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WHY TO TEACH MATHEMATICS SO AS TO BE USEFUL

My first task at this moment is to welcome you who have come here from various countries to sacrifice one week of your holidays for the benefit of mathematical education all over the world. I trust this meeting will be as useful as according to the general theme of this conference mathematical education should be held to be. I trust we all will learn as much from each other's experiences and arguments as we like to do and often have done at such opportunities. With great satisfaction I remember the meeting of December 1964 at Utrecht and I hope the few among you who have participated in that conference will share my feelings of gratitude. But whenever I shall remember those pleasant days and evenings, and lively discussions, I will never forget the man whom I met first and last on that occasion, the liveliest among all of us, the much regretted Wittenberg, this fiery nature who died much too early as though he had burnt himself in his own fire. Though I admit there was none among us who shared his opinions, I am sure everybody was impressed by his honest search into the *truth* of our educational philosophy. To my mind, his definitive absence overclouds the bright sky of this day.

The present colloquium is an activity of the ICMI sponsored by the government of the Netherlands and by IMU. It is not the first in this new period of ICMI and in this year. In January we met in Lausanne with the physicists, in a meeting sponsored by UNESCO, which was attended by some among you. In my opinion the resolutions adopted at Lausanne are a mile-stone in the philosophy of mathematical education.* If I substitute my wishes and hopes for my opinion, I would say they should be so. It is evident that the use of mathematics has been a key criterion in all arguments on mathematics at that meeting.

In this introductory address I feel I have to justify the general theme of the present conference rather than to tell you about techniques of teaching useful mathematics. This means that I will not speak about how to teach mathematics so as to be useful but about why we should teach mathematics so as to be useful, or rather about why we should teach mathematics so as to be more useful.

Of course this is a question of educational philosophy, and as such it will

* See pp. 244–245.

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be answered in a different way according to which philosophy we adhere to. Yet educational philosophy is not an abstract system. It depends on the real educational system in which we live, and on our, positive or negative, attitude with respect to that system. Is the variety of national educational philosophies really a drawback to international talks on mathematical education or should I say that there is no better opportunity to test them than to have them bump against each other? Are not we too often and too readily inclined, when reading or hearing about the educational experiences in another country, under another educational system, to sigh: it is just a pity, but this does not apply to our situation? I would say whenever this happens, then something is wrong either in the one system or in the other, or, most likely, in both.

It is generally admitted that there is a wide gap between the educational philosophy of the U.S.A. and the Socialist European countries on one hand and the continental Western European countries on the other hand, though this gap has been narrowing to a considerable extent. On the one side one has for long times pursued the ideal of one kind of education for all youth, on the other side one has always overstressed that part of the educational system which provides educational facilities for a small group of students selected more on social than on intellectual grounds. I have to admit, and I do it with shame and distress, that in the Western countries of continental Europe, if we speak about mathematical education, we more often than not, mean the gymnasiums and lycées, and tacitly forget the about more than 90% who do not attend this type of schools. I agree that a more balanced educational system can be as bad if its highest level is too low to do justice to the most gifted students. But instead of discussing the question which kind of injustice is the least evil, I would rather try to do the most justice to all people and to the society they belong to.

I need not explain to you *why* mathematics can be useful though the fact itself is one of the most recent and most astonishing features of the history of civilization. It would be more difficult to tell *how* mathematics can be useful provided that we do not limit ourselves to counting up instances of the all-pervading influence of mathematics in our culture, but ask what happens in the individual if he applies mathematics or if he tries to. Much has been done to investigate the learning process, though it is a fact that most of this research has been rather laboratory than classroom-oriented. Very little, if anything, is known about how the individual manages to apply what he has learned, though such a knowledge would be the key to understanding why most people never succeed in putting their theoretical knowledge to practical use.

Since mathematics has proved indispensable for the understanding and the technological control not only of the physical world but also of the social structure, we can no longer keep silent about teaching mathematics so as to be useful. In educational philosophies of the past, mathematics often figures as the paragon of a disinterested science. No doubt it still is, but we can no longer afford to stress this point if this keeps our attention off the widespread use of mathematics and the fact that mathematics is needed not by a few people, but virtually by everybody.

Mathematics is distinguished from other teaching subjects by the fact that, even in its actual totality, it is a comparatively small body of knowledge, of such a generality that it applies to a richer variety of situations than any other teaching subject. Modern mathematics can be seen as an effort to reduce this body of knowledge even more and to enhance the flexibility of what remains to be taught. At the same time this fact about mathematics is the source of the principal dilemma in teaching mathematics so as to be useful. In an objective sense the most abstract mathematics is without doubt the most flexible. In an objective sense, but not subjectively, since it is wasted on individuals who are not able to avail themselves of this flexibility. On the other hand, teaching applied mathematics is as bad, if it means mathematics in a specialized context, which does not account for the greatest virtue of mathematics, its flexibility.

Though it might look different, I am still busy with the question *why* mathematics has to be taught so as to be useful, after we had agreed that it *is* useful and that students are expected to use it. There are two extreme attitudes: to teach mathematics with no other relation to its use than the hope that students will be able to apply it whenever they need to. If anything, this hope has proved idle. The huge majority of students are not able to apply their mathematical classroom experiences, neither in the physics or chemistry school laboratory nor in the most trivial situations of daily life. The opposite attitude would be to teach useful mathematics. It has not been tried too often, and you understand that this is not what I mean when speaking about mathematics being taught to be useful. The disadvantage of useful mathematics is that it may prove useful as long as the context does not change, and not a bit longer, and this is just the contrary of what true mathematics should be. Indeed it is the marvellous power of mathematics to eliminate the context, and to put the remainder into a mathematical form in which it can be used time and again.

Between two extreme attitudes one may be inclined to try compromising. If this means teaching pure mathematics and afterwards to show how to apply it, I am afraid we are no better off. I think this is just the wrong order. I have always considered it a remarkbale fact that people are able to apply simple arithmetic, but not quadratic equations or even linear functions. Do not object that arithmetic is so easy. It is not. Take such problems as:

If I have got ten marbles and I give three away, how many are left?

If I have got ten marbles, and John has three less, how many does he have?

If there are ten students in the room and three are girls, how many are boys?

If I am ten years old now, how old was I three years ago?

If B is between A and C, B is at a distance of 7 miles from A, and C is at a distance of 10 miles from A, how far is B from C?

It is not so easy to learn that in all these and a hundred other situations the same arithmetical operation applies. It takes some time, but finally everybody succeeds in understanding it. Why? I daresay, because arithmetic starts in a concrete context and patiently returns to concrete contexts as often as needed. The counterexample is fractions. In its traditional teaching the concrete context is no more than a ceremony which is hurried through in a jiffy. If afterwards the abstract theory of fractions has to be applied, its comes too late, on too high a level, and is not connected to any previous experience on a level where fractions should have been introduced. What is the reason for this change of attitude of the teacher? Is the patience of the schoolmaster exhausted when fractions turn up? I believe the answer is rather that the schoolmaster himself does not know fractions in a concrete context, and that for this reason he is not able to teach them in a more responsible way than he is used to do.

I am afraid this answer applies to the greater part of our mathematics teaching. Even the fact that a teacher applies mathematics himself, does not necessarily imply that he knows how he is able to do so and to use such a knowledge in his teaching.

The problem, however, is still much more serious. In the past, and mostly even now, textbook writing has been dominated by quite other aims than by the goal of a mathematics that could be useful. Mathematics is a peculiar subject. Arithmetic and geometry have sprung from mathematizing part of reality. But soon, at least from the Greek antiquity onwards, mathematics itself has become the object of mathematizing. Arranging and rearranging the subject matter, turning definitions into theorems and theorems into definitions, looking for more general approaches from which all can be derived by specialization, unifying several theories into one – this has been a most fruitful activity of the mathematician, and no doubt our students are entitled to enjoy these fruits. No doubt modern mathematics is both much more flexible and much simpler than the mathematics of fifty years ago. No doubt our students have to learn the most modern mathematics. Teachers are more and more prepared and more and more inclined to bridge the gap between school mathematics and grown-up mathematics which had become wider from year to year.

However, this is not the whole story. The problem is not what kind of mathematics, but how mathematics has to be taught. In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect of mathematics, too. For a few ones this activity extends to mathematizing mathematics itself. The result can be a paper, a treatise, a textbook. A systematic textbook is a thing of beauty, a joy for its author, who knows the secret of its architecture and who has the right to be proud of it. Look how such an author would justify his construction: Why have you defined addition on page 10 in such a circumstantial way? – because this more general definition will prove useful on p. 110. Why have you proved this geometrical theorem in such an unnatural manner? – because at this stage I restrict myself to affine notions which have to precede metric notions. Why do not you mention forces as an instance of vectors? – because mechanics has to be based upon vector algebra and not the other way round.

Systematization is a great virtue of mathematics, and if possible, the student has to learn this virtue, too. But then I mean the activity of systematizing, not its result. Its result is a system, a beautiful closed system, closed, with no entrance and no exit. In its highest perfection it can even be handled by a machine. But for what can be performed by machines, we need no humans. What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics.

New mathematics has been met with criticism. People who apply mathematics often feel uneasy when observing that the mathematics they have been used to apply is replaced by something they judge less suited for applications. It is a fact that biologists, economists, sociologists are better prepared to apply modern mathematics than physicists who carry the burden of a longer tradition. In the universities the gap between the mathematics of mathematicians and that of physicists has become terrifying. It is a habit of physicists to treat any particular subject with that kind of mathematics which prevailed at the time when that subject turned up in the history of physics. For instance, though physicists know eigenvalues of symmetric matrices because Laplace introduced them in a physical context, they still deal with orthogonal matrices with such oddities as Eulerian angles, because Euler was not yet acquainted with eigenvalues.

It would be a disaster if this lag would become permanent, though I hope it will not. Time ago I eavesdropped on a talk between a physics professor and his assistants, criticizing his course and particularly such a subject as Lagrange multipliers: this is not physics, one of them said, this is plain linear algebra.

Probably we will have to wait for the next generation to have physicists reconciled with modern mathematics teaching.

It is a pity that most of the criticism against modern mathematics is made with no knowledge about what modern mathematics really is. It is a pity, because there is ample reason for such criticism as long as mathematicians care so little about how people can use mathematics. We are not entitled to reproach physicists for identifying modern mathematics with a preposterous educational philosophy, since this identification is of our own making. I am convinced that, if we do not succeed in teaching mathematics so as to be useful, users of mathematics will decide that mathematics is too important a teaching matter to be taught by the mathematics teacher. Of course this would be the end of all mathematical education.

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