Computational Optimization and Applications, 5, 39–48 (1996)

A New Algorithm for Solving the General Quadratic Programming Problem

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Received January 18, 1994; Revised November 4, 1994

Abstract. For the general quadratic programming problem (including an equivalent form of the linear complementarity problem) a new solution method of branch and bound type is proposed. The branching procedure uses a well-known simplicial subdivision and the bound estimation is performed by solving certain linear programs.

Keywords: general quadratic programming problem, linear complementarity problem, global optimization, branch and bound algorithm

1. Introduction

We consider the following general quadratic programming problem

$$\min\{f(x) = xQx + qx : Ax \le b,\ x \ge 0\}, \tag{QP}$$

where Q is an arbitrary real $(n \times n)$-matrix, $q \in \mathbb{R}^n$, A is a real $(m \times n)$-matrix and $b \in \mathbb{R}^m$. We assume that the feasible set $D = \{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}$ of Problem (QP) is bounded.

Quadratic programming has many diverse applications and includes as special cases the equivalent formulations of many important and well studied combinatorial optimization problems, e.g., the linear zero-one programming problem, the assignment problem, the maximum clique problem, the linear complementarity problem, etc. (cf., e.g. [9, 6, 1, 4, 7]).

When Q is positive semi-definite Problem (QP) can be solved by many efficient algorithms which are discussed in almost all classical books on nonlinear mathematical programming. If Q is negative semi-definite, then the function f is concave, and hence Problem (QP) can be solved by each algorithm developed for concave minimization programming problems (cf. e.g. [6, 1, 7]). Algorithms for the case where the matrix Q is indefinite are presented e.g., in Pardalos and Rosen [9], Muu and Oettli [8], Bomze and Danninger [2], Floudas and Visweswaran [4] and references given there.

One of the most successful approach for handling global optimization problems is the well known branch and bound scheme. Algorithms of the branch and bound type consist of two basic operations: branching and bounding. Every new procedure for branching and/or bounding operation leads to a new branch and bound algorithm. The purpose of this article is to propose a new branch and bound algorithm for solving the general problem (QP) in which the branching procedure uses the well-known radial simplex subdivision, and the lower bounding procedure is performed by solving certain ordinary linear programs.

In the next section we describe the algorithm formally. A detailed implementation of this algorithm is given in Section 3. Section 4 contains some illustrative examples and preliminary computational experiments.

2. The algorithm

The basic scheme of our approach is a standard simplicial branch and bound algorithm of the following form.

Initialization:

Construct an n-simplex $S \subset \mathbb{R}^n$ containing the feasible set D of (QP);
Compute a lower bound $\mu(S)$ for $f(x)$ in $S \cap D$;
Determine a finite set $F(S) \subset S \cap D$;
Set $F \leftarrow F(S)$; $\gamma \leftarrow \min\{f(x) : x \in F\}$;
Choose a point $u \in F$ such that $f(u) = \gamma$;
Set $\mu \leftarrow \mu(S)$; $\mathcal{S} \leftarrow \{S\}$; stop $\leftarrow$ false; $k \leftarrow 1$.

while stop = false **do**
  If $\gamma = \mu$ **then**
    stop $\leftarrow$ true (u is optimal solution and γ is optimal value of (QP)).
  else
    Divide S in r n-subsimplices $S_1, \ldots, S_r$ satisfying
    $\bigcup_{i=1}^{r} S_i = S$, int $S_i \cap$ int $S_j = \emptyset$ for $i \neq j$;
    For each $i = 1, \ldots, r$ compute a lower bound $\mu(S_i)$ for f in $S_i \cap D$
    satisfying $\mu(S_i) \geq \mu$, and determine a finite set $F(S_i) \subset S_i \cap D$
    (comment: $F(S_i)$ can be empty);
    Set $F \leftarrow F \cup \{F(S_i) : i = 1, \ldots, r\}$;
    $\gamma \leftarrow \min\{f(x) : x \in F\}$; Choose $u \in F$ such that $f(u) = \gamma$;
    Set $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S\} \cup \{S_i : i = 1, \ldots, r,\ \mu(S_i) < \gamma\}$,
    $\mu \leftarrow \min\{\mu(S) : S \in \mathcal{S}\}$; Choose $S \in \mathcal{S}$ such that $\mu(S) = \mu$.
  end if
    $k \leftarrow k + 1$
end while

In order to investigate the convergence of the algorithm let us assign the index k to every quantity dealt with at the beginning of iteration k. If the procedure terminates at some iteration j, then, obviously the point u^j is an optimal solution and γ_j is the optimal value of problem (QP). If the algorithm is infinite, then it generates at least one infinite "decreasing" sequence $\{S^q\}$ of simplices, i.e., $S^{q+1} \subset S^q\ \forall\ q$. For this case, the convergence of the above algorithm is stated below.

Theorem 2.1 *If for any infinite decreasing sequence* $\{S^q\}$, *the condition*

$$\lim_{q \to \infty} (\gamma_q - \mu_q) = 0, \tag{1}$$

is fulfilled, then

$$\mu = \lim_{k \to \infty} \mu_k = \lim_{k \to \infty} f(u^k) = \lim_{k \to \infty} \gamma_k = \gamma, \tag{2}$$

and every accumulation point u^* of the sequence $\{u^k\}$ is an optimal solution of problem (QP).

Proof: This result follows from convergence conditions of a branch and bound scheme discussed, e.g., in Horst and Tuy [6]. For the sake of completeness of the paper, we present a proof, however, since it is quite short.

Let u^* be an accumulation point of $\{u^k\}$. Then it is easy to see that an infinite decreasing subsequence $\{S^q\}$ of $\{S^k\}$ exists such that $\lim_{q\to\infty} u^q = u^*$. From the continuity of the quadratic function f, it follows that $\lim_{q\to\infty} f(u^q) = f(u^*)$. Let μ and γ be the limits of $\{\mu_k\}$ and $\{\gamma_k\}$, respectively. Then μ and γ exist, since $\{\mu_k\}$ is monotonically increasing and bounded by the optimal value f^* of (QP) and $\{\gamma_k\}$ is monotonically decreasing and bounded by f^*. Obviously, we have $\mu \le f^* \le \lim_{k\to\infty} f(u^k) = \gamma$. Therefore condition (1) implies (2), and u^* is an optimal solution to (QP). □

3. Implementation

In order to implement the above branch and bound algorithm we have to perform two basic operations: the simplicial division and the (lower and upper) bound estimation. These basic operations are discussed in this section.

Initial simplex and radial simplex division

An inital simplex S which tightly encloses the feasible polytope D can be constructed in several ways (cf. [6] and references there). For example, if a nondegenerate vertex v^0 of D is known and let $I(v^0) = \{i \, : \, A_i v^0 = b_i\}$, where A_i are columns of A, then one can choose $S = \{x \, : \, A_i x \le b_i, i \in I(v^0), dx \le \gamma\}$, where $d = -\frac{1}{n}\sum_{i\in I(v^0)} A_i$, $\gamma = \max\{dx \, : \, x \in D\}$. An other simple choice is $S = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_j \le \gamma\}$, where $\gamma = \max\{\sum_{i=1}^n x_j : x \in D\}$.

The subdivision of simplices is defined in the following way.

Let $S = [v^1, \ldots, v^{n+1}]$ be a n-simplex in $\mathbb{R}^n$ and $v \in S$. Then v is uniquely represented by

$$v = \sum_{i=1}^{n+1} \lambda_i v^i, \; \lambda_i \ge 0, \; \sum_{i=1}^{n+1} \lambda_i = 1.$$

For each $i \in \{1, \ldots, n+1\}$ such that $\lambda_i > 0$ let us define an n-simplex $S_i = [v^1, \ldots, v^{i-1}, v, v^{i+1}, \ldots, v^{n+1}]$. Then we have $S = \bigcup_{\lambda_i > 0} S_i$ and $S_i \cap S_j = \emptyset$ for $i \ne j$, $\lambda_i, \lambda_j > 0$. This kind of division is called the radial simplex division and is used in many algorithms in global optimization (cf. [6] and references there). A special case where v is chosen in the middle point of an edge of S with the largest length is called simplicial bisection.

Lower bound estimation

For each simplex $S = [v^1, \ldots, v^{n+1}] \subset \mathbb{R}^n_+$ we intend to compute a lower bound $\mu(S)$ for the function f in the set $S \cap D$. This operation is the essentially new part of our algorithm and is based on the following result.

Theorem 3.1 *Let U be the matrix with columns $v^1, \ldots, v^{n+1}$, $e = (1, \ldots, 1) \in \mathbb{R}^{n+1}$, and for each $i \in \{1, \ldots, n+1\}$ let d_i be the optimal value of the following linear program (in the variables $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ and $y = (y_1, \ldots, y_n)$):*

$$\begin{aligned} &\min v^i y \\ s.t.\quad &AU\lambda \le b \\ &U\lambda \ge 0 \\ &QU\lambda - y = 0 \\ &e\lambda = 1 \\ &\lambda \ge 0. \end{aligned} \tag{LP$_i$}$$

Then a lower bound $\mu(S)$ for f on $S \cap D$ can be computed by

$$\begin{aligned} \mu(S) = \min &\sum_{i=1}^{n+1} c_i\lambda_i \\ s.t.\quad &AU\lambda \le b \\ &U\lambda \ge 0 \\ &e\lambda = 1 \\ &\lambda \ge 0, \end{aligned} \tag{LP}$$

where $c_i = d_i + qv^i$ $(i = 1, \ldots, n+1)$. We understand that $\mu(S) = +\infty$ if the feasible set of (LP) *is empty.*

Proof: Let $d : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$d(x) = qx + \min_{\xi, y}\{xy : \xi \in S \cap D,\ Q\xi - y = 0\}. \tag{3}$$

Then d is a concave function whenever $S \cap D \neq \emptyset$, since it is the sum of a linear function and the pointwise minimum of a family of linear functions. Let $\delta(x)$ be the convex envelope of $d(x)$ over the simplex S defined by $\delta(x) = \sum_{i=1}^{n+1} d(v^i)\lambda_i$, where $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ satisfies $U\lambda = x,\ e\lambda = 1,\ \lambda \ge 0$ (cf., e.g. [3, 6]).

Then we have $\delta(x) \le d(x)$ for each $x \in S$, and therefore it follows that

$$\min\{qx + xQx : x \in S \cap D\} = \min\{qx + xy : x \in S \cap D,\ Qx - y = 0\}$$
$$\ge \min_{x \in S\cap D}\left\{qx + \min_{\xi, y}\{xy : \xi \in S \cap D,\ Q\xi - y = 0\}\right\} = \min_{x\in S\cap D} d(x) \ge \min_{x \in S\cap D} \delta(x).$$

Moreover, the constraint $x \in S \cap D$ is equivalent to the constraint $\lambda \in \{\lambda : AU\lambda \le b, U\lambda \ge 0, e\lambda = 1,\ \lambda \ge 0\}$, and the constraint $Q\xi - y = 0$ is equivalent to $QU\lambda - y = 0$. Therefore, it follows that

$$\mu(S) = \min_{x\in S\cap D} \delta(x) = \min\left\{\sum_{i=1}^{n+1} d(v^i)\lambda_i : AU\lambda \le b,\ e\lambda = 1,\ \lambda \ge 0\right\},$$

where, for each $i \in \{1, \ldots, n+1\}$,

$$d(v^i) = qv^i + \min\{v^i y : AU\lambda \le b,\ QU\lambda - y = 0,\ e\lambda = 1,\ \lambda \ge 0\} = qv^i + d_i = c_i.$$

□

Remark 3.1

(a) If $S \subset \mathbb{R}^n_+$, then obviously constraint $U\lambda \geq 0$ in $(LP)_i$ and (LP) is trivially fulfilled, because $v^i \geq 0$ $(i = 1, \dots, n+1)$ and $\lambda \geq 0$.

(b) For computing the values c_i $(i = 1, \dots, n+1)$ we have to solve $n+1$ linear programs $(LP)_i$. Note, however, all these programs have a common feasible set, so that they can be solved efficiently by the simplex algorithm.

The following property ensures that the sequence $\{\mu_k\}$ of lower bounds computed throughout the algorithm is monotonically increasing.

Theorem 3.2 *It holds $\mu(S) \leq \mu(\bar{S})$ for each pair of n-simplices $S, \bar{S}$ satisfying $\bar{S} \subseteq S$.*

Proof: Let $S = [v^i, \dots, v^{n+1}]$ and $\bar{S} = [\bar{v}^1, \dots, \bar{v}^{n+1}]$. If $\bar{S} \cap D = \emptyset$, then the theorem is true, because $\mu(\bar{S}) = +\infty$. So, we assume that $\bar{S} \cap D \neq \emptyset$. Let $d(x),\ \bar{d}(x)$ be the concave functions defined by (3) according to $S, \bar{S}$, respectively, and let $\delta(x),\ \bar{\delta}(x)$ be the convex envelope of $d(x),\ \bar{d}(x)$, respectively. Since $\bar{S} \subseteq S$, it follows that $\bar{d}(x) \geq d(x)$ for all $x \in \bar{S}$, and hence $\bar{\delta}(x) \geq \delta(x)$ for all $x \in \bar{S}$. Thus, $\mu(\bar{S}) = \min\{\bar{\delta}(x) : x \in \bar{S} \cap D\} \geq \min\{\delta(x) : x \in \bar{S} \cap D\} \geq \min\{\delta(x) : x \in S \cap D\} = \mu(S)$. □

Upper bound estimation

At every iteration of the algorithm the upper bound γ of the optimal value is improved by using a set of new feasible points which are generated while computing the lower bounds. For each simplex S, let $(\lambda^i,\ y^i)$ be an optimal solution of problem $(LP)_i$, $(i = 1, \dots, n+1)$. Then, obviously, the points $x^i = U\lambda^i$ $(i = 1, \dots, n+1)$ are feasible. In addition, let λ^* be an optimal solution of problem (LP), then the point $x^* = U\lambda^*$ is feasible, too. So, while computing the lower bound $\mu(S)$ we obtain a set $F(S) = \{x^1, \dots, x^{n+1}, x^*\}$ which is used to update the upper bound for the optimal value of problem (QP).

Convergence of the algorithm

Theorem 3.3 *Assume that the algorithm, implemented as above, is infinite and assume that each infinite decreasing subsequence $\{S^q\}$ of simplices generated throughout the algorithm satisfies $\bigcap_{q=1}^{\infty} S^q = \{s^*\}$, where $s^* \in D$. Then every accumulation point of the sequence $\{u^k\}$ is an optimal solution of problem* (QP).

Proof: In view of Theorem 2.1 we need to show that $\lim_{q\to\infty}(\gamma_q - \mu_q) = 0$ for each decreasing sequence $\{S^q\}$. For each q, let x^{qi} $(i = 1, \dots, n+1)$ and x^{*q} be the feasible points generated by solving $(LP)_i$ $(i = 1, \dots, n+1)$ and (LP) according to the simplex S^q. Since S^q shrinks to s^* as $q \to \infty$ it follows that $x^{qi} \to s^*$ $(i = 1, \dots, n+1)$ and $x^{*q} \to s^*$ as $q \to \infty$.

Thus, letting v^{qi} $(i = 1, \dots, n+1)$ denote the vertices of S^q, we have $v^{qi} \to v^{*i} = s^*$ $(i = 1, \dots, n+1)$, and hence $\lim_{q\to\infty} d(v^{qi}) = d(v^{*i}) = f(s^*)$ $(i = 1, \dots, n+1)$. Therefore, $\lim_{q\to\infty}\mu_q = \lim_{q\to\infty}\mu(S^q) = \sum_{i=1}^{n+1} d(v^{*i}\lambda_i = f(s^*)\sum_{i=1}^{n+1}\lambda_i = f(s^*)$, and hence $\lim_{q\to\infty}(\gamma_q - \mu_q) \leq \lim_{q\to\infty}\gamma_q - f(s^*) \leq f(s^*) - f(s^*) = 0$. This implies that $\lim_{q\to\infty}(\gamma_q - \mu_q) = 0$, since $\gamma_q - \mu_q \geq 0\ \forall\, q$. □

Remark 3.2

(a) A decreasing subsequence $\{S^q\}$ satisfying $\bigcap_{q=1}^{\infty} S^q = \{s^*\}$ is usually called exhaustive. A simplex division procedure is called exhaustive if each decreasing subsequence of simplices generated throughout the algorithm is exhaustive. An exhaustive sequence $\{S^q\}$ can be constructed if, e.g., for each q the simplex S^q is generated from S^{q-1} by a simplicial bisection. The exhaustiveness is needed for proving the convergence of the algorithm. However, in the implementation the following (heuristic) subdivision procedure yields better results (cf. also [7] and references given there).
Let $(\lambda_1^*, \ldots, \lambda_{n+1}^*)$ *be an optimal value of problem* (LP) *according to a simplex S. If* $\min\{\lambda_i^* : \lambda_i^* > 0\} \geq \sigma$ *(where* σ *is a chosen positive number, e.g.,* $\sigma = 1/2n^2$*), then perform a radial simplex division of S using the point* $x^* = U\lambda^*$*. Otherwise, perform a simplicial bisection.*

(b) If in the algorithm, the stopping criterion $\gamma = \mu$ is replaced by $\gamma - \mu \leq \varepsilon$, where $\varepsilon > 0$ is a given tolerance, then whenever the algorithm terminates, the point u is an approximate optimal solution of Problem (QP) in the sense that $f(u) \leq f(x) - \varepsilon$ for every feasible point x. Usually, this kind of approximate solutions is called ε-optimal solution. From Theorem 3.3 we obtain immediately the following result.

Corollary *Assume that throughout the algorithm an exhaustive simplex division procedure is used. Then for each given* $\varepsilon > 0$*, whenever the stopping criterion* $\gamma - \mu \leq \varepsilon$ *is used, the algorithm terminates after finitely many iterations yielding an* ε*-optimal solution of Problem* (QP).

(c) If we formulate Problem (QP) in the following equivalent biconcave minimization problem

$$\min\{g(x, y) = xy + qx : Ax \leq b, Qx - y = 0, x \geq 0\}, \tag{4}$$

then the algorithm presented in this article for solving Problem (QP) can be interpreted as a specific realization of the general biconcave minimization algorithm developed in Horst and Thoai [5] for the case of Problem (4).

4. Illustrative examples and preliminary computational experiments

In the first part of this section we present two numerical examples for illustrating our algorithm. Some preliminary computational experiments are reported subsequently.

Example 1 We consider Problem (QP) with following input data:

$m = 7, n = 3,$

$$Q = \begin{pmatrix} 0.992934 & -0.640117 & 0.337286 \\ -0.640117 & -0.814622 & 0.960807 \\ 0.337286 & 0.960807 & 0.500874 \end{pmatrix}, \qquad q = \begin{pmatrix} -0.992372 \\ -0.046466 \\ 0.891766 \end{pmatrix},$$

$$A = \begin{pmatrix} 0.488509 & 0.063565 & 0.945686 \\ -0.578592 & -0.324014 & -0.501754 \\ -0.719203 & 0.099562 & 0.445225 \\ -0.346896 & 0.637939 & -0.257623 \\ -0.202821 & 0.647361 & 0.920135 \\ -0.983091 & -0.886420 & -0.802444 \\ -0.305441 & -0.180123 & -0.515399 \end{pmatrix}, \quad b = \begin{pmatrix} 2.865062 \\ -1.491608 \\ 0.519588 \\ 1.584087 \\ 2.198036 \\ -1.301853 \\ -0.738290 \end{pmatrix}.$$

Iteration 1 A first simplex containing the feasible set is $S^1 = [v^{11}, v^{12}, v^{13}, v^{14}]$ with

$$v^{11} = (0.0, 0.0, 0.0)^T, v^{12} = (20.0, 0.0, 0.0)^T,$$
$$v^{13} = (0.0, 20.0, 0.0)^T, v^{14} = (0.0, 0.0, 20.0)^T.$$

Solving the linear programs $(\mathrm{LP})_i$ $(i = 1, \ldots, 4)$ we obtain

$d_1 = 0.0, c_1 = 0.0,$
$(\lambda^1, y^1) = (0.488071, 0.260534, 0.251395, 0.000000, 3.156646, 0.000000, 5.718817),$
$x^{11} = (5.210677, 5.027908, 0.000000), f(x^{11}) = -31.528496,$

$d_2 = -24.024761, c_2 = -43.839029,$
$(\lambda^2, y^2) = (0.803216, 0.027475, 0.147788, 0.021521, 0.0000005.085268, 2.371314),$
$x^{12} = (0.549500, 2.955757, 0.430420), f(x^{12}) = -5.707612,$

$d_3 = -148.625705, c_3 = -145.409224,$
$(\lambda^3, y^3) = (0.488071, 0.260534, 0.251395, 0.000000, 3.1566460.000000, 5.718817),$
$x^{13} = (5.210677, 5.027908, 0.000000), f(x^{13}) = -31.528496,$

$d_4 = 17.390460, c_4 = 43.450860,$
$(\lambda^4, y^4) = (0.871100, 0.128900, 0.000000, 0.000000, 3.7610165.781067, 0.000000)$
$x^{14} = (2.577995, 0.000000, 0.000000), f(x^{14}) = 4.045042)$

Solving the linear program (LP) we obtain

$\lambda^* = (0.488071, 0.260534, 0.251395, 0.000000),$
$x^{*1} = (5.210677, 5.027908, 0.000000),$
$f(x^{*1}) = -31.528496$ lower bound $\mu(S^1) = -47.976758$

Current best feasible point at Iteration 1: $u^1 = (5.210677, 5.027908, 0.000000)$, current best function value: $\gamma_1 = -31.528496$, lower bound: $\mu_1 = -47.976758$.

Iteration 2 S^1 is divided into three subsimplices S_1^1, S_2^1 and S_3^1 by a radial simplex division using the point (5.210677, 5.027908, 0.000000).

Computing lower bounds we have

$$\mu(S_1^1) = -31.528496, \quad \mu(S_2^1) = -31.528496, \quad \mu(S_3^1) = -31.528496.$$

Algorithm terminates yielding an optimal solution (5.210677, 5.027908, 0.000000) with the optimal value −31.528496.

Example 2 We consider the quadratic problem formulation of the linear complementarity problem which seeks a solution of the system

$$x \geq 0, Mx + p \geq 0, x(Mx + p) = 0, \tag{5}$$

where M is a real $(n \times n)$-matrix and $p \in \mathbb{R}^n$.

Obviously, Problem (5) has a solution in the simplex $S^0 := \{x : x \geq 0, ex \leq L\}$, where $e = (1, \ldots, 1) \in \mathbb{R}^n$ and L is a positive number, if and only if the optimal value of the program

$$\min\{f(x) = xMx + px : x \geq 0, Mx + p \geq 0, x \in S^0\} \tag{6}$$

does not exceed zero.

Thus, to find a solution of Problem (5) in the simplex S^0, we can apply our algorithm with the starting simplex S^0 for solving the quadratic problem (QP) with $Q = M, A = -M$ and $q = b = p$. Notice, in addition, that the algorithm will terminate immediately whenver a point u is found satisfying $f(u) = 0$, or the lower bound becomes positive. In the first case, u is a solution of the underlying complementarity problem; in the second case, it is indicated that the complementarity problem has no solution in the simplex S^0.

Input data:

$$n = 5,$$

$$M = \begin{pmatrix} 0.022983 & 0.872869 & -0.891371 & 0.578592 & 0.324014 \\ 0.501754 & 0.719203 & -0.099562 & -0.445225 & 0.346896 \\ -0.637939 & 0.257623 & 0.202821 & -0.647361 & -0.920135 \\ 0.983091 & 0.886420 & 0.802444 & 0.305441 & 0.180123 \\ 0.515399 & 0.424820 & -0.897498 & -0.187268 & 0.591515 \end{pmatrix},$$

$$p = \begin{pmatrix} -0.073726 \\ 0.347034 \\ 2.007665 \\ -2.723395 \\ -0.581514 \end{pmatrix}$$

Iteration 1 Choose $L = 100$, and set $S^1 = S^0$; Solving linear programs $(\text{LP})_i (i = 1, \ldots, 5)$ and (LP) we obtain $\mu_1 = 0.0$, $\lambda^* = (0.966423, 0.021762, 0.0, 0.004489, 0.007326, 0.0)$, current best feasible point $u^1 = (2.176197, 0.0, 0.448921, 0.732582, 0.0)$ with $\gamma_1 = 0.106030$.

Iteration 2 S^1 is divided into two simplices S_1^1 and S_2^1 by a simplicial bisection.

Computing lower bounds we obtain $\mu(S_1^1) = 0.0$, $\mu(S_2^1) = 0.552728$, current best feasible point $u^2 = (1.963712, 0.0, 0.607922, 0.744995, 0.430322)$ with $\gamma_2 = 0.0$. Algorithm terminates yielding u^2 as an optimal solution of the resulting quadratic problem.

It is worth noting that actually, while computing $\mu(S_1^1)$ the point u^2 was found with $f(u^2) = 0.0$, and therefore, the algorithm could immediately terminate yielding a solution of the underlying linear complementarity problem.

Next, we present some preliminary computational experiments on our algorithm. We consider three types of the square matrix Q in Problem (QP): Type 1 consists of arbitrary

Table 1. Preliminary computational results.

Matrix Q	m	n	ITER	SIM	TIME
Type 1	6	5	3	6	0.12
	10	10	9	27	6.21
	13	10	12	46	13.35
	11	14	112	454	290.87
Type 2	11	14	18	70	60.47
	12	20	67	262	534.36
	11	25	99	365	1207.12
	10	30	160	438	1954.28
Type 3	7	7	25	51	1.75
	8	8	220	657	33.2*
	9	9	55	120	10.98
	10	10	323	993	150.8*
	12	12	111	318	82.27
	15	15	401	1952	609.8*
	18	18	154	472	104.92
	20	20	202	594	194.67

Notation:
ITER: Avarage number of iterations,
SIM: Avarage number of simplices generated throughout the algorithm,
TIME: Avarage CPU-Time in seconds.
Notice that in problems with Q of Type 3 we have $m = n$, and for problems marked by *, it was indicated that no solution of the original linear complementarity problems exists.

real $(n \times n)$-matrices, Type 2 consists of matrices of the form $Q = -R^T R$ with R being a real $(n \times n)$- matrix, and Type 3 consists of matrices resulting from linear complementarity problems as considered in Example 2. Obviously, functions with Q of Type 2 are concave quadratic, so that minima are achieved at some vertices of feasible polytopes. Although in the quadratic problem formulation of linear complementarity problems the objective functions are in general nonconcave, the vertex optimal solution property does also hold whenever the optimal value is equal to zero. This property follows from the fact that whenever the linear complementarity problem (5) has a solution, there is a solution in the vertex set of the corresponding polyhedral set $\{x : x \geq 0, Mx + p \geq 0\}$.

Some computational results on a large set of randomly generated problems with $m \leq 20$ and $n \leq 30$ are given in Table 1. For all test problems, the starting simplex is defined by $\{x : x \geq 0, ex \leq 100.\}$, the simplex subdivision procedure and the way to determine ε-optimal solutions are as in Remark 3.2. For problems with Q of Types 1 and 2, the algorithm terminates if the stopping criterion $\gamma - \mu \leq \gamma/100$ is fulfilled. As a result, we obtain an ε-optimal solution with $\varepsilon = 1\%$ of the optimal value.

Finally, we notice that for solving all linear subproblems generated throughout the algorithm we used a subroutine based on the simplex method, and test problems are run on a Sun SPARC station 10 Model 20 workstation.

Acknowledgment

The authors would like to thank two anonymous reviewers for their constructive suggestions that helped to improve the first version of this article.

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