorem A. In Section 3 we prove a basic global upper bound for solutions of Problem (I), and finally in Section 4 we prove Proposition B and the asymptotic estimates. The main ingredients here are the upper bound of Section 3 and the *Pohozaev Identity* which says [P] that if u is a solution of Problem (P), then

$$N \int_\Omega F(u) - \frac{N-2}{2} \int_\Omega u f(u) = \tfrac{1}{2} \int_{\partial\Omega} (x-y, n) \left(\frac{\partial u}{\partial n} \right)^2, \tag{1.25}$$

where $F(u) = \int_0^u f(t)\,dt$, y any point in Ω and n the outward pointing normal vector on $\partial\Omega$.

2. Existence and basic properties

There are many ways to prove the existence of a solution u_ε of Problem (I) for ε sufficiently small. Here we shall use a shooting technique. However, we first derive a general property for (I).

For convenience we rescale the variables and write

$$y = \gamma^{(p-1)/2} x, \qquad v(y) = \gamma^{-1} u(x). \tag{2.1}$$

This yields the following problem for v:

$$-\Delta v = v^p - cv^q, \qquad c \geqq 0 \tag{2.2}$$

$$v(0) = 1. \tag{2.3}$$

where $q > p \geqq p_N$ and

$$c = \varepsilon\gamma^{q-p}. \tag{2.4}$$

Note that (2.2) and (2.3) imply that

$$-\Delta v(0) = 1 - c,$$

Since v takes on its maximum value at the origin, this means that $c \leqq 1$.

Lemma 2.1. *Suppose v is a radial solution of* (2.2)–(2.3), *and*

$$c \leqq \frac{c(p, N)}{c(q, N)}, \tag{2.5}$$

where $c(s, N)$ is given by (1.17). *Then*

$$v > 0 \quad \text{in } \mathbf{R}^N.$$

Proof. We argue by contradiction. Suppose there exists a radius $R > 0$ such that $v > 0$ in B_R and $v = 0$ on ∂B_R. Then writing the Pohozaev Identity (1.25) for (2.2) on B_R we obtain

$$-c(p, N) \int_{B_R} v^{p+1} + cc(q, N) \int_{B_R} v^{q+1} = \tfrac{1}{2} \int_{\partial B_R} (x, n) \left(\frac{\partial v}{\partial n}\right)^2$$

and so, by the Boundary Point Lemma,

$$c(p, N) \int_{B_R} v^{p+1} < cc(q, N) \int_{B_R} v^{q+1}$$

$$< cc(q, N) \int_{B_R} v^{p+1}$$

because $v \leqq 1$ in B_R. This would imply that $c \geqq c(p, N)/c(q, N)$, contradicting (2.5), whence we may conclude that $v > 0$ in $\mathbf{R}^N$.

Set $r = |y|$ and write $\tilde{v}(r) = v(y)$. Then, omitting the tilde again we obtain the initial value problem

$$v'' + \frac{N-1}{r} v' + v^p - cv^q = 0 \tag{2.6}$$

$$v(0) = 1, \quad v'(0) = 0. \tag{2.7}$$

Plainly, for each $c \in [0, 1]$ there exists a unique local solution of (2.6)–(2.7) which we denote by $v(r, c)$ and which can be continued as long as it is bounded.

Define

$$R(c) = \sup\{r > 0 : v(\cdot, c) > 0 \quad \text{on } (0, r)\}.$$

Note that by Lemma 2.1, $R(c) = \infty$ if $c \leqq c(p, N)/c(q, N)$.

Lemma 2.2. *Suppose that* $0 \leqq c < 1$. *Then*

(a) $$v'(r, c) < 0 \quad \textit{for } 0 < r < R(c);$$

(b) $$R(c) = \infty \;\Rightarrow\; \lim_{r\to\infty} v(r, c) = 0.$$

Proof. (a) Because $c < 1$ it follows that $v''(0, c) < 0$ and so that $v'(c, r) < 0$ for small values of r. Suppose to the contrary that for some $r_0 \in (0, R(c))$, $v(r, c)$ ceases to be decreasing, i.e. $v'(r_0, c) = 0$ and $v(r, c) > 0$ on $(0, r_0)$. Then we would have $v''(r_0, c) \geqq 0$, which is incompatible with the differential equation (2.6).

(b) Because $v'(r, c) < 0$ and $v(r, c) > 0$ for all $r > 0$, it follows that $\lim_{r\to\infty} v(r, c)$ exists. It is readily seen that this limit can only be zero.

In the following lemma we establish some further properties of $R(c)$.

Lemma 2.3. *There is a number* $\hat{c} \in [0, 1]$ *such that*

(a) $$R(c) < \infty \quad \textit{when } \hat{c} < c < 1,$$

(b) $$\lim_{c\uparrow 1} R(c) = \infty \quad \textit{and} \quad \lim_{c\downarrow \hat{c}} R(c) = \infty,$$

and $R(c)$ *is continuous on* $(0, \hat{c})$.

Proof. (a) Suppose to the contrary that there is a sequence $\{c_n\} \in (0, 1)$ such that $c_n \to 1$ as $n \to \infty$ and $R(c_n) = \infty$ for every $n \geqq 1$. Then, since $v(0, c_n) = 1$ an by Lemma 2.2 $v(r, c_n) \to 0$ as $r \to \infty$, for every $n \geqq 1$ there is a radius $\varrho_n > 0$ such that $v(\varrho_n, c_n) = \frac{1}{2}$. Thus if we set

$$s = r - \varrho_n \quad \text{and} \quad w_n(s) = v(r, c_n),$$

we obtain for w_n the problem

$$w_n'' + \frac{N-1}{s+\varrho_n} w_n' + w_n^p - cw_n^q = 0, \tag{2.8}$$

$$w_n(0) = \tfrac{1}{2}, \quad 0 < w_n(s) \leqq \tfrac{1}{2} \quad \text{for } 0 \leqq s < \infty. \tag{2.9}$$

Plainly, the sequence $\{w_n\}$ is uniformly bounded, but also the sequence of derivatives $\{w_n'\}$ is uniformly bounded. To see this we multiply (2.8) by w_n' and integrate over $(0, s)$. This yields

$$\tfrac{1}{2}(w_n')^2 = \int_{w_n(s)}^{1} f(t)\,dt - (N-1)\int_0^s \frac{(w_n')^2}{t+\varrho_n}\,dt$$

$$< \int_0^1 f(t)\,dt,$$

where $f(t) = t^p - ct^q$.

We now let $n \to \infty$. Then, by a standard compactness argument there exists a subsequence, which we denote by w_n again, which converges to a function W which, taking the limit in (2.8)–(2.9), satisfies

$$W'' + W^p - W^q = 0,$$

$$W(0) = \tfrac{1}{2}, \quad 0 \leqq W(s) \leqq \tfrac{1}{2} \quad \text{for } 0 \leqq s < \infty.$$

Because this problem has no solution, we have arrived at a contradiction and we must conclude that $R(c) < \infty$ in a left-neighbourhood of $c = 1$.

(b) Since $v(r, 1) = 1$ is a solution of equation (2.6) the first limit follows from the continuous dependence of solutions of (2.6)–(2.7) on c. As to the second limit, we conclude from Lemma 2.1 that

$$\hat{c} = \inf\{c < 1 : R(t) < \infty \text{ for } c < t < 1\} \geqq \frac{c(p, N)}{c(q, N)}.$$

Invoking the continuous dependence of $v(r, c)$ on c again, we conclude that $R \in C(\hat{c}, 1)$ and that $R(c) \to \infty$ as $c \to \hat{c}$.

We now return to Problem (I). The function $v(r, c)$ will correspond to a solution of this problem if

$$R(c) = \gamma^{(p-1)/2} \quad \text{and} \quad c = \varepsilon\gamma^{q-p},$$

or, when we eliminate γ, if

$$c = \varepsilon\{R(c)\}^{2(q-p)/(p-1)}. \tag{2.10}$$

In view of the properties of the function $R(c)$ established in Lemma 2.3, there exist for ε sufficiently small, two solutions $c^+(\varepsilon)$ and $c^-(\varepsilon)$ of equation (2.10) such that

$$c^+(\varepsilon) \to 1, \quad c^-(\varepsilon) \to \hat{c} \quad \text{as } \varepsilon \to 0.$$

They correspond to two solutions u_ε^+ and u_ε^- of Problem (I) with

$$\varepsilon\,\|u_\varepsilon\|_{L^\infty}^{q-p} \to 1, \quad \varepsilon\,\|u_\varepsilon\|_{L^\infty}^{q-p} \to \hat{c} \quad \text{as } \varepsilon \to 0. \tag{2.11}$$

This completes the proof of Theorem A with the exception of the assertion that $\hat{c} = c^*(p, q, N)$ which will be proved in Section 4.

3. An upper bound for u_ε

In this section we exhibit a surprising property of solutions of Problem (I), in that they can be globally bounded means of a solution of the Yamabe equation

$$-\Delta u = K u^{p_N} \quad \text{in } \mathbf{R}^N$$

for some appropriately chosen constant K.

In [AP1] this property had been observed for solutions of Problem (P) in spherical domains, when the nonlinearity f is subcritical or critical, *i.e.*, when

$$s f'(s) \leqq p_N f(s).$$

Here we shall show that for the functons f_ε defined in (1.10), which are neither critical nor subcritical for all $s > 0$, this property is still true.

Theorem 3.1. *Let u be a solution of Problem* (I) *in which $q > p \geqq p_N$ and let $u(0) = \gamma$. Then*

$$u(x) \leqq W_\gamma(x) \quad \textit{in} \quad B_1,$$

where

$$W_\gamma(x) = \left(\frac{\gamma^{2/(N-2)}}{1 + \alpha_\varepsilon \gamma^{p-1} |x|^2}\right)^{(N-2)/2} \tag{3.1}$$

and

$$\alpha_\varepsilon = \frac{1}{N(N-2)}(1 - \varepsilon\gamma^{q-p}). \tag{3.2}$$

Remark. The function W_γ is the solution of the problem

$$-\Delta W = (1 - \varepsilon\gamma^{q-p})\, \gamma^{p-p_N} W^{p_N} \quad \text{in } \mathbf{R}^N \tag{3.3}$$

$$W(0) = \gamma, \quad 0 < W \leqq \gamma \quad \text{in } \mathbf{R}^N. \tag{3.4}$$

In what follows we shall often work with the function $v(y) = \gamma^{-1} u(x)$ introduced in Section 2. If u is a solution of Problem (I), then v is a solution of the problem

(II)

$$-\Delta v = v^p - \varepsilon\gamma^{q-p} v^q \quad \text{in } B_\varrho \tag{3.5}$$

$$v > 0 \quad \text{in } B_\varrho \tag{3.6}$$

$$v(0) = 1, \quad v = 0 \quad \text{on } \partial B_\varrho, \tag{3.7}$$

where $\varrho = \gamma^{(p-1)/2}$. For v, Theorem 3.1 states that

$$v(y) \leqq W_1(y),$$

where α_ε is still defined by (3.2).

The proof of Theorem 3.1 proceeds along the lines of that of the corresponding result for critical or subcritical nonlinearities [AP1]. Thus, we first introduce the new variables

$$t = \left(\frac{N-2}{|y|}\right)^{N-2}, \quad y(t) = v(y). \tag{3.8}$$

Then Problem (II) becomes

$$y'' + t^{-k} f(y) = 0, \qquad T < t < \infty \tag{3.9}$$

(III)
$$y > 0 \qquad T < t < \infty \tag{3.10}$$

$$y(T) = 0, \quad \lim_{t \to \infty} y(t) = 1, \tag{3.11}$$

where $k = (2N-2)/(N-2)$, $T = \{(N-2)\gamma^{-(p-1)/2}\}^{N-2}$ and

$$f(y) = y^p - \varepsilon\gamma^{q-p} y^q. \tag{3.12}$$

Note that in this context the critical power is given by $p_N = 2k - 3$.

In the proof the functional

$$H(t) = t(y')^2 - yy' + \frac{1}{k-1} t^{1-k} y(f(y) \tag{3.13}$$

plays a central rôle.

Lemma 3.2. *Let $y(t)$ be the solution of Problem* (III), *in which $q > p \geqq p_N$. Then*

$$H(t) > 0 \quad \textit{for } T < t < \infty.$$

Proof. As a first observation we note that

$$H(T) = 0. \tag{3.14}$$

and we deduce from (3.10) that $y'(t) = O(t^{1-k})$ as $t \to \infty$ and so, because $k > 2$, that

$$H(t) \to 0 \quad \text{as } t \to \infty. \tag{3.15}$$

Differentiating H and using the equation, we obtain

$$H'(t) = \frac{1}{k-1} t^{1-k} y' \{(p - 2k + 3)\, y^p - \varepsilon\gamma^{q-p}(q - 2k + 3)\, y^q\}. \tag{3.16}$$

Because $y' > 0$, it is clear that if $p = p_N = 2k - 3$, then $H'(t) < 0$ and it follows from (3.15) alone that $H(t) > 0$ on $[T, \infty)$.

To deal with the case $p > p_N$, we inspect $H'(t)$ more closely, writing it as

$$H'(t) = \frac{q - 2k + 3}{k - 1} t^{1-k} y^p y' Q(t),$$

where

$$Q(t) = \frac{p - 2k + 3}{q - 2k + 3} - \varepsilon\gamma^{q-p} y^{q-p}(t). \tag{3.17}$$

Plainly, the sign of $H'(t)$ is determined by the sign of $Q(t)$. At $t = T$ we have

$$Q(T) = \frac{p - 2k + 3}{q - 2k + 3} > 0$$

and as $t \to \infty$,

$$\lim_{t\to\infty} Q(t) = \frac{p-2k+3}{q-2k+3} - \varepsilon\gamma^{q-p}$$
$$< \frac{(N-2)p-(N+2)}{(N-2)q-(N+2)} - \frac{c(p,N)}{c(q,N)}$$
$$= -\frac{q-p}{p+1}\frac{(N-2)p-(N+2)}{(N-2)q-(N+2)}$$
$$< 0,$$

where we have used Theorem 2.4. Thus, in view of (3.14), $H(t)$ starts positive near $t = T$, and, in view of (3.15), decreases to zero as $t \to \infty$. Since $y(t)$ is strictly increasing it follows from (3.17) that $H'(t)$ can only *once* change sign on (T, ∞), and so $H(t) > 0$ on (T, ∞). This completes the proof.

Lemma 3.3. *Let $y(t)$ be the solution of Problem* (III). *Then*

$$y(t) < z(t),$$

where

$$z(t) = \left(1 + \frac{1-\varepsilon\gamma^{q-p}}{k-1} t^{2-k}\right)^{-1/(k-2)}. \tag{3.18}$$

Remark. The function $z(t)$ is the solution of the problem

$$z'' + (1-\varepsilon\gamma^{q-p})\, t^{-k} z^{p_N} = 0 \quad \text{for } 0 < t < \infty,$$
$$\lim_{t\to\infty} z(t) = 1.$$

Proof. As in [AP1] we observe that

$$(t^{k-1}y^{1-k}y')' = -(k-1)\, t^{k-2}y^{-k} H(t) < 0 \quad \text{on } (T, \infty).$$

Integrating this inequality from $t > T$ to $t = \infty$, we obtain

$$y^{1-k}(t)\, y'(t) > \frac{1}{k-1}(1-\varepsilon\gamma^{q-p})\, t^{1-k} \quad \text{for } T < t < \infty.$$

Carrying out another integration from $t > T$ to $t = \infty$ finally yields

$$y^{2-k}(t) > 1 + \frac{1}{k-1}(1-\varepsilon\gamma^{q-p})\, t^{2-k} \quad \text{for } T \leqq t < \infty, \tag{3.19}$$

which proves the lemma.

Returning to the original variables y and u we find that $z(t) = W_1(y)$, and thus that Lemmas 3.2 and 3.3 together prove Theorem 3.1.

The upper bound for u_ε supplied by Theorem 3.1 is not uniform in ε because of the factor α_ε in the denominator of W_γ. In particular we have

(a) If u_ε is a *large solution* of Problem (I), then $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$ by (1.14) and so

$$\lim_{\varepsilon\to 0} \gamma^{-1} W_\gamma(x) = 1$$

uniformly in sets $\{x \in \mathbf{R}^N : \gamma^{p-1} |x|^2 \text{ bounded}\}$.

(b) If u_ε is a *small solution* of Problem (I), then

$$\limsup_{\varepsilon\to 0} \gamma^{-1} W_\gamma(x) \leqq \left(\frac{1}{1+\alpha_0 \gamma^{p-1} |x|^2}\right)^{(N-2)/2},$$

where

$$\alpha_0 = \frac{1}{N(N-2)}(1-c^*)$$

uniformly on sets $\{x \in \mathbf{R}^N : \gamma^{p-1} |x|^2 \text{ bounded}\}$.

Thus for small solutions we have the following uniform upper bound.

Theorem 3.4. *Suppose* $\{u_\varepsilon\}$ *is a family of small solutions. Then there are numbers* $\varepsilon_0 > 0$ *and* $\nu > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$,

$$u_\varepsilon(x) \leqq \left(\frac{\gamma^{2/(N-2)}}{1+\nu\gamma^{p-1}|x|^2}\right)^{(N-2)/2} \quad \text{in } B_1.$$

4. Asymptotic behaviour of u_ε

In this Section we finally turn to a description of the small solution u_ε defined in Theorem 2.4 as $\varepsilon \to 0$. Thus, we assume that along a subsequence

$$\varepsilon\gamma^{q-p} \to c \quad \text{as } \varepsilon \to 0, \tag{4.1}$$

where c is some constant which satisfies $0 \leqq c < 1$.

As in [AP2] and [BP] the Pohozaev Identity (1.25) plays a central rôle here. For solutions of Problem (I) it becomes

$$-c(p,N)\int_{B_1} u_\varepsilon^{p+1} + \varepsilon c(q,N)\int_{B_1} u_\varepsilon^{q+1} = \tfrac{1}{2}\int_{\partial B_1} (x,n)\left(\frac{\partial u_\varepsilon}{\partial n}\right)^2, \tag{4.2}$$

where

$$c(k,N) = \frac{(N-2)k-(N+2)}{2(k+1)}. \tag{4.3}$$

Some basic elliptic estimates will also be needed. They are supplied by the following lemma which we take from [BP].

Lemma 4.1. *Suppose* u *is the solution of the problem*

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. Then there is a constant $C > 0$, which depends only on Ω, such that

$$\|u\|_{W^{1,s}(\Omega)} + \|\nabla u\|_{C^{0,\alpha}(\partial\Omega)} \leqq C(\|f\|_{L^1(\Omega)} + \|f\|_{L^\infty(\omega)}) \tag{4.4}$$

for any $s < N/(N-1)$, any $\alpha \in (0, 1)$ and any neighbourhood ω of $\partial\Omega$.

As a consequence of the compact embedding of $W_0^{1,s}(\Omega)$ in $L^m(\Omega)$ when $s < N$ and $m < sN/(N-s)$, Lemma 4.1 has the following corollary.

Corollary 4.2. *In the notation of Lemma 4.1 we have*

(a) $$\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\partial\Omega)} \leqq C(\|f\|_{L^1(\Omega)} + \|f\|_{L^\infty(\omega)}).$$

(b) *If $\{f_n\}$ is a bounded sequence in $L^1(\Omega)$ and in $L^\infty(\omega)$, then the corresponding sequence of solutions $\{u_n\}$ has compact closure in $L^2(\Omega)$, whilst the sequence $\{\nabla u_n\}$, restricted to $\partial\Omega$, has compact closure in $L^2(\partial\Omega)$.*

As before, we introduce the rescaled variables y and v_ε defined by (2.1) to determine the behaviour of small solutions u_ε near the origin.

Lemma 4.3. *We have*

$$\varepsilon\gamma^{q-p} \to c^*(p, q, N) \quad \text{as } \varepsilon \to 0$$

and

$$v_\varepsilon(y) \to V(y) \quad \text{as } \varepsilon \to 0$$

uniformly on $\mathbf{R}^N$, where the pair (c^*, V) is the unique radially symmetric solution of the problem

$$\text{(IV)} \qquad \begin{aligned} -\Delta V &= V^p - c^* V^q && \text{in } \mathbf{R}^N, \\ V(0) &= 1, \quad 0 < V \leqq 1 && \text{in } \mathbf{R}^N, \\ V(y) &= O(|y|^{-(N-2)}) && \text{as } |y| \to \infty. \end{aligned}$$

Proof. Because the family $\{v_\varepsilon\}$ is uniformly bounded in $\mathbf{R}^N$, it follows from elliptic regularity theory that there exists a sequence, also denoted by $\{v_\varepsilon\}$, which converges uniformly on compact sets to some radial function V. Since the functions v_ε are solutions of Problem (II) and $\varepsilon\gamma^{q-p} \to c$ as $\varepsilon \to 0$ according to (4.1) it follows that V satisfies

$$-\Delta V = V^p - cV^q \quad \text{in } \mathbf{R}^N \tag{4.5}$$

$$V(0) = 1, \quad 0 \leqq V \leqq 1 \quad \text{in } \mathbf{R}^N. \tag{4.6}$$

By Theorem 3.4 there is a constant $K > 0$ which does not depend on ε such that

$$v_\varepsilon(y) \leqq K|y|^{-(N-2)} \quad \text{in } \mathbf{R}^N \tag{4.7}$$

for ε small enough. This implies that the convergence of v_ε to V is actually uniform in the whole for $\mathbf{R}^N$ and that

$$V(y) \leqq K|y|^{-(N-2)} \quad \text{in } \mathbf{R}^N. \tag{4.8}$$

Thus V is a solution of (4.5), (4.6) and (4.8), which means that $c = c^*(p, q, N)$ [KMPT] and hence that (c^*, V) is the solution of Problem (IV).

Finally, we note that by the uniqueness of the solution of Problem (IV) the entire family $\{v_\varepsilon\}$ converges to V as $\varepsilon \to 0$ and that $\varepsilon\gamma^{q-p}$ converges to c^*.

For future reference we note the following limit.

Lemma 4.4. *Suppose* $m > N/(N-2)$. *Then*

$$\lim_{\varepsilon\to 0} \gamma^{-m+N(p-1)/2} \int_{B_1} u_\varepsilon^m(x)\,dx = \int_{\mathbf{R}^N} V^m(y)\,dy.$$

Proof. Transforming to the variables y and v_ε, we obtain

$$\int_{B_1} u_\varepsilon^m(x)\,dx = \gamma^{m-N(p-1)/2} \int_{B_\varrho} v_\varepsilon^m(y)\,dy,$$

where $\varrho = \gamma^{(p-1)/2}$. For ε sufficiently small, it follows from (4.7) that

$$v_\varepsilon(y) \leqq \hat{v}(y) = \min\{1, K|y|^{-(N-2)}\}.$$

Since $\hat{v} \in L^m(\mathbf{R}^N)$ if $m > N/(N-2)$ it follows from Lemma 4.3 and the dominated convergence theorem that

$$\int_{B_\varrho} v_\varepsilon^m(y)\,dy \to \int_{\mathbf{R}^N} V^m(y)\,dy,$$

from which the assertion follows.

In what follows we shall write

$$J_m = \int_{\mathbf{R}^N} V^m(y)\,dy. \tag{4.9}$$

The limiting behaviour of the left-hand side of the Pohozaev Identity (4.2) now readily follows from Lemma 4.4:

$$\lim_{\varepsilon\to 0} \gamma^\beta \left(-c(p, N) \int_{B_1} u_\varepsilon^{p+1} + \varepsilon c(q, N) \int_{B_1} u_\varepsilon^{q+1}\right) = -c(p, N)\, J_{p+1} + c^* c(q, N)\, J_{q+1}, \tag{4.10}$$

where

$$\beta = \tfrac{1}{2}\{(N-2)\,p - (N+2)\}. \tag{4.11}$$

To determine the behaviour of u_ε away from the origin and to estimate the right-hand side of (4.2) we define, following [BP], the function

$$w_\varepsilon(x) = \gamma^{\beta+1}\, u_\varepsilon(x). \tag{4.12}$$

By (1.9) w_ε is as solution of the problem

$$\begin{aligned} -\Delta w_\varepsilon &= h_\varepsilon(x) \quad \text{in } B_1 \\ w_\varepsilon &= 0 \qquad\quad \text{on } \partial B_1, \end{aligned}$$

where

$$h_\varepsilon(x) = \gamma^{\beta+1}\{u_\varepsilon^p(x) - \varepsilon u_\varepsilon^q(x)\}. \tag{4.13}$$

According to Theorem 3.1 and (4.7) we have for $x \neq 0$,

$$h_\varepsilon(x) \leqq \gamma^{\beta+1} W_\gamma^p(x) \leqq K^p \gamma^{-(\beta+1)(p-1)} |x|^{-p(N-2)}$$

and so, if $x \neq 0$, then

$$h_\varepsilon(x) \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.14}$$

On the other hand,

$$\int_{B_1} h_\varepsilon(x)\,dx = \gamma^{\beta+1} \int_{B_1} u_\varepsilon^p(x)\,dx - \varepsilon\gamma^{\beta+1} \int_{B_1} u_\varepsilon^q(x)\,dx$$

and β has been chosen so that

$$\beta + 1 = -p + \frac{N}{2}(p-1).$$

Hence, by Lemma 4.4 and (4.1)

$$\lim_{\varepsilon\to 0} \int_{B_1} h_\varepsilon(x)\,dx = J_p - c^* J_q. \tag{4.15}$$

We note that

$$\begin{aligned} J_p - c^* J_q &= \int_{\mathbf{R}^N} (V^p - c^* V^q) \\ &> (1 - c^*) \int_{\mathbf{R}^N} V^p \\ &> 0. \end{aligned}$$

From (4.14) and (4.15) we conclude that

$$h_\varepsilon \to \mu\,\delta_0 \quad \text{as } \varepsilon \to 0,$$

where δ_0 is the Dirac mass centered at the origin and

$$\mu = J_p - c^* J_q. \tag{4.16}$$

This implies, according to Corollary 4.2, that

$$w_\varepsilon \to \mu G_0 \quad \text{as } \varepsilon \to 0 \tag{4.17}$$

in $L^2(B_1)$, as well as in $L^\infty(\omega)$, where ω is any compact subset of B_1 which does not contain the origin. Here $G_0 = G(\cdot, 0)$, where G is the Green's function of $-\Delta$ with zero Dirichlet boundary conditions in B_1, defined in the Introduction. It is given by

$$G_0(x) = \frac{1}{(N-2)\,\sigma_N}\left(\frac{1}{|x|^{N-2}} - 1\right). \tag{4.18}$$

In addition we conclude from Corollary 4.2 that on the boundary ∂B_1

$$\nabla w_\varepsilon \to \mu\,\nabla G_0 \quad \text{as } \varepsilon \to 0 \text{ in } L^2(\partial B_1). \tag{4.19}$$

This yields for the right-hand side of (4.2)

$$\gamma^{2(\beta+1)} \int_{\partial B_1} (x, n)\left(\frac{\partial u_\varepsilon}{\partial n}\right)^2 \to \mu^2 \int_{\partial B_1} (x, n)\left(\frac{\partial G_0}{\partial n}\right)^2 \quad \text{as } \varepsilon \to 0. \tag{4.20}$$

To simplify the right-hand side of (4.2), we recall a result about the Green's function from [BP, Theorem 4.3].

Lemma 4.5. *Let $G(x, y)$ be the Green's function defined by* (1.4)–(1.5). *Then for every* $y \in \Omega$,

$$\int_{\partial\Omega} (x - y, n) \left(\frac{\partial G}{\partial n}(x, y) \right)^2 dx = -(N-2)\, g(y, y),$$

where $n = n(x)$ *denotes the outward normal to* $\partial\Omega$ *at* x.

Thus, setting $g_0 = g(\cdot, 0)$, we can write (4.20) as

$$\gamma^{2(\beta+1)} \int_{\partial B_1} (x, n) \left(\frac{\partial u_\varepsilon}{\partial n} \right)^2 \quad \to \quad -\mu^2 (N-2)\, g_0(0) = \frac{\mu^2}{\sigma_N}, \tag{4.21}$$

where we have used the explicit expression for G_0 given in (4.18).

We now equate the estimates (4.10) and (4.21) for respectively the left-hand and the right-hand side of the Pohozaev Identity (4.2). To begin with this yields the relation

$$c^* c(q, N)\, J_{q+1} = c(p, N)\, J_{p+1}. \tag{4.22}$$

and thus proves Proposition B.

We distinguish two cases

$$\text{I: } p > p_N, \qquad \text{II: } p = p_N.$$

Case I. If $p > p_N$, then $c(p, N) > 0$ and we deduce from (4.22) that $c^* > 0$. Thus we conclude from (4.1), (4.12), (4.17) and (4.22) the following limiting behaviour of small solutions of Problem (I).

Theorem 4.6. *Let u_ε be a small solution of Problem* (I) *in which $p > p_N$ such that* (4.1) *is satisfied. Then*

(a) $$\|u_\varepsilon\|_{L^\infty} \asymp c^* \varepsilon^{-1/(q-p)} \quad \text{as } \varepsilon \to 0;$$

(b) $$\varepsilon^{-\theta} u_\varepsilon(x) \to (c^*)^{-\theta} (J_p - c^* J_q)\, G_0(x) \quad \text{as } \varepsilon \to 0,$$

where

$$\theta = \frac{(N-2)p - N}{2(q-p)}, \qquad c^* = \frac{c(p, N)}{c(q, N)} \cdot \frac{J_{p+1}}{J_{q+1}}$$

and G_0 is the Green's function given by (4.18).

Case II. If $p = p_N$, then $c(p, N) = 0$. Therefore, according to (4.22), $c^* = 0$ and so, by (4.1),

$$\|u_\varepsilon\|_{L^\infty} = o(\varepsilon^{-1/(q-p)}) \quad \text{as } \varepsilon \to 0.$$

To establish the precise behaviour of $\|u_\varepsilon\|_{L^\infty}$ as $\varepsilon \to 0$ we return to the Pohozaev Identity (4.2). Note that $\beta = 0$ in this case, and so we multiply (4.2) by γ^2 and let ε tend to zero. Using Lemma 4.4 in the left-hand side and (4.21) in the right-hand side, we obtain

$$\varepsilon\gamma^{2+q+1-N(p-1)/2} \to \frac{1}{c(q, N)\, J_{q+1}} \cdot \frac{\mu^2}{2\sigma_N}. \tag{4.23}$$

Because $p = p_N$ we have

$$2 + q + 1 - \tfrac{1}{2} N\,(p-1) = q - p + 2$$

and, since $c^* = 0$ in this case, $\mu = J_p$, whence (4.23) can be written as

$$\varepsilon\gamma^{q-p+2} \to \frac{1}{2\sigma_N\, c(q, N)} \cdot \frac{J_p^2}{J_{q+1}}. \tag{4.24}$$

However, when $c^* = 0$, V is given by (1.9) and J_p and J_{q+1} can be computed explicitly. We have

$$J_m = \int_{\mathbf{R}^N} \left(1 + \frac{|y|^2}{N(N-2)}\right)^{-m(N-2)/2} dy,$$

which we can write with $|y|^2 = N(N-2)\,t$ as

$$J_m = \tfrac{1}{2}\{N(N-2)\}^{N/2}\, \sigma_N \int_0^\infty t^{(N-2)/2}(1+t)^{-m(N-2)/2}\, dt$$

$$= \tfrac{1}{2}\{N(N-2)\}^{N/2}\, \sigma_N B\left(\frac{N}{2}, m\frac{N-2}{2} - \frac{N}{2}\right),$$

where

$$B(a, b) = \int_0^\infty t^{a-1}(1+t)^{-a-b}\, dt$$

is the beta function [AS]. Recall that

$$B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}.$$

In particular, for J_p and J_{q+1} we obtain

$$J_p = \frac{1}{N}\{N(N-2)\}^{N/2}\, \sigma_N,$$

$$J_{q+1} = \tfrac{1}{2}\{N(N-2)\}^{N/2}\, \sigma_N B\left(\frac{N}{2}, q\frac{N-2}{2} - 1\right).$$

Using these expressions in (4.24) and (4.17) we can formulate the asymptotic behaviour of small solutions of Problem (I) in the critical case.

Theorem 4.7. *Let* u_ε *be a small solution of Problem* (I) *in which* $p = p_N$ *such that* (4.1) *holds. Then*

(a) $$\|u_\varepsilon\|_{L^\infty} \asymp A(q, N)\, \varepsilon^{-1/(q-p+2)} \quad \textit{as } \varepsilon \to 0,$$

(b) $$\varepsilon^{-1/(q-p+2)}\, u_\varepsilon(x) \to \frac{\{N(N-2)\}^{N/2}\, \sigma_N}{NA(q, N)}\, G_0(x) \quad \textit{as } \varepsilon \to 0,$$

where

$$A(q, N) = \left\{ \frac{N^2 c(q, N)}{\{N(N-2)\}^{N/2}}\, B\left(\frac{N}{2}, q\frac{N-2}{2} - 1\right)\right\}^{-1/(q-p+2)}$$

and G_0 *is the Green's function given by* (4.18).

Remark. Comparing the critical and the supercritical case we find that

$$\varepsilon\, \|u_\varepsilon\|^{q-p} \asymp A^{q-p}(q, N)\, \varepsilon^{2/(q-p+2)} \quad \text{as } \varepsilon \to 0.$$

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