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L_p-Continuous Extreme Selectors of Multifunctions with Decomposable Values: Existence Theorems

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Abstract. The existence theorems of L_p-continuous selectors that values are extreme points are proved for a class of multivalued maps. Applications to multivalued maps appearing in multivalued differential equations are presented.

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0. Introduction

The results of Antosievicz and Cellina [1] appeared to be fundamental to study an existence problem for continuous selectors of multivalued maps with closed, nonconvex, decomposable values [5, 23–26, 34]. In this paper, we develop the method based on the Baire category theorem to prove the existence of continuous selectors the values of which are extreme points of a multivalued map with values in a Banach space.

The contents of the paper can be represented by the following theorems formulated for simplicity in a special case.

Let T be a metrisable compact with a positive, nonatomic Radon measure μ_0, $(X, \|\cdot\|)$ be a separable Banach space, M be a σ-compact metric space, $L_p(T, X)$, $1 \leqslant p < +\infty$, be the Banach space of eqiuvalence classes of Bochner-integrable functions v: $T \mapsto X$ with the norm

$$\|v\|_p = \left(\int_T \|v(t)\|^p \mathrm{d}\mu_0 \right)^{1/p}$$

THEOREM 0.1. *Let* Γ: $M \mapsto L_p(T, X)$, $1 \leqslant p < \infty$, *be a multifunction the values of which are decomposable, convex, weakly compact subsets of* $L_p(T, X)$.

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If Γ *is continuous in the Hausdorff metric generated by the topology of* $L_p(T, X)$ *and bounded on* M, *then there exists a continuous selector* u: $M \mapsto L_p(T, X)$ *of the multifunction* Γ, *such that*

$$u(\xi) \in \operatorname{ext} \Gamma(\xi), \quad \xi \in M.$$

THEOREM 0.2. *Let* F: $M \mapsto L_p(T, X)$, $1 \leqslant p < \infty$, *be a multifunction the values of which are decomposable, closed, bounded subsets of* $L_p(T, X)$.

If the multifunction

$$\Gamma(\xi) = \overline{\operatorname{co}} F(\xi), \quad \xi \in M,$$

satisfies all the assumptions of Theorem 0.1, *then there exists a continuous selector* u: $M \mapsto L_p(T, X)$ *of the multifunction* F, *such that*

$$u(\xi) \in \operatorname{ext} \overline{\operatorname{co}} F(\xi), \quad \xi \in M.$$

(About the definitions of ext, $\overline{\operatorname{co}}$, etc., see Section 1.)

As is well known, in the general case, the set of extreme points of a closed, convex set is nonconvex and nonclosed. Theorem 0.1 can be considered to be one of the results on the existence of continuous selectors of multifunctions with nonconvex, nonclosed values.

In Theorem 0.2, it is not assumed that the multifunction F possesses any properties similar to continuity or lower semicontinuity, as is usually required [1, 5, 23]. The multifunction $\overline{\operatorname{co}} F$ must have such properties. However, in this case, the multifunction F has a continuous selector.

It should be mentioned that Theorem 0.2 does not follow from Theorem 0.1, since under the assumptions of Theorem 0.2,

$$\operatorname{ext} \overline{\operatorname{co}} F(\xi) \not\subset F(\xi), \quad \xi \in M.$$

The present results supplement the well-known theorems on the existence of continuous selectors for multivalued maps with closed, nonconvex, decomposable values [1, 5, 23, 24, 25, 26, 34].

This paper is organized as follows.

Section 1 contains notations and terminology.

In Section 2, some nonstandard results concerning a continuous partition of the space T and the connections between various convergence types in $L_p(T, X)$, $p \geqslant 1$, are given.

In Section 3, a class of functions for choosing extreme points is introduced, and their properties are established.

In Section 4, we present some new results concerning decomposable sets, in particular, the properties of the Hausdorff metric generated by the topology of $L_p(T, X)$.

Section 5 is devoted to functions for choosing extreme points of decomposable sets, their properties and their connections with the functions introduced in Section 3.

In Section 6, we prove an auxiliary approximate theorem.

The main results are given in Section 7.

In Section 8, the results from Theorems 0.1 and 0.2 are reformulated for multifunctions often appearing in multivalued differential equations.

1. Notations and Definitions

Let

- $(X, \|\cdot\|)$ be a separable Banach space,
- $(X', \|\cdot\|)$ be its topological dual,
- $\sigma - X'$ be a space X' endowed with the weak $\sigma(X', X)$ topology [2],
- R be the numerical line,
- M be a separable metric space,
- T be a metrisable compact with a positive, nonatomic Radon measure μ_0 and a σ-algebra Σ of μ_0-measurable subsets of T,
- $L_p(T, X)$, $1 \leqslant p < +\infty$, be the Banach space of eqiuvalence classes of Bochner-integrable functions v: $T \mapsto X$ with the norm

$$\|v\|_p = \left(\int_T \|v(t)\|^p \mathrm{d}\mu_0\right)^{1/p},$$

- $L'_p(T, X)$ be its topological dual.

Let, for a normed space Y,

- $c(Y)$ be the family of nonempty, closed subsets of Y,
- $cb(Y)$ be the family of nonempty, closed, bounded subsets of Y,
- $cc(Y)$ be the family of nonempty, closed, convex subsets of Y,
- $ccb(Y)$ be the family of nonempty, closed, convex, bounded subsets of Y,
- $cwk(Y)$ be the family of nonempty, convex, weakly compact subsets of Y,
- $ck(Y)$ be the family of nonempty, convex, compact subsets of Y,
- $\chi(A)$ be the characteristic function of some subset A of a given set,
- $d(x, K)$ be the distance of a point $x \in X$ to a subset $K \subset X$,
- $d_p(v, Q)$ be the distance of a point $v \in L_p(T, X)$ to a subset $Q \subset L_p(T, X)$.

If A and B are subsets of X, then $e(A, B) = \sup\{d(a, B); a \in A\}$ is the excess of A over B, and $h(A, B) = \max\{e(A, B), e(B, A)\}$ is the Hausdorff distance between A and B.

Let

- B, $\overline{B}$ be the open and closed unit balls of X, respectively,
- B', $\overline{B}'$ be the open and closed unit balls of X', respectively.

If A and B are subsets of $L_p(T, X)$, then $e_p(A, B) = \sup\{d_p(a, B); a \in A\}$ is the excess of A over B and $h_p(A, B) = \max\{e_p(A, B), e_p(B, A)\}$ is the Hausdorff distance.

Let $\|A\|_p = \sup\{\|v\|_p;\ v \in A\}$, where A is a subset of $L_p(T, X)$, $1 \leqslant p < \infty$, and let B_p, $\overline{B}_p$ be the open and closed unit balls of $L_p(T, X)$, respectively.

A set $A \subset L_p(T, X)$ is said to be decomposable if, for any $u, v \in A$ and $E \in \Sigma$, the element $\chi(E)u + \chi(T \setminus E)v$ belongs to A.

For a set $A \in L_p(T, X)$, we denote by $\operatorname{dec} A$ the decomposable hull of A, i.e. the smallest (with respect to inclusion) decomposable set containing A.

Let

- $\overline{\operatorname{dec}}\, A$ be the closure of $\operatorname{dec} A$ in $L_p(T, X)$,
- $\mathrm{d}cL_p(T, X)$ be the set of nonempty, decomposable, closed subsets of $L_p(T, X)$,
- $\mathrm{d}cbL_p(T, X)$ be the set of nonempty, decomposable, closed, bounded subsets of $L_p(T, X)$,
- $\mathrm{d}ccL_p(T, X)$ be the set of nonempty, decomposable, closed, convex subsets of $L_p(T, X)$,
- $\mathrm{d}ccbL_p(T, X)$ be the set of nonempty, decomposable, closed, convex, bounded subsets of $L_p(T, X)$,
- $\mathrm{d}cwkL_p(T, X)$ be the set of nonempty, decomposable, convex, weakly compact subsets of $L_p(T, X)$.

A multifunction F: $T \mapsto c(X)$ is called measurable if, for any closed subset $U \subset X$, the set $\{t \in T;\ F(t) \cap U \neq \emptyset\}$ is measurable.

If F: $T \mapsto c(X)$ is measurable, then the function $\|F(t)\|$ is measurable [28].

A multifunction F: $T \mapsto cb(X)$ is called p-integrally bounded if there exists a function $\lambda \in L_p(T, R)$ such that $\|F(t)\| \leqslant \lambda(t)$ a.e. on T.

A multifunction F from a topological space Y into a topological space Z is called lower semicontinuous at a point $y_0 \in Y$ if, for any open set $V \subset Z$, $F(y_0) \cap V \neq \emptyset$, there exists a neighbourhood $U(y_0)$ of y_0 such that $F(y) \cap V \neq \emptyset$ for every $y \in U(y_0)$.

A multifunction F from a topological space Y into a topological space Z is called upper semicontinuous at a point $y_0 \in Y$ if, for any open set $V \subset Z$, $F(y_0) \subset V$, there exists a neighbourhood $U(y_0)$ of y_0 such that $F(y) \subset V$ for every $y \in U(y_0)$.

A multifunction F from a topological space Y into a Hausdorff locally convex space Z is called Hausdorff lower semicontinuous at a point $y_0 \in Y$ if, for any open neighbourhood V about zero in Z, there exists a neigbourhood $U(y_0)$ of y_0 such that $F(y_0) \subset F(y) + V$ for every $y \in U(y_0)$.

A multifunction F from a topological space Y into a Hausdorff locally convex space Z is called Hausdorff upper semicontinuous at a point $y_0 \in Y$ if, for any open neighbourhoods V about zero in Z, there exists a neigbourhood $U(y_0)$ of y_0 such that $F(y) \subset F(y_0) + V$ for every $y \in U(y_0)$.

A family K of measurable p-integrally bounded multifunctions $F\colon T \mapsto cb(X)$ is called uniformly p-integrable if, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\int_E \|F(t)\|^p \mathrm{d}\mu_0 < \epsilon$$

for every subset $E \in \Sigma$ with $\mu_0(E) \leqslant \delta$ and for every $F \in K$.

For a set $A \subset Y$, $\operatorname{co} A$ is its convex hull and $\overline{\operatorname{co}}\, A$ is its closed convex hull.

For a set $A \subset ccb(Y)$, $\operatorname{ext} A$ is the set of all extreme points of A.

A set $K \subset M$ is σ-compact if $K = \bigcup_{n=1}^{\infty} K_n$, where K_n, $n \geqslant 1$, are some compact sets.

2. Preliminaries

We recall some results that are applied in the next sections.

Denote by $\mathcal{M}$ the space of numerical measures $\mu\colon \Sigma \mapsto R$ of bounded variation, absolutely continuous with respect to the measure μ_0, with the norm $\|\mu\|_{\mathcal{M}} = |\mu|(T)$, where $|\mu|(T)$ means the total variation of μ on T.

Let $\mu_0(A \triangle B)$ be the pseudometric on Σ. Here $\triangle$ stands for the symmetric difference of sets A and B.

PROPOSITION 2.1 ([26]). *Let $\mathcal{P}_n\colon M \mapsto \mathcal{M}$, $n \geqslant 1$, be Hausdorff upper semicontinuous multivalued mappings with nonempty relatively compact values, $\{V_n\}_1^{\infty}$ be a locally finite open covering of the space M, and $\{e_n\}_1^{\infty}$ be a continuous partition of unity subordinated to $\{V_n\}_1^{\infty}$ with their supports* $\operatorname{supp} e_n \subset V_n$, $n \geqslant 1$.

Then, for any $\epsilon > 0$, there exists a sequence of mappings $\mathcal{B}_n\colon M \mapsto \Sigma$, $n \geqslant 1$, continuous with respect to the pseudometric $\mu_0(A \triangle B)$, possessing the following properties:

(i) *for any $\xi \in M$ the sets $\{\mathcal{B}_n(\xi)\}_1^{\infty}$ are disjoint and $\mu_0(\bigcup_{n=1}^{\infty} \mathcal{B}_n(\xi)) = \mu_0(T)$;*

(ii) *for any $\xi \in M$, $n \geqslant 1$, the set $\mathcal{B}_n(\xi) = \emptyset$ if and only if $e_n(\xi) = 0$;*

(iii) *for any $\xi \in M$, $n \geqslant 1$, $\mu \in \mathcal{P}_n(\xi)$*

$$|\mu(\mathcal{B}_n(\xi)) - e_n(\xi)\mu(T)| < \epsilon.$$

PROPOSITION 2.2. *Let $F\colon M \mapsto \mathrm{d}cL_p(T, X)$, $1 \leqslant p < \infty$, be a lower semicontinuous multifunction. Then F has a continuous selector.*

Proof. For $p = 1$, it is well known [5]. In the case where $p > 1$, one can prove this analogously by the obvious transformation.

PROPOSITION 2.3. *Let $H \subset L_1(T, R)$. Then the following properties are equivalent:*

(i) *H is weakly relatively compact;*
(ii) *H is uniformly integrable.*

If u_k, $k \geqslant 1$, is a uniformly integrable sequence from $L_1(T,R)$, $u_k(t) \to u(t)$, $u \in L_1(T,R)$ a.e. on T, then the convergence in $L_1(T,R)$ holds:

$$\lim_{k\to\infty} \int_T |u_k(t) - u(t)|\, \mathrm{d}\mu_0 = 0.$$

Proof. From the famous Lyapunov theorem, we know that the set

$$\Re = \{\mu_0(E);\ E \in \Sigma\}$$

is compact and convex. In particular, this theorem implies the following: for arbitrary $A \in \Sigma$ there is $B \in \Sigma$ such that $B \subset A$ and $\mu_0(B) = \frac{1}{2}\mu_0(A)$. Hence, we obtain that, for any $\delta > 0$, there exists a family $A_n \in \Sigma$, $1 \leqslant n \leqslant N$, of disjoint sets such that $T = \bigcup_{n=1}^{N} A_n$, $\mu_0(A_n) \leqslant \delta$. Therefore, if H is uniformly integrable, then H is a bounded subset of $L_1(T,R)$. Now, Proposition 2.3 follows from the Dunford–Pettis theorem [22].

PROPOSITION 2.4. *Let $f_n, f \in L_p(T,X)$, $n \geqslant 1$, $1 \leqslant p < \infty$.*

If the sequence f_n, $n \geqslant 1$, is uniformly p-integrable and $f_n(t)$ converges to $f(t)$ a.e., then the strong convergence in $L_p(T,X)$ also takes place:

$$\lim_{n\to\infty} \left(\int_T \|f_n(t) - f(t)\|^p \mathrm{d}\mu_0 \right)^{1/p} = 0.$$

Proof. Consider the sequence $z_n(t) = \|f_n(t) - f(t)\|^p$, $n \geqslant 1$, that converges pointwisely a.e. to zero. From the inequality

$$\begin{aligned} \|f_n(t) - f(t)\|^p &\leqslant (\|f_n(t)\| + \|f(t)\|)^p \\ &\leqslant 2^{p-1}\|f_n(t)\|^p + 2^{p-1}\|f(t)\|^p, \end{aligned}$$

it follows that the sequence $z_n \in L_1(T,R)$, $n \geqslant 1$, is uniformly integrable. Having used Proposition 2.3, one obtains that

$$\lim_{n\to\infty} \int_T z_n(t)\mathrm{d}\mu_0 = \lim_{n\to\infty} \int_T \|f_n(t) - f(t)\|^p \mathrm{d}\mu_0 = 0.$$

Therefore, the sequence f_n, $n \geqslant 1$, converges to f in $L_p(T,X)$.

3. Functions for Choosing the Extreme Points and their Properties

Let $\{x'_s\}_1^\infty$ be a countable, dense in weak topology, balanced subset of the set $\overline{B}'$. For any $A \in ccb(X)$, $u \in A$, and x'_s define the functions

$$d^s(A,u) = \sup\{\langle y - x, x'_s \rangle;\ y, x \in A,\ u = (x+y)/2\}, \quad s \geqslant 1. \tag{3.1}$$

The following lemma is the infinite-dimensional version of Lemma 1 in [32].

LEMMA 3.1. *For every* $s \geqslant 1$

(i) *the function* $u \mapsto d^s(A,u)$ *is nonnegative, concave and, if* $u \in A \subset B \in ccb(X)$, *then*

$$d^s(A,u) \leqslant d^s(B,u); \tag{3.2}$$

(ii) $u \in \operatorname{ext} A$ *if and only if*

$$d^s(A,u) = 0 \quad \text{for all } s \geqslant 1; \tag{3.3}$$

(iii) *if* Y *is a topological space,* A: $Y \mapsto ccb(X)$ *is a Hausdorff continuous multifunction, and* $u(y)$ *is one of its continuous selectors, then the function* $y \mapsto d^s(A(y),u(y))$ *is upper semicontinuous.*

Proof. (i), (ii) can be proved analogously [32]. Let us show (iii).

Put $B(y) = A(y) - u(y)$. Then B: $Y \mapsto ccbX$ is the Hausdorff continuous multifunction.

Denote by X'' the topological dual space of the space $(X', \|\cdot\|)$, and let $\sigma - X''$ be the space X'' endowed with the weak $\sigma(X'',X')$ topology [2]. Consider X as a subspace of X''. Then $B(y) \subset X''$, $y \in Y$, is relatively compact in the space $\sigma - X''$. Denote by $\overline{B}(y)$ the closure of the set $B(y)$ in the space $\sigma - X''$. It is easy to prove that the multifunction $\overline{B}(y)$ is continuous in the Hausdorff metric $h(\cdot,\cdot)$, generated by the norm of space X''. Then $\overline{B}$: $Y \mapsto cwk(X'')$ is Hausdorff upper semicontinuous in the topology of $\sigma - X''$.

By using the compactness of sets $\overline{B}(y)$, $y \in Y$, in the $\sigma(X'',X')$ topology, one can get that the multifunction $\overline{B}(y)$ is upper semicontinuous in the $\sigma(X'',X')$ topology. So the multifunction $C(y) = \overline{B}(y) \cap (-\overline{B}(y))$, $y \in Y$, is upper semicontinuous in the $\sigma(X'',X')$ topology. It is easy to prove that

$$\begin{aligned} d^s(A(y),u(y)) &= \sup\{\langle z-x, x'_s\rangle;\ x,z \in \overline{B}(y),\ z+x=0\} \\ &= \sup\{\langle 2z, x'_s\rangle;\ z \in C(y)\}. \end{aligned}$$

From this and the upper semicontinuity of $C(y)$ in the $\sigma(X'',X')$ topology, it follows that the function $d^s(A(y),u(y))$ is upper semicontinuous.

For a set $A \in ccb(X)$ denote that

$$g^s(A) = \sup\{\langle x-y, x'_s\rangle;\ x,y \in A\}. \tag{3.4}$$

Obviously, $0 \leqslant d^s(A,u) \leqslant g^s(A)$ for any $u \in A$. Let

$$L^s(A,u) = \{r \in R^+;\ d^s(A,u) \leqslant r \leqslant g^s(A)\}, \tag{3.5}$$

$$A_n \in ccb(X), \quad u_n \in A_n, \quad n \geqslant 1,$$

$$A \in ccb(X), \quad u \in A, \quad r \in L^s(A,u).$$

Take the point $r_n \in L^s(A_n,u_n)$ such that

$$|r_n - r| = \min\{|r-k|;\ k \in L^s(A_n,u_n)\}.$$

LEMMA 3.2. *Let $u_n \in A_n \in ccb(X)$, $n \geqslant 1$. Suppose that $A_n \to A$ in the Hausdorff metric $h(\cdot,\cdot)$ and $u_n \to u$ in X.*

Then, for fixed $s \geqslant 1$, $r \in L^s(A,u)$, the sequence $r_n \in L^s(A_n,u_n)$, $n \geqslant 1$, converges to r.

Proof. Let us show that $g^s(A_n) \to g^s(A)$. Take an arbitrary $\epsilon > 0$. Then for some number N,

$$A \subset A_n + \epsilon \cdot B, \qquad A_n \subset A + \epsilon \cdot B, \quad n \geqslant N. \tag{3.6}$$

Fix $x, y \in A$. According to the left-hand inclusion in (3.6), there exist $x_n, y_n \in A_n$, $v_n, w_n \in \epsilon \cdot B$ such that

$$x = x_n + v_n, \qquad y = y_n + w_n.$$

Since

$$\langle x - y, x'_s \rangle = \langle x_n - y_n, x'_s \rangle + \langle v_n - w_n, x'_s \rangle,$$

the definition of $g^s(A_n)$ implies that

$$g^s(A_n) \leqslant g^s(A) + 2\epsilon, \quad n \geqslant N. \tag{3.7}$$

Analogously, by using the right-hand inclusion of (3.6), we obtain

$$g^s(A) \leqslant g^s(A_n) + 2\epsilon, \quad n \geqslant N.$$

Then (3.6), (3.7) yield that $g^s(A_n) \to g^s(A)$.

Now, by virtue of Lemma 3.1(iii), for an arbitrary $\epsilon > 0$ there exists a number N such that for any $n \geqslant N$

$$d^s(A_n,u_n) < d^s(A,u) + \epsilon, \qquad g^s(A) < g^s(A_n) + \epsilon.$$

Thus, for each $n \geqslant N$

$$L^s(A,u) \subset L^s(A_n,u_n) + \epsilon \cdot C,$$

where $C = \{r \in R;\ |r| < 1\}$. This implies that $|r_n - r| < \epsilon$, $n \geqslant N$. Consequently, $r_n \to r$.

LEMMA 3.3. *Let F: $T \mapsto ccb(X)$ be a measurable map, and $u(t)$ be one of its measurable selectors.*

Then the functions $d^s(F(t),u(t))$, $g^s(F(t))$ and the multivalued map $L^s(F(t), u(t))$ are measurable.

Lemma 3.3 is proved by using standard arguments and Lemma 2.1 [27]. We refer to [28] for details concerning measurable multifunctions.

From Lemma 3.3 and the properties of measurable multivalued maps with compact values, it follows [28] that, if F_n, F: $T \mapsto ccb(X)$ and $u_n(t) \in F_n(t)$, $u(t) \in F(t)$, $r(t) \in L^s(F(t),u(t))$, $n \geqslant 1$, are measurable, then there exists the sequence of measurable selectors $r_n(t) \in L^s(F_n(t),u_n(t))$, $t \in T$, such that

$$|r_n(t) - r(t)| = \min\{|r(t) - k|\ k \in L^s(F_n(t),u_n(t))\}. \tag{3.8}$$

LEMMA 3.4. *Let a sequence of measurable multifunctions F_n: $T \mapsto ccb(X)$ converge in the Hausdorff metric $h(\cdot,\cdot)$ pointwisely a.e. to a measurable multifunction F: $T \mapsto ccb(X)$, and a sequence $u_n(t)$ of measurable selectors of $F_n(t)$ converge a.e. to a measurable selector $u(t)$ of $F(t)$.*

Then for any measurable selector $r(t)$ of the map $L^s(F(t), u(t))$, the sequence of measurable selectors $r_n(t)$ of $L^s(F_n(t), u_n(t))$, satisfying (3.8), *converges a.e. to $r(t)$.*

The result follows from Lemma 3.2.

4. Some Properties of Decomposable Sets

Let F: $T \mapsto c(X)$ be a measurable multifunction. For $1 \leqslant p < \infty$ define the set

$$S^p(F) = \{f \in L_p(T, X);\ f(t) \in F(t) \text{ a.e.}\}. \tag{4.1}$$

As is readily verified, $S^p(F)$ is a closed subset of $L_p(T, X)$.

In short, we call a function $f \in L_p(T, X)$ p-integrable.

It is well known [27] that, for $\Gamma \in \mathrm{d}cL_p(T, X)$, there exists a unique (up to a set of zero measure) measurable multifunction F^Γ: $T \mapsto c(X)$ such that $\Gamma = S^p(F^\Gamma)$.

Put for $\Gamma \in \mathrm{d}cbL_p(T, X)$, $1 \leqslant p < \infty$,

$$\|\Gamma\|_p = \sup_{f\in\Gamma} \|f\|_p = \sup_{f\in S^p(F^\Gamma)} \|f\|_p. \tag{4.2}$$

PROPOSITION 4.1. *Let $\Gamma \in \mathrm{d}cbL_p(T, X)$, $1 \leqslant p < \infty$.*

Then the multifunction F^Γ: $T \mapsto c(X)$ is p-integrally bounded and

$$\|\Gamma\|_p = \left(\int_T \|F^\Gamma(t)\|^p \,\mathrm{d}\mu_0\right)^{1/p} \tag{4.3}$$

Proof. For $p = 1$, equality (4.3) is well-known [27]. If $1 < p < \infty$, then

$$\begin{aligned} \|\Gamma\|_p &= \sup_{f\in\Gamma} \left(\int_T \|f(t)\|^p \mathrm{d}\mu_0\right)^{1/p} \\ &= \left(-\inf_{f\in\Gamma} \int_T (-\|f(t)\|^p)\,\mathrm{d}\mu_0\right)^{1/p} \end{aligned} \tag{4.4}$$

In virtue of Theorem 2.2 [27]

$$\begin{aligned} &\inf_{f\in S^p(F^\Gamma)} \int_T (-\|f(t)\|^p)\,\mathrm{d}\mu_0 \\ &= \int_T \inf_{x\in F^\Gamma(t)} (-\|x\|^p)\,\mathrm{d}\mu_0 \\ &= -\int_T (\sup_{x\in F^\Gamma(t)} \|x\|^p)\,\mathrm{d}\mu_0. \end{aligned} \tag{4.5}$$

From (4.2), (4.4), (4.5) it follows that (4.3) holds.

PROPOSITION 4.2. *Let* $\Gamma_1, \Gamma_2 \in \mathrm{d}cbL_p(T, X)$, $1 \leqslant p < \infty$.
Then

$$h_p(\Gamma_1, \Gamma_2) \leqslant \left(\int_T h^p(F^{\Gamma_1}(t), F^{\Gamma_2}(t))\,\mathrm{d}\mu_0\right)^{1/p} \leqslant 2^{1/p} h_p(\Gamma_1, \Gamma_2). \tag{4.6}$$

Proof. From the well known inequality

$$\begin{aligned} &h_p(F^{\Gamma_1}(t), F^{\Gamma_2}(t)) \\ &\quad\leqslant (\|F^{\Gamma_1}(t)\| + \|F^{\Gamma_2}(t)\|)^p \\ &\quad\leqslant 2^{p-1}\|F^{\Gamma_1}(t)\|^p + 2^{p-1}\|F^{\Gamma_2}(t)\|^p, \end{aligned}$$

Proposition 4.1, the measurability of the function $h(F^{\Gamma_1}(t), F^{\Gamma_2}(t))$ [27] and Theorem 2.2 [27], it follows that

$$\begin{aligned} d_p(x, \Gamma_2) &= \inf_{y \in \Gamma_2} \left(\int_T \|x(t) - y(t)\|^p \mathrm{d}\mu_0\right)^{1/p} \\ &= \left(\inf_{y \in \Gamma_2} \int_T \|x(t) - y(t)\|^p \mathrm{d}\mu_0\right)^{1/p} \\ &= \left(\int_T \left\{\inf_{z \in F^{\Gamma_2}(t)} \|x(t) - z\|\right\}^p \mathrm{d}\mu_0\right)^{1/p} \\ &\leqslant \left(\int_T h^p(F^{\Gamma_1}(t), F^{\Gamma_2}(t))\,\mathrm{d}\mu_0\right)^{1/p} \end{aligned} \tag{4.7}$$

Analogously,

$$d_p(y, \Gamma_1) \leqslant \left(\int_T h^p(F^{\Gamma_1}(t), F^{\Gamma_2}(t))\,\mathrm{d}\mu_0\right)^{1/p} \tag{4.8}$$

By joining (4.7) and (4.8), we obtain the left-hand inequality of (4.6).

Let us show that the right-hand inequality of (4.6) is true. From the definition of the function $h(F^{\Gamma_1}(t), F^{\Gamma_2}(t))$, it follows that there exists a set $E \in \Sigma$ such that

$$h(F^{\Gamma_1}(t), F^{\Gamma_2}(t)) = \sup\{d(x, F^{\Gamma_1}(t);\ x \in F^{\Gamma_2}(t)\}, \quad t \in E, \tag{4.9}$$

$$h(F^{\Gamma_1}(t), F^{\Gamma_2}(t)) = \sup\{d(y, F^{\Gamma_2}(t);\ y \in F^{\Gamma_1}(t)\}, \quad t \in T \setminus E. \tag{4.10}$$

Since the function $d(x, F^{\Gamma_1}(t))$ is measurable in t for every x and continuous in x for almost every t, without loss of generality, one may assume that $d(x, F^{\Gamma_1}(t))$

is continuous in x for every t. Then Theorem 2.2 [27] is applicable and

$$\begin{aligned}
&\left(\int_E \sup_{x\in F^{\Gamma_2}(t)} \{d^p(x, F^{\Gamma_1}(t))\}\,\mathrm{d}\mu_0\right)^{1/p}\\
&\quad = \left(-\int_E \inf_{x\in F^{\Gamma_2}(t)} \{-d^p(x, F^{\Gamma_1}(t))\}\,\mathrm{d}\mu_0\right)^{1/p}\\
&\quad = \left(-\inf_{y\in\Gamma_2}\left(-\int_E d^p(y(t), F^{\Gamma_1}(t))\,\mathrm{d}\mu_0\right)\right)^{1/p}\\
&\quad = \left(\sup_{y\in\Gamma_2}\int_E d^p(y(t), F^{\Gamma_1}(t))\,\mathrm{d}\mu_0\right)^{1/p}\\
&\quad \leqslant \sup_{y\in\Gamma_2}\left(\int_E d^p(y(t), F^{\Gamma_1}(t))\,\mathrm{d}\mu_0\right)^{1/p}\\
&\quad \leqslant \sup_{y\in\Gamma_2} d_p(y, \Gamma_1) \leqslant h_p(\Gamma_1, \Gamma_2).
\end{aligned} \tag{4.11}$$

In the same way, we obtain

$$\left(\int_{T\setminus E} \sup_{y\in F^{\Gamma_1}(t)} \{d^p(y, F^{\Gamma_2}(t))\}\,\mathrm{d}\mu_0\right)^{1/p} \leqslant h_p(\Gamma_1, \Gamma_2). \tag{4.12}$$

From (4.9)–(4.12) we obtain the right-hand inequality of (4.6).

PROPOSITION 4.3. *Let $\Gamma_n, \Gamma \in \mathrm{d}cbL_p(T, X)$ and $\Gamma_n \to \Gamma$ in the Hausdorff metric $h_p(\cdot,\cdot)$.*

Then the sequence $\|F_n(t)\|$, $n \geqslant 1$, where $F_n(t) = F^{\Gamma_n}(t)$, $n \geqslant 1$, is uniformly p-integrable.

Proof. For any $E \in \Sigma$ from the inequality

$$\begin{aligned}
&\int_E \|F_n(t)\|^p \mathrm{d}\mu_0\\
&\quad \leqslant 2^{p-1}\int_E \big|\|F_n(t)\| - \|F(t)\|\big|^p \mathrm{d}\mu_0 + 2^{p-1}\int_E \|F(t)\|^p \mathrm{d}\mu_0\\
&\quad \leqslant 2^{p-1}\int_T h^p(F_n(t), F(t))\,\mathrm{d}\mu_0 + 2^{p-1}\int_E \|F(t)\|^p \mathrm{d}\mu_0,
\end{aligned}$$

Propositions 4.1 and 4.2 we obtain that the sequence $F_n(t)$, $n \geqslant 1$, is uniformly p-integrable.

THEOREM 4.4. *Let F: $T \mapsto cwk(X)$ be a measurable, p-integrally bounded map, $1 \leqslant p < \infty$.*

Then $S^p(F)$ is the decomposable, convex, weakly compact subset of $L_p(T, X)$.

Proof. For $p = 1$, this result is well known [31]. Setting $F(t) = \{0\}$ on the exceptional μ_0-zero set, we can assume, without loss of generality, that $F(t) \in cwk(X)$ for all $t \in T$. From [22], it is known that each element g of the space $L_p'(T, X)$, $1 < p < \infty$, can be represented in the form

$$\langle f, g \rangle = \int_T \langle f(t), g(t) \rangle \, \mathrm{d}\mu_0,$$

where g: $T \mapsto X'$ is a weakly measurable function [22] such that

$$\|g\|_q = \left(\int_T \|g(t)\|^q \mathrm{d}\mu_0 \right)^{1/q} < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Clearly, $S^p(F)$ is a closed, convex, bounded subset of $L_p(T, X)$. Taking $g \in L_p'(T, X)$, we have

$$\sup_{f \in S^p(F)} \langle f, g \rangle = \sup_{f \in S^p(F)} \int_T \langle f(t), g(t) \rangle \, \mathrm{d}\mu_0.$$

Since $\langle x, g(t) \rangle$ is measurable in t for every x and continuous in x for t a.e., by changing values of g on the exceptional μ_0-zero set, we can assume, without loss of generality, that $\langle x, g(t) \rangle$ is continuous in x for every t. Hence, Theorem 2.2 [27] is applicable for the function $\langle x, g(t) \rangle$. Consequently, we obtain

$$\sup_{f \in S^p(F)} \int_T \langle f(t), g(t) \rangle \, \mathrm{d}\mu_0 = \int_T \sup_{x \in F(t)} \langle x, g(t) \rangle \, \mathrm{d}\mu_0.$$

Let

$$R(t) = \left\{ y \in F(t); \ \sup_{x \in F(t)} \langle x, g(t) \rangle = \langle y, g(t) \rangle \right\}.$$

Since $F(t) \in cwk(X)$ for all $t \in T$, we have $R(t) \neq \emptyset$ for all $t \in T$. Put

$$\phi(t, y) = \sigma(t) - \langle y, g(t) \rangle,$$

where $\sigma(t) = \sup\{\langle x, g(t) \rangle; \ x \in F(t)\}$. Clearly, $\phi(t, y)$ is measurable in t for every y and continuous in y for every t. Then, for the graph, we have

$$\mathrm{Gr}\, R = \{(t, y) \in T \times X; \ \phi(t, y) = 0\} \cap \mathrm{Gr}\, F \in \Sigma \otimes \mathcal{B}_X,$$

where $\mathcal{B}_X$ is the Borel field of X. So applying Aumann's selection theorem [28], we can find a measurable function f^*: $T \mapsto X$, $f^*(t) \in R(t)$ for all $t \in T$. Notice that $f^* \in S^p(F)$. Hence, we have

$$\begin{aligned} &\int_T \sup_{x \in F(t)} \langle x, g(t) \rangle \, \mathrm{d}\mu_0 \\ &\quad = \int_T \langle f^*(t), g(t) \rangle \, \mathrm{d}\mu_0 \\ &\quad = \sup_{f \in S^p(F)} \int_T \langle f(t), g(t) \rangle \, \mathrm{d}\mu_0 = \langle f^*, g \rangle. \end{aligned}$$

Since $g \in L'_p(T, X)$ is arbitrary, using James's Theorem [20] we conclude that $S^p(F)$ is the weak compact in $L_p(T, X)$.

5. Functions for Choosing the Extreme Points of Decomposable Sets and their Properties

Let $\Gamma \in \mathrm{d}ccbL_p(T, X)$ and $u \in \Gamma$. Put

$$D^s(\Gamma, u) = \sup\{\langle y - z, x'_s \cdot \chi(T)\rangle;\ y, z \in \Gamma,\ u = (y + z)/2\},$$

$$G^s(\Gamma) = \sup\{\langle y - z, x'_s \cdot \chi(T)\rangle;\ y, z \in \Gamma\},$$

where

$$\langle y - z, x'_s \cdot \chi(T)\rangle = \int_T \langle y(t) - z(t), x'_s \cdot \chi(T)(t)\rangle \,\mathrm{d}\mu_0.$$

According to Proposition 4.1, there exists a unique (up to a set of zero measure) p-integrally bounded, measurable multifunction F^Γ: $T \mapsto ccb(X)$ such that (4.3) is true. So one is able to define the functions $d^s(F^\Gamma(t), u(t))$, $g^s(F^\Gamma(t))$ and the multifunction $L^s(F^\Gamma(t), u(t))$ (see (3.1), (3.5), (3.6)). From Lemma 3.3, it follows that $d^s(F^\Gamma(t), u(t))$, $g^s(F^\Gamma(t))$ are measurable, p-integrally bounded, and the multifunction $L^s(F^\Gamma(t), u(t))$ is measurable, and their values belong to $ck(R)$.

LEMMA 5.1. *Let* $\Gamma \in \mathrm{d}ccbL_p(T, X)$ *and* $u \in \Gamma$, $F(t) = F^\Gamma(t)$.
Then for every $s \geqslant 1$*:*

(i)

$$D^s(\Gamma, u) = \int_T d^s(F(t), u(t)) \,\mathrm{d}\mu_0; \tag{5.1}$$

(ii)

$$G^s(\Gamma) = \int_T g^s(F(t)) \,\mathrm{d}\mu_0; \tag{5.2}$$

(iii)

$$u \in \operatorname{ext}\Gamma \textit{ if and only if } D^s(\Gamma, u) = 0 \textit{ for every } s \geqslant 1; \tag{5.3}$$

(iv) *If* $\Gamma_n, \Gamma \in \mathrm{d}ccbL_p(T, X)$, $n \geqslant 1$, Γ_n *converges to* Γ *in the Hausdorff metric* $h_p(\cdot,\cdot)$, *and* $u_n \in \Gamma_n$, $n \geqslant 1$, *converges to* u *in* $L_p(T, X)$, *then*

$$\lim_{n\to\infty} \sup D^s(\Gamma_n, u_n) \leqslant D^s(\Gamma, u). \tag{5.4}$$

Proof. Put

$$\Gamma_0 = \{y \in L_p(T, X);\ y \in \Gamma - u,\ -y \in \Gamma - u\}$$

and

$$F_0(t) = \{x \in X;\ x \in F(t) - u(t),\ -x \in F(t) - u(t)\}.$$

Obviously, $\Gamma_0 \in \mathrm{d}ccbL_p(T, X)$, F_0: $T \mapsto ccb(X)$ is measurable, p-integrally bounded and $\Gamma_0 = S^p(F_0)$. One may verify that

$$D^s(\Gamma, u) = \sup\{\langle 2y, x'_s \cdot \chi(T)\rangle;\ y \in \Gamma_0\} \tag{5.5}$$

and

$$d^s(F(t), u(t)) = \sup\{\langle 2z, x'_s\rangle;\ z \in F_0(t)\}. \tag{5.6}$$

Since, for every $y \in \Gamma_0$,

$$\langle 2y, x'_s \cdot \chi(T)\rangle = \int_T \langle 2y(t), x'_s\rangle \,\mathrm{d}\mu_0, \tag{5.7}$$

by using (5.5)–(5.7) and Theorem 2.2 [27], we obtain equalities (5.1), (5.2).

We now pass to the proof of statement (iii). If $u \in \operatorname{ext}\Gamma$, then equality (5.3) follows from the definition of an extreme point.

Let $D^s(\Gamma, u) = 0$ for every $s \geqslant 1$. It should be mentioned that the set

$$\{x_k \cdot \chi(T)\}_1^\infty \subset L'_p(T, X)$$

does not separate points of the space $L_p(T, X)$, therefore statement (iii) does not follows directly from Lemma 3.1(ii).

Suppose that $u \notin \operatorname{ext}\Gamma$. Then there exist $y, z \in \Gamma$, $y \neq z$, such that $u = (y+z)/2$. Fix $\epsilon > 0$. By using (5.1), take a compact subset $T_\epsilon \subset T$, $\mu_0(T \backslash T_\epsilon) \leqslant \epsilon$ such that $u(t)$, $y(t)$, $z(t)$ are continuous on T_ϵ, $y(t) \neq z(t)$, $u(t) = (y(t) + z(t))/2$, $u(t), y(t), z(t) \in F(t)$, and $d^s(F(t), u(t)) = 0$, $t \in T_\epsilon$, $s \geqslant 1$.

Since the set $\{x'_s\}_1^\infty$ separates points of the space X, then for every $t \in T_\epsilon$, there exists a number $s(t) \geqslant 1$, depending on t, such that

$$\langle y(t) - z(t), x'_{s(t)}\rangle > 0.$$

By the continuity of $y(t), z(t)$ on T_ϵ, there exists a neighbourhood $V(t)$ in T_ϵ of the point t such that

$$\langle y(\tau) - z(\tau), x'_{s(t)}\rangle > 0 \quad \text{for every } \tau \in V(t).$$

The family $\{V(t)\}$, $t \in T_\epsilon$, is the open covering of the compact T_ϵ. Then there exists some $t^* \in T_\epsilon$ with $\mu_0(V(t^*)) > 0$. From the definition of the Radon measure, it follows that there exists a compact $T_0 \subset V(t^*)$ with $\mu_0(T_0) > 0$. Then for $s^* = s(t^*)$

$$0 < \int_{T_0} \langle y(\tau) - z(\tau), x'_{s^*} \rangle \, \mathrm{d}\mu_0 \leqslant \int_T d^{s^*}(F(\tau), u(\tau)) \, \mathrm{d}\mu_0 = D^{s^*}(\Gamma, u) = 0.$$

But the last inequality gives us the contradiction. Statement (iii) is proved.

Statement (iv) follows from Lemma 3.1(iii).

COROLLARY 5.2. *Let* $\Gamma \in \mathrm{d}ccbL_p(T, X)$.

Then $u \in \operatorname{ext}\Gamma$ *if and only if* $u(t) \in \operatorname{ext} F(t)$ *a.e. on* T.

The result follows from Lemma 5.1(i), (iii) and Lemma 3.1(ii).

Let us take a separable metric space M and a multifunction $\Gamma\colon M \mapsto \mathrm{d}ccbL_p(T, X)$, $1 \leqslant p < \infty$, continuous in the Hausdorff metric $h_p(\cdot,\cdot)$. From the famous Michael theorem [30], we know that there exists a continuous selector $u\colon M \mapsto L_p(T, X)$ of the multifunction Γ.

For every $\xi \in M$, denote by $F(\xi)(t) = F^{\Gamma(\xi)}(t)$ the measurable, p-integrable function $F(\xi)\colon T \mapsto ccb(X)$ such that $\Gamma(\xi) = S^p(F(\xi))$. Then the functions

$$D^s(u(\xi)) = D^s(\Gamma(\xi), u(\xi)), \qquad G^s(\xi) = G^s(\Gamma(\xi)),$$

$$D^s(u(\xi)) \leqslant G^s(\xi),$$

$$d^s(u(\xi)(t)) = d^s(F(\xi)(t), u(\xi)(t)),$$

$$g^s(\xi)(t) = g^s(F(\xi)(t)), \quad \xi \in M,$$

and the multifunctions

$$L^s(u(\xi)(t)) = L^s(F(\xi)(t), u(\xi)(t))$$

are defined.

As we know, for every $\xi \in M$, the functions $d^s(u(\xi)(t))$, $g^s(\xi)(t)$ are measurable, p-integrable and the multifunction $L^s(u(\xi)(t))$ is measurable, p-integrally bounded, and its values belong to $ck(R)$. According to Lemma 5.1, for every $s \geqslant 1$, $\xi \in M$, we have

$$D^s(u(\xi)) = \int_T d^s(u(\xi)(t)) \, \mathrm{d}\mu_0, \tag{5.8}$$

$$G^s(\xi) = \int_T g^s(\xi)(t) \, \mathrm{d}\mu_0. \tag{5.9}$$

For every $\xi \in M$, put

$$\mathcal{L}^s(u(\xi)) = S^p(L^s(u(\xi))). \tag{5.10}$$

THEOREM 5.3. *Let a multifunction* $\Gamma\colon\ M \mapsto \mathrm{dccb}L_p(T,X)$, $1 \leqslant p < \infty$, *be continuous in the Hausdorff metric* $h_p(\cdot,\cdot)$. *Then* $\xi \mapsto \mathcal{L}^s(u(\xi))$ *is a lower semicontinuous in multifunction with values in* $cwkL_p(T,R)$.

Proof. For fixed $\xi \in M$ and every measurable selector $r(t)$ of $L^s(u(\xi)(t))$, the inequality

$$0 \leqslant d^s(u(\xi)(t)) \leqslant r(t) \leqslant g^s(\xi)(t) \leqslant 2\|F(\xi)(t)\| \tag{5.11}$$

is true. From (5.11) and Theorem 4.4, we immediately obtain that $\mathcal{L}^s(u(\xi)) \in cwkL_p(T,R)$, $\xi \in M$.

Fix $\xi_0 \in M$. To prove the lower semicontinuity of $\mathcal{L}^s(u(\xi))$ at the point t_0, one has to show that for any sequence $\xi_n \to \xi_0$, $n \geqslant 1$, and any $r_0 \in \mathcal{L}^s(u(\xi_0))$, there exists a sequence $r_n \in \mathcal{L}^s(u(\xi_n))$, $n \geqslant 1$, converging to r_0 in $L_p(T,R)$.

Since $r_0(t) \in L^s(u(\xi_0)(t))$, $t \in T$, then there exists a sequence $r_n(t) \in L^s(u(\xi_n)(t))$, $n \geqslant 1$, of measurable selectors such that

$$|r_0(t) - r_n(t)| = \min\{|r_0(t) - k|;\ k \in L^s(u(\xi_n)(t))\},\ \ t \in T.$$

It is clear that $r_n \in \mathcal{L}^s(u(\xi_n))$, $n \geqslant 1$. Using Proposition 4.2, we find a subsequence ξ_{n_k}, $k \geqslant 1$, of the sequence ξ_n, $n \geqslant 1$, such that $F(\xi_{n_k})(t)$, $k \geqslant 1$, a.e. converges to $F(\xi_0)(t)$, and $u(\xi_{n_k})(t)$, $k \geqslant 1$, a.e. converges to $u(\xi_0)(t)$. Then, according to Lemma 3.4, the sequence $r_{n_k}(t)$, $k \geqslant 1$, a.e. converges to $r_0(t)$. Inequality (5.11) and Proposition 4.3 give us that the sequence $r_{n_k} \in L_p(T,R)$ is uniformly p-integrable. Then, from Proposition 2.4, it follows that the sequence r_{n_k}, $k \geqslant 1$, converges to r_0 in $L_p(T,R)$.

Let us show that the sequence $r_n \in \mathcal{L}^s(u(\xi_n))$, $n \geqslant 1$, itself converges to $r_0 \in \mathcal{L}^s(u(\xi_0))$ in $L_p(T,R)$. We assume the opposite. Then there exists a subsequence r_{n_m}, $m \geqslant 1$, of the sequence r_n, $n \geqslant 1$, such that every subsequence of the sequence r_{n_m}, $m \geqslant 1$, does not converge to r_0 in $L_p(T,R)$. Since the sequence ξ_{n_m}, $m \geqslant 1$, converges to ξ_0, one should repeat the above arguments in order to get the contradiction that proves the lower semicontinuity of the multifunction $\mathcal{L}^s(u(\xi))$.

Denote by $\Lambda^s(u(\cdot))$ the family of all continuous selectors of the multifunction $\mathcal{L}^s(u(\cdot))$. From Theorem 5.3 and the Michael theorem, it follows that $\Lambda^s(u(\cdot)) \neq \emptyset$ and for every $r_0 \in \mathcal{L}^s(u(\xi_0))$ there exists a continuous in $L_p(T,R)$ selector $r(u(\xi))$ of the multifunction $\mathcal{L}^s(u(\xi))$, $\xi \in M$, such that $r(u(\xi_0)) = r_0$. From this and (5.8), (5.9) follows the following lemma.

LEMMA 5.4. *Let a multifunction* $\Gamma\colon\ M \mapsto \mathrm{dccb}L_p(T,X)$, $1 \leqslant p < \infty$, *satisfy all the assumptions of Theorem* 5.1. *Then*

(i)

$$\mathcal{L}^s(u(\xi)) = \{r(u(\xi));\ r(u(\cdot)) \in \Lambda^s(u(\cdot))\},\ \ \xi \in M;$$

(ii)

$$D^s(u(\xi)) \leqslant \int_T r(u(\xi))(t)\,\mathrm{d}\mu_0 \leqslant G^s(\xi), \quad \xi \in M,\ r(u(\cdot)) \in \Lambda^s(u(\cdot));$$

(iii) *for every* $\xi_0 \in M$ *there exists* $r(u(\cdot)) \in \Lambda^s(u(\cdot))$ *such that*

$$D^s(u(\xi_0)) = \int_T r(u(\xi_0))(t)\,\mathrm{d}\mu_0.$$

6. Auxiliary Results

Let $A \in ccb(X)$, $x' \in X'$, $x' \neq 0$, and $C(A, x') = \sup\{\langle x, x'\rangle;\ x \in A\}$. If $\alpha > 0$, then

$$C(A, x', \alpha) = \{x \in A;\ \langle x, x'\rangle > C(A, x') - \alpha\}.$$

DEFINITION 6.1 ([3]). Let $A \in ccb(X)$ and $x \in A$. The point x is a strongly exposed point of A if there is $x' \in X$ such that $\langle x, x'\rangle > \langle y, x'\rangle$, whenever $y \neq x$, $y \in A$ and $\{C(A, x', \alpha);\ \alpha > 0\}$ is a neighbourhood base for x in A in the norm topology (or, equivalently, such that $\lim_{\alpha\to 0+}$ (norm diameter $C(A, x', \alpha)) = 0$).

Denote the set of strongly exposed points of A by $\operatorname{st} A$. It is well known [3] that if $A \in cwk(X)$, then $\operatorname{st} A \neq 0$, $\operatorname{st} A \subset \operatorname{ext} A$ and $\overline{\operatorname{co}}\operatorname{st} A = \overline{\operatorname{co}}\operatorname{ext} A = A$.

PROPOSITION 6.2. *Let* $A \in cb(X)$ *be such that* $\operatorname{st}\overline{\operatorname{co}}\,A \neq \emptyset$ *and* $\overline{\operatorname{co}}\operatorname{st}\overline{\operatorname{co}}\,A = \overline{\operatorname{co}}\,A$.

Then $\operatorname{st}\overline{\operatorname{co}}\,A \subset A$.

Proof. If the statement is not true, then there exists $x_0 \in \operatorname{st}\overline{\operatorname{co}}\,A$ and an open neighbourhood $\mathrm{O}(x_0)$ of the point x_0 such that $A \cap \mathrm{O}(x_0) = \emptyset$. Take $x' \in X'$ satisfying $\langle x_0, x'\rangle > \langle x, x'\rangle$, whenever $x \neq x_0$, $x \in \overline{\operatorname{co}}A$.

Let $\{C(\overline{\operatorname{co}}\,A, x', \alpha);\ \alpha > 0\}$ be a neighbourhood base for x_0 in $\overline{\operatorname{co}}\,A$ in the norm topology. Then there exists $\alpha_0 > 0$ such that $C(\overline{\operatorname{co}}\,A, x', \alpha_0) \subset \mathrm{O}(x_0)$. Since, for every $x \in A$,

$$\langle x, x'\rangle \leqslant \sup\{\langle y, x'\rangle;\ y \in \overline{\operatorname{co}}\,A\} - \alpha_0,$$

then

$$\overline{\operatorname{co}}\,A \neq \overline{\operatorname{co}}\operatorname{st}\overline{\operatorname{co}}\,A.$$

This contradiction concludes the proof.

For short, any continuous in the topology of the space $L_p(T, X)$ function $u\colon M \mapsto L_p(T, X)$ will be called L_p-continuous.

Denote by $\overline{\operatorname{dec}}\operatorname{st}\Gamma$ the closure of $\operatorname{dec}\operatorname{st}\Gamma$ in $L_p(T, X)$.

PROPOSITION 6.3. *Let* $\Gamma\colon\ M \mapsto L_p(T,X)$ *be a lower semicontinuous multifunction.*

Then the multifunction $\overline{\operatorname{dec}}\,\Gamma$, $(\overline{\operatorname{dec}}\,\Gamma)(\xi) = \overline{\operatorname{dec}}\,\Gamma(\xi)$, $\xi \in M$, *is lower semicontinuous with closed, decomposable values.*

Proof. Fix $\xi_0 \in M$. Let $\{u_n\}_1^\infty \subset \Gamma(\xi_0)$ be a countable dense subset and $F\colon T \mapsto X$, $F(t) = \overline{\bigcup_{n\geqslant 1} u_n(t)}$. F is a measurable multifunction with closed values. According to Theorem 3.1 and Lemma 1.3 in [27], $\overline{\operatorname{dec}}\,\Gamma(\xi_0) = S^p(F)$. So $\overline{\operatorname{dec}}\,\Gamma$ is a multifunction with closed, decomposable values. For any $u \in \operatorname{dec}\Gamma(\xi_0)$, we have $u(t) \in F(t)$ a.e. on T. Then, according to Lemma 1.3 [27], for any $\epsilon > 0$, there exists a finite measurable partition $\{E_1,\dots,E_m\}$ of T such that

$$\left\| u - \sum_{i=1}^{m} \chi(E_i)\cdot u_i \right\|_p < \epsilon/2.$$

Since Γ is lower semicontinuous, there exists a neighbourhood $V(\xi_0)$ of ξ_0 such that for any $\xi \in V(\xi_0)$ there are elements $v_i(\xi) \in \Gamma(\xi)$, $i = 1,\dots,m$, satisfying the inequalities

$$\|u_i - v_i(\xi)\|_p < \epsilon/2m, \quad i = 1,\dots,m.$$

Consider the element $v(\xi) \in \operatorname{dec}\Gamma(\xi)$,

$$v(\xi) = \sum_{i=1}^{m} \chi(E_i)\cdot v_i(\xi).$$

Then $\|u - v(\xi)\|_p < \epsilon$ for any $\xi \in V(\xi_0)$. This means that the multifunction $\operatorname{dec}\Gamma$ is lower semicontinuous. Therefore, the multifunction $\overline{\operatorname{dec}}\,\Gamma$ is lower semicontinuous.

Define for any $x \in L_p(T,X)$ the numerical measure

$$\mathcal{K}(x)(E) = \int_E \|x(t)\|^p \,\mathrm{d}\mu_0, \quad E \in \Sigma.$$

PROPOSITION 6.4. *Let* $x_n, x \in L_p(T,X)$, $n \geqslant 1$, *and the sequence* x_n *converge in* $L_p(T,X)$ *to* x. *Then the sequence* $\mathcal{K}(x_n)$, $n \geqslant 1$, *converges to* $\mathcal{K}(x)$ *in the topology of the space* $\mathcal{M}$.

Proof. Take a subsequence $x_{n_k}(t)$, $k \geqslant 1$, of the sequence $x_n(t)$, $n \geqslant 1$, converging a.e. to $x(t)$. Then the sequence $\|x_{n_k}(t)\|^p$, $k \geqslant 1$, a.e. converges to $\|x(t)\|^p$. From Proposition 4.3 we obtain that the sequence $\|x_{n_k}(t)\|^p$, $k \geqslant 1$, is uniformly integrable. Using Proposition 2.3, we obtain [21]

$$\lim_{k\to\infty} \|\mathcal{K}(x_{n_k}) - \mathcal{K}(x)\|_{\mathcal{M}} = \lim_{k\to\infty} \int_T \big|\, \|x_{n_k}(t)\|^p - \|x(t)\|^p \,\big| \,\mathrm{d}\mu_0 = 0.$$

This means that the sequence $\mathcal{K}(x_{n_k}),\ k \geqslant 1$, converges to $\mathcal{K}(x)$ in the topology of the space $\mathcal{M}$.

By arguments that we have used repeatedly, it is proved that the sequence $\mathcal{K}(x_n),\ k \geqslant 1$, itself converges to $\mathcal{K}(x)$ in the topology of the space $\mathcal{M}$.

HYPOTHESIS. A multifunction Γ: $M \mapsto \mathrm{d}ccbL_p(T, X)$, $1 \leqslant p < \infty$, possesses the property **(H1)** if $\mathrm{st}\,\Gamma(\xi) \neq \emptyset$, $\Gamma(\xi) = \overline{\mathrm{co}}\,\mathrm{st}\,\Gamma(\xi)$ for every $\xi \in M$.

It should be mentioned that every multifunction Γ: $M \mapsto \mathrm{d}cwkL_p(T, X)$, $1 \leqslant p < \infty$, possesses the property **(H1)**.

PROPOSITION 6.5. *Let a continuous in the Hausdorff metric $h_p(\cdot,\cdot)$ multifunction Γ: $M \mapsto \mathrm{d}ccbL_p(T, X)$ possess the property* **(H1)**.

Then the multifunction $\xi \mapsto \overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma(\xi)$ is lower semicontinuous and for any $\xi_0 \in M$, $w_0 \in \overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma(\xi)$ there exists a L_p-continuous selector $w(\xi)$ of the multifunction $\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma(\xi)$ satisfying $w(\xi) = w_0$.

Proof. From Lemma 2 [33] it follows that the multifunction $\mathrm{st}\,\Gamma(\xi)$ is lower semicontinuous. By using Proposition 6.3, we obtain that the multifunction $\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma(\xi)$ is lower semicontinuous. Now the statement follows from Proposition 2.2.

THEOREM 6.6. *Let a continuous in Hausdorff metric $h_p(\cdot,\cdot)$ multifunction*

$$\Gamma\colon\ M \mapsto \mathrm{d}ccbL_p(T, X), \quad 1 \leqslant p < \infty,$$

possess the property **(H1)**, *ϕ_i: $M \mapsto (0, +\infty)$, $i = 1, 2$, be lower semicontinuous and $w(\xi)$ be a L_p-continuous selector of the multifunction $\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma(\xi)$.*

Then, for any $s \geqslant 1$, there exists a L_p-continuous selector $v(\xi)$ of the multifunction $\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma(\xi)$ such that

$$\|w(\xi) - v(\xi)\|_p < \phi_1(\xi), \quad \xi \in M, \tag{6.1}$$

$$D^s(v(\xi)) < \phi_2(\xi), \quad \xi \in M. \tag{6.2}$$

Proof. Denote by $\mathcal{G}$ the family of all L_p-continuous selectors of the multifunction $\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma(\xi)$. From Proposition 6.5, it follows that $\mathcal{G} \neq \emptyset$. For every $v(\cdot) \in \mathcal{G}$ take the collection $\Lambda^s(v(\cdot))$ of all continuous selectors of the multifunction $\mathcal{L}_s(v(\cdot))$ (see Lemma 5.4).

Set

$$c(\xi) = \min\{\phi_1(\xi), \phi_2(\xi)\}, \quad \xi \in M.$$

According to Lemma 3.6 [15], there exists a continuous function ν: $M \mapsto (0, +\infty)$ such that $0 < \nu(\xi) < c(\xi)$, $\xi \in M$. For every $(v(\cdot), r(v(\cdot)))$, $v(\cdot) \in \mathcal{G}$, $r(v(\cdot)) \in \Lambda^s(v(\cdot))$ put

$$\mathcal{U}_{v,r(v)} = \Big\{\xi \in M;\ v(\xi) \in w(\xi) + \frac{\phi_1(\xi) - \nu(\xi)}{2} \cdot B_p,$$
$$\int_T r(v(\xi))(t)\,\mathrm{d}\mu_0 < \phi_2(\xi) - \nu(\xi)\Big\}.$$

Since the functions $\phi_1(\xi) - \nu(\xi)$, $\phi_2(\xi) - \nu(\xi)$ are lower semicontinuous, $v(\xi)$, $w(\xi)$ are L_p-continuous and the function $r(v(\xi))$ is continuous in topology of $L_p(T, R)$, then every set $\mathcal{U}_{v,r(v)}$ is open.

Let us show that

$$M = \left\{ \bigcup \mathcal{U}_{v,r(v)};\ v(\cdot) \in \mathcal{G},\ r(v(\cdot)) \in \Lambda^s(v(\cdot)) \right\}.$$

Take $\xi_0 \in M$. Then, for $w(\xi_0)$, there exists $v_0 \in \operatorname{dec}\operatorname{st}\Gamma(\xi_0)$ such that

$$v_0 \in w(\xi_0) + \frac{\phi_1(\xi_0) - \nu(\xi_0)}{2} \cdot B_p.$$

According to Proposition 6.5, we find a L_p-continuous selector $v^*(\xi)$ of the multifunction $\overline{\operatorname{dec}}\operatorname{st}\Gamma(\xi)$ satisfying $v^*(\xi_0) = v_0$. Since $v_0 \in \operatorname{ext}\Gamma(\xi_0)$, from Lemma 5.1, it follows that $D^s(v^*(\xi_0)) = 0$. Using Lemma 5.4, we obtain that there is $r^*(v^*(\cdot)) \in \Lambda^s(v^*(\cdot))$ satisfying

$$D^s(v^*(\xi_0)) = \int_T r^*(v^*(\xi_0))(t)\, \mathrm{d}\mu_0 = 0.$$

Therefore, $\xi_0 \in \mathcal{U}_{v^*,r^*(v^*)}$.

Let $\{V_n\}_1^\infty$ be a countable, locally finite, open refinement of the covering $\{\mathcal{U}_{v,r(v)}\}$, and $\{e_n\}_1^\infty$ be a continuous, locally finite partition of the unity subordinated to $\{V_n\}_1^\infty$ such that $\operatorname{supp} e_n \subset V_n$, where $\operatorname{supp} e_n$ is the support of the function e_n [29]. Then, for every $n \geqslant 1$, there exist $v_n(\cdot) \in \mathcal{G}$, $r_n(v_n(\cdot)) \in \Lambda^s(v_n(\cdot))$ such that $V_n \subset \mathcal{U}_{v_n,r(v_n)}$ [29]. Since $\operatorname{supp} e_n$ is a closed subset of the space M, there exists a continuous function $h[\operatorname{supp} e_n, V_n]$ satisfying

$$0 \leqslant h[\operatorname{supp} e_n, V_n] \leqslant 1, \qquad h[\operatorname{supp} e_n, V_n](\xi) = 1, \quad \xi \in \operatorname{supp} e_n,$$

and

$$h[\operatorname{supp} e_n, V_n](\xi) = 0, \quad \xi \in M \setminus V_n.$$

Since $\operatorname{supp} e_n \subset V_n$, $n \geqslant 1$, and $\{V_n\}_1^\infty$ is a locally finite covering of the space M, we are able to define a continuous function k: $M \mapsto (0, +\infty)$ by

$$k(\xi) = \sum_{n=1}^{\infty} h[\operatorname{supp} e_n, V_n](\xi).$$

For every $x \in L_p(T, X)$, define the measure

$$\mathcal{K}_1(x)(E) = \int_E \|x(t)\|^p\, \mathrm{d}\mu_0, \quad E \in \Sigma.$$

Similarly, for every $(v(\cdot), r(v(\cdot))) \in \mathcal{G} \times \Lambda^s(v(\cdot))$ and $\xi \in M$, the measure

$$\mathcal{K}_2(v(\xi), r(v(\xi)))(E) = \int_E r(v(\xi))(t)\, \mathrm{d}\mu_0, \quad E \in \Sigma,$$

is defined.

From Proposition 6.4, it follows that for every $n \geqslant 1$, the measure $\mathcal{K}_1(v_n(\xi) - w(\xi))$ is continuous on M, in the topology of the space $\mathcal{M}$.

By using the equality

$$\|\mathcal{K}_2(v_n(\xi_1), r_n(v_n(\xi_1))) - \mathcal{K}_2(v_n(\xi_2), r_n(v_n(\xi_2)))\|_{\mathcal{M}} \\ = \int_T |r_n(v_n(\xi_1))(t) - r_n(v_n(\xi_2))(t)| \, d\mu_0,$$

we obtain that, for every $n \geqslant 1$, the measure $\mathcal{K}_2(v_n(\xi), r_n(v_n(\xi)))$ is continuous on M in the topology of the space $\mathcal{M}$.

Consider the multifunctions

$$\mathcal{P}_n(\xi) = \left\{ \frac{k(\xi)}{\nu^p(\xi)} \cdot \mathcal{K}_1(v_n(\xi) - w(\xi)) \right\} \cup \left\{ \frac{k(\xi)}{\nu(\xi)} \cdot \mathcal{K}_2(v_n(\xi), r_n(v_n(\xi))) \right\}.$$

Since all the multipliers in the above-defined functions $\mathcal{P}_n(\xi)$ are continuous and $k(\xi), \nu(\xi) > 0$, $\xi \in M$, then the multifunctions $\mathcal{P}_n\colon M \mapsto \mathcal{M}$, $n \geqslant 1$, are Hausdorff upper semicontinuous in the topology of the space $\mathcal{M}$, and their values are compact sets in $\mathcal{M}$ consisting of two elements. As follows from Proposition 2.1, there exists a sequence of continuous (with respect to the pseudometric $\mu_0(\cdot\triangle\cdot)$) maps $\mathcal{B}_n\colon M \mapsto \Sigma$ satisfying Proposition 2.1(i), (ii), and

$$|\mu(\mathcal{B}_n(\xi)) - e_n(\xi)\mu(T)| < \tfrac{1}{4}, \quad \xi \in M, \ \mu \in \mathcal{P}_n(\xi), \ n \geqslant 1. \tag{6.3}$$

According to Proposition 2.1(i), the function

$$v(\xi) = \sum_{n=1}^{\infty} v_n(\xi) \cdot \chi(\mathcal{B}_n(\xi))$$

is well defined. From the inequality

$$\begin{aligned} &\|v_n(\xi_0) \cdot \chi(\mathcal{B}_n(\xi_0)) - v_n(\xi) \cdot \chi(\mathcal{B}_n(\xi))\|_p \\ &\quad \leqslant \|v_n(\xi) \cdot \chi(\mathcal{B}_n(\xi)) - v_n(\xi_0) \cdot \chi(\mathcal{B}_n(\xi))\|_p + \\ &\qquad + \|v_n(\xi_0) \cdot \chi(\mathcal{B}_n(\xi)) - v_n(\xi_0) \cdot \chi(\mathcal{B}_n(\xi_0))\|_p \\ &\quad \leqslant \|v_n(\xi) - v_n(\xi_0)\|_p + \left(\int_{C(\xi)} \|v_n(\xi_0)(t)\|^p \, d\mu_0 \right)^{1/p}, \end{aligned}$$

where $C(\xi) = \mathcal{B}_n(\xi) \triangle \mathcal{B}_n(\xi_0)$, we get that every function $v_n(\xi) \cdot \chi(\mathcal{B}_n(\xi))$ is continuous.

Since $\{e_n(\cdot)\}_1^{\infty}$ is a continuous, locally finite partition of unity, from Proposition 2.1(ii), it follows that the function $v(\xi)$ is continuous. Moreover, the decomposability of the sets $\overline{\operatorname{dec}} \operatorname{st} \Gamma(\xi)$, $\xi \in M$, implies $v(\xi) \in \overline{\operatorname{dec}} \operatorname{st} \Gamma(\xi)$, $\xi \in M$.

Let us show that inequalities (6.1) and (6.2) hold. Fix $n \geqslant 1$, $\xi \in M$. Then for the measure

$$\mu = \frac{k(\xi)}{\nu^p(\xi)} \cdot \mathcal{K}_1(v_n(\xi) - w(\xi))$$

the representation

$$\mu(E) = \frac{k(\xi)}{\nu^p(\xi)} \int_E \|v_n(\xi)(t) - w(\xi)(t)\|^p \mathrm{d}\mu_0, \quad E \in \Sigma$$

holds. By using (6.3), we obtain

$$\int_{\mathcal{B}_n(\xi)} \|v_n(\xi)(t) - w(\xi)(t)\|^p \mathrm{d}\mu_0 \leqslant e_n(\xi) \cdot \int_T \|v_n(\xi)(t) - w(\xi)(t)\|^p \mathrm{d}\mu_0 + \frac{\nu^p(\xi)}{4k(\xi)}. \tag{6.4}$$

If $e_n(\xi) > 0$, then $\xi \in V_n \subset \mathcal{U}_{v_n, r_n(v_n)}$. Therefore

$$\|v_n(\xi) - w(\xi)\|_p < \frac{\phi_1(\xi) - \nu(\xi)}{2},$$

or

$$\int_T \|v_n(\xi)(t) - w(\xi)(t)\|^p \mathrm{d}\mu_0 < \frac{(\phi_1(\xi) - \nu(\xi))^p}{2^p}. \tag{6.5}$$

From (6.4), (6.5), we have

$$\begin{aligned} &\int_{\mathcal{B}_n(\xi)} \|v_n(\xi)(t) - w(\xi)(t)\|^p \mathrm{d}\mu_0 \\ &\quad < e_n(\xi) \cdot \frac{(\phi_1(\xi) - \nu(\xi))^p}{2^p} + \frac{\nu^p(\xi)}{4k(\xi)} \\ &\quad < e_n(\xi) \cdot \frac{\phi_1^p(\xi)}{2^p} + \frac{\phi_1^p(\xi)}{4k(\xi)}. \end{aligned} \tag{6.6}$$

If $e_n(\xi) = 0$, then according to Proposition 2.1(ii), we have $\mathcal{B}_n(\xi) = \emptyset$. Therefore, in any case inequality (6.6) holds.

Let $\{1_n\}$ be the sequence $\{1, 1, \ldots, 1, \ldots\}$. For every $\xi \in M$, put

$$m(\xi) = \sum_{\{n;\ \mathcal{B}_n(\xi) \neq \emptyset\}} 1_n.$$

Then, as follows from Proposition 2.1(ii), we have that $m(\xi) \leqslant k(\xi)$, $\xi \in M$. Now, from (6.6), we obtain

$$\begin{aligned} &\int_T \|v(\xi)(t) - w(\xi)(t)\|^p \mathrm{d}\mu_0 \\ &\quad < \sum_{\{n;\ \mathcal{B}_n(\xi) \neq \emptyset\}} \frac{e_n(\xi) \cdot \phi_1^p(\xi)}{2^p} + \frac{m(\xi) \cdot \phi_1^p(\xi)}{4k(\xi)} \\ &\quad < \frac{\phi_1^p(\xi)}{2^p} + \frac{1}{4}\phi_1^p(\xi) < \phi_1^p(\xi). \end{aligned}$$

Hence,

$$\|v(\xi) - w(\xi)\|_p < \phi_1(\xi), \quad \xi \in M.$$

We now pass to the proof of (6.2). Fix $n \geqslant 1$, $\xi \in M$. Then for the measure

$$\mu = \frac{k(\xi)}{\nu(\xi)} \cdot \mathcal{K}_2\left(v_n(\xi), r_n(v_n(\xi))\right),$$

we get

$$\mu(E) = \frac{k(\xi)}{\nu(\xi)} \int_E r_n(v_n(\xi))(t)\, \mathrm{d}\mu_0, \quad E \in \Sigma.$$

From (6.3) it follows

$$\int_{\mathcal{B}_n(\xi)} r_n(v_n(\xi))\, \mathrm{d}\mu_0 \leqslant e_n(\xi) \cdot \int_T r_n(v_n(\xi))(t)\, \mathrm{d}\mu_0 + \frac{\nu(\xi)}{4k(\xi)}.$$

If $e_n(\xi) > 0$, then there is $\xi \in V_n \subset \mathcal{U}_{v_n, r_n(v_n)}$. Therefore,

$$\int_T r_n(v_n(\xi))(t)\, \mathrm{d}\mu_0 < \phi_2(\xi) - \mu(\xi)$$

and

$$\int_{\mathcal{B}_n(\xi)} r_n(v_n(\xi))(t)\, \mathrm{d}\mu_0 < e_n(\xi) \cdot (\phi_2(\xi) - \nu(\xi)) + \frac{\nu(\xi)}{4k(\xi)}. \tag{6.7}$$

If $e_n(\xi) = 0$, then according to Proposition 2.1(ii) $\mathcal{B}_n(\xi) = \emptyset$. Therefore, in any case, inequality (6.7) is true.

From the inequality $d^s(v_n(\xi)(t)) \leqslant r_n(v_n(\xi))(t)$ a.e., it follows that

$$\int_{\mathcal{B}_n(\xi)} d^s(v_n(\xi)(t)) d\mu_0 < e_n(\xi)(\phi_2(\xi) - \nu(\xi)) + \frac{\nu(\xi)}{4k(\xi)}. \tag{6.8}$$

By using (5.8) and (6.8), we obtain

$$\begin{aligned} D^s(v(\xi)) &= \sum_{\{n;\ \mathcal{B}_n(\xi) \neq \emptyset\}} \int_{\mathcal{B}_n(\xi)} d^s(v_n(\xi)(t))\, \mathrm{d}\mu_0 \\ &< \phi_2(\xi) - \nu(\xi) + \frac{\nu(\xi)}{4} < \phi_2(\xi), \quad \xi \in M. \end{aligned}$$

Therefore, inequality (6.2) is proved.

7. Main Results

Let $C(M, L_p)$ be the space of all continuous, bounded functions from M to $L_p(T, X)$ with the topology of uniform convergence on M. Take a continuous in the Hausdorff metric $h_p(\cdot, \cdot)$, bounded on M multifunction Γ: $M \mapsto \mathrm{d}ccbL_p(T, X)$, possessing the property **(H1)**.

Denote by $C(\overline{\text{dec}}\,\text{st}\,\Gamma)$ and $C(\Gamma)$ the families of all continuous functions from M to $L_p(T, X)$ that are selectors of maps $\overline{\text{dec}}\,\text{st}\,\Gamma$ and Γ respectively with the topology of uniform convergence on M. As it follows from Proposition 6.5, $C(\overline{\text{dec}}\,\text{st}\,\Gamma)$ and $C(\Gamma)$ are the nonempty sets.

It is trivial that $C(\overline{\text{dec}}\,\text{st}\,\Gamma)$ and $C(\Gamma)$ are the closed subsets of the space $C(M, L_p)$. So $C(\overline{\text{dec}}\,\text{st}\,\Gamma)$ and $C(\Gamma)$ are the complete metric spaces.

Fix a dense, σ-compact subset K of M.

THEOREM 7.1. *Let a continuous in the Hausdorff metric $h_p(\cdot,\cdot)$, bounded on M multifunction Γ: $M \mapsto \text{dccb}L_p(T, X)$ possess the property* **(H1)**.

Then for any $u(\cdot) \in C(\overline{\text{dec}}\,\text{st}\,\Gamma)$ and any lower semicontinuous function ϕ: $M \mapsto (0, +\infty)$ there exists $v(\cdot) \in C(\overline{\text{dec}}\,\text{st}\,\Gamma)$ such that

$$\|u(\xi) - v(\xi)\|_p < \phi(\xi), \quad \xi \in M, \tag{7.1}$$

$$v(\xi) \in \text{ext}\,\Gamma(\xi), \quad \xi \in K. \tag{7.2}$$

If M is σ-compact, then inclusion (7.2) *is true for all $\xi \in M$.*

Let us subdivide the proof of Theorem 7.1 into several steps.

For an L_p-continuous selector $u(\cdot) \in C(\overline{\text{dec}}\,\text{st}\,\Gamma)$ and a function ϕ: $M \mapsto (0, +\infty)$ denote by $\mathcal{H}_{u,\phi}$ the closure in $C(\overline{\text{dec}}\,\text{st}\,\Gamma)$ of the set of all L_p-continuous selectors $v(\cdot)$ of the multifunction $\overline{\text{dec}}\,\text{st}\,\Gamma(\xi)$, satisfying the inequality

$$\|u(\xi) - v(\xi)\|_p < \frac{\phi(\xi)}{2}, \quad \xi \in M. \tag{7.3}$$

Obviously, $\mathcal{H}_{u,\phi}$ is a complete closed subset of $C(\overline{\text{dec}}\,\text{st}\,\Gamma)$. Fix $\eta > 0$ and $s \geqslant 1$. Set

$$\mathcal{H}_\eta^s = \{v(\cdot) \in \mathcal{H}_{u,\phi};\ D^s(v(\xi)) < \eta,\ \xi \in K\}.$$

LEMMA 7.2. *The set $\mathcal{H}_\eta^s$ is a G_δ subset of the set $\mathcal{H}_{u,\phi}$.*

Proof. According to Theorem 6.6, the set $\mathcal{H}_\eta^s$ is nonempty. Let $K = \bigcup_{n=1}^\infty K_n$. Denote by $\mathcal{H}_\eta^s(n)$ the set

$$\mathcal{H}_\eta^s(n) = \{v(\cdot) \in \mathcal{H}_{u,\phi};\ D^s(v(\xi)) < \eta,\ \xi \in K_n\}, \quad n \geqslant 1.$$

Then

$$\mathcal{H}_\eta^s = \bigcap_{n=1}^\infty \mathcal{H}_\eta^s(n).$$

The lemma will be proved if we show that $\mathcal{H}_\eta^s(n)$, $n \geqslant 1$, are open subsets of $\mathcal{H}_{u,\phi}$.

Fix $n \geqslant 1$. It is enough to prove that the set $\mathcal{H}_{u,\phi} \backslash \mathcal{H}_\eta^s(n)$ is closed in $\mathcal{H}_{u,\phi}$. Let $u_k(\cdot) \in \mathcal{H}_{u,\phi} \setminus \mathcal{H}_\eta^s(n)$ be an arbitrary sequence converging to $v(\cdot) \in \mathcal{H}_{u,\phi}$. Then, for each $k \geqslant 1$, there exists a point $\xi_k \in K_n$ such that $D^s(u_k(\xi_k)) \geqslant \eta$. Since the set K_n is compact, passing to a subsequence (without change of notation), we can assume that the sequence ξ_k, $k \geqslant 1$, converges to a point $\xi \in K_n$. As $u_k(\xi_k)$, $k \geqslant 1$, converges to $v(\xi)$, then according to Lemma 5.1, $D^s(u_k(\xi_k)) \geqslant \eta$. Therefore, $v(\cdot) \in \mathcal{H}_{u,\phi} \setminus \mathcal{H}^s_\eta(n)$ and the set $\mathcal{H}^s_\eta(n)$ is open in $\mathcal{H}_{u,\phi}$.

LEMMA 7.3. *The set $\mathcal{H}^s_\eta$ is a dense subset of the set $\mathcal{H}_{u,\phi}$.*

Proof. Let $w(\cdot) \in \mathcal{H}_{u,\phi}$ and $\epsilon > 0$ be arbitrary. According to the definition of $\mathcal{H}_{u,\phi}$ there exists $v_1(\cdot) \in C(\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma)$ such that

$$\|u(\xi) - v_1(\xi)\|_p < \phi(\xi)/2, \qquad \|v_1(\xi) - w(\xi)\|_p < \epsilon, \quad \xi \in M.$$

Set

$$d(\xi) = \min\{\phi(\xi)/2 - \|u(\xi) - v_1(\xi)\|_p, \epsilon - \|v_1(\xi) - w(\xi)\|_p\}.$$

The function $d(\xi)$ is lower semicontinuous and $d(\xi) > 0$, $\xi \in M$. By Lemma 3.6 [15], there exists a continuous function $c(\xi)$, $0 < c(\xi) < d(\xi)$, $\xi \in M$. As follows from Theorem 6.6, there exists an L_p-continuous selector $v(\cdot)$ of the multifunction $\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma$ such that

$$\|v_1(\xi) - v(\xi)\|_p < c(\xi), \qquad D^s(v(\xi)) < \eta, \quad \xi \in M.$$

Then

$$\begin{aligned}
&\|u(\xi) - v(\xi)\|_p \\
&\quad \leqslant \|u(\xi) - v_1(\xi)\|_p + \|v_1(\xi) - v(\xi)\|_p < \|u(\xi) - v_1(\xi)\|_p + c(\xi) \\
&\quad < \|u(\xi) - v_1(\xi)\|_p + \phi(\xi)/2 - \|u(\xi) - v_1(\xi)\|_p < \phi(\xi)/2, \quad \xi \in M.
\end{aligned}$$

Analogously, one can obtain that

$$\|w(\xi) - v(\xi)\|_p < \epsilon, \quad \xi \in M.$$

Therefore, $v(\cdot) \in \mathcal{H}^s_\eta$ and $\mathcal{H}^s_\eta$ is a dense subset of the set $\mathcal{H}_{u,\phi}$.

LEMMA 7.4. *The set $\bigcap_{n\geqslant 1}\bigcap_{s\geqslant 1}\mathcal{H}^s_{1/n}$ is a dense subset of the set $\mathcal{H}_{u,\phi}$.*

The result follows from Lemmas 7.2, 7.3 and the Baire category theorem [29].

Proof of Theorem 7.1. Let $v(\cdot) \in \bigcap_{n\geqslant 1}\bigcap_{s\geqslant 1}\mathcal{H}^s_{1/n}$. Then $D^s(v(\xi)) < 1/n$ for any $\xi \in K$, $s \geqslant 1$. Therefore, $D^s(v(\xi)) = 0$, $\xi \in K$, $s \geqslant 1$, and, as follows from Lemma 5.1, $v(\xi) \in \mathrm{ext}\,\Gamma(\xi)$, $\xi \in K$. Since $v(\cdot) \in \mathcal{H}_{u,\phi}$, then

$$\|u(\xi) - v(\xi)\|_p \leqslant \phi(\xi)/2 < \phi(\xi), \quad \xi \in M$$

and $v(\cdot) \in C(\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma)$.

If M is σ-compact, we can put $K = M$. This concludes the proof.

COROLLARY 7.5. *Let a multifunction* $\Gamma\colon M \mapsto \mathrm{dccb}L_p(T,X)$ *satisfy all the assumptions of Theorem* 7.1.

Then there exists an L_p*-continuous selector* $v(\cdot)$ *of* Γ *such that*

$$v(\xi) \in \operatorname{ext}\Gamma(\xi), \quad \xi \in K, \tag{7.4}$$

$$v(\xi) \in \overline{\operatorname{ext}}\,\Gamma(\xi), \quad \xi \in M \setminus K. \tag{7.5}$$

If M *is* σ*-compact, then inclusion* (7.4) *holds for every* $\xi \in M$.

Since

$$\overline{\operatorname{dec}}\,\operatorname{st}\Gamma(\xi) \subset \overline{\operatorname{ext}}\,\Gamma(\xi), \quad \xi \in M,$$

the result follows immediately from Theorem 7.1.

THEOREM 7.6. *Let a multifunction* $F\colon M \mapsto \mathrm{dcb}L_p(T,X)$ *be such that the multifunction* $\Gamma(\xi) = \overline{\operatorname{co}}\,F(\xi)$, $\xi \in M$, *satisfies all the assumptions of Theorem* 7.1.

Then there exists an L_p*-continuous selector* $u(\cdot)$ *of the multifunction* F, *such that*

$$u(\xi) \in \operatorname{ext}\overline{\operatorname{co}}\,F(\xi), \quad \xi \in K, \tag{7.6}$$

$$u(\xi) \in \overline{\operatorname{ext}}\,\overline{\operatorname{co}}\,F(\xi), \quad \xi \in M \setminus K.$$

If M *is* σ*-compact, then inclusion* (7.6) *holds for every* $\xi \in M$.

Proof. From Theorem 7.1 and Corollary 7.5, it follows that there exists a continuous selector $u(\cdot)$ of the multifunction $\overline{\operatorname{dec}}\,\operatorname{st}\Gamma$ such that inclusions (7.4), (7.5) are true. As is proved in Proposition 6.2, $\operatorname{st}\Gamma(\xi) \subset F(\xi)$, $\xi \in M$. Since F has decomposable, closed values, then $\overline{\operatorname{dec}}\,\operatorname{st}\Gamma(\xi) \subset F(\xi)$, $\xi \in M$. This proves Theorem 7.6.

In Theorem 7.1, it is assumed that the multifunction $\Gamma\colon M \mapsto \mathrm{dccb}L_p(T,X)$ is bounded on M. If M is a locally compact, separable metric space, the condition of the boundedness of Γ on M can be omitted.

THEOREM 7.7. *Let a continuous in the Hausdorff metric* $h_p(\cdot,\cdot)$ *multifunction* $\Gamma\colon M \mapsto \mathrm{dccb}L_p(T,X)$ *have the property* **(H1)**, *and* M *be a locally compact, separable metric space.*

Then for any $u(\cdot) \in C(\overline{\operatorname{dec}}\,\operatorname{st}\Gamma)$ *and any lower semicontinuous function* $\phi\colon M \mapsto (0,+\infty)$, *there exists* $v(\cdot) \in C(\overline{\operatorname{dec}}\,\operatorname{st}\Gamma)$ *such that*

$$\|u(\xi) - v(\xi)\|_p < \phi(\xi), \quad \xi \in M,$$

$$v(\xi) \in \operatorname{ext}\Gamma(\xi), \quad \xi \in M.$$

Proof. Denote by $C_c(M, L_p)$ the space of all continuous functions from M to $L_p(T, X)$ with the topology of uniform convergence on compact subsets of M. Then $C_c(M, L_p)$ is a metrisable, complete metric space. Later, all the sets that are used in the proof of Theorem 7.1 (in particular, the sets $C(\overline{\mathrm{dec}\,\mathrm{st}\,\Gamma})$, $C(\Gamma)$) should be considered as subsets of $C_c(M, L_p)$. Since M is σ-compact space, Theorem 7.7 can be proved analogously to Theorem 7.1 with obvious transformations.

8. Some Examples

In this section we present some examples of multifunctions which are used in the theory of multivalued equations.

A multifunction $F\colon T \times X \mapsto cb(X)$ is said to be of the Caratheodory type if it is measurable with respect to t for every $x \in X$ and continuous with respect to x for almost every $t \in T$.

Let $C(T, X)$ be the space of all continuous functions from T to X with the topology of the uniform convergence on T. Suppose that, a.e. on T, for every $x \in X$

$$\|F(t, x)\| \leqslant m(t) + n(t)\|x\|, \tag{8.1}$$

where $m, n \in L_p(T, R)$.

Take a fixed compact $M \subset C(T, X)$.

PROPOSITION 8.1. *Let $F\colon T \times X \mapsto cwk(X)$ be a multifunction of the Caratheodory type, satisfying inequality* (8.1). *Then there exists a continuous function $g\colon M \mapsto L_p(T, X)$ such that for every $x(\cdot) \in M$ a.e. on T*

$$g(x)(t) \in \operatorname{ext} F(t, x(t)). \tag{8.2}$$

Proof. It is easy to prove that the multifunction $F(t, x(t))$ is measurable for any $x(\cdot) \in M$. Set

$$\Gamma(x(\cdot)) = \{f(\cdot) \in L_p(T, X);\ f(t) \in F(t, x(t)) \text{ a.e.}\}, \quad x(\cdot) \in M.$$

From Proposition 4.2, Theorem 4.4 and inequality (8.6), it follows that $\Gamma(x(\cdot))$ is a continuous in the Hausdorff metric $h_p(\cdot, \cdot)$ multifunction from M to $\mathrm{d}cwkL_p(T, X)$. By Theorem 7.1, there exists a L_p-continuous selector $g(\cdot)$ of multifunction $\Gamma(\cdot)$ satisfying

$$g(x(\cdot)) \in \operatorname{ext} \Gamma(x(\cdot)), \quad x \in M. \tag{8.3}$$

Using Corollary 5.2, one proves the result.

PROPOSITION 8.2. *Let* $F\colon\ T \times X \mapsto cb(X)$ *be a multifunction such that* $\overline{\mathrm{co}}\, F(t,x)$ *satisfies all the assumptions of Proposition* 8.1.

If for every $x(\cdot) \in M$ *the multifunction* $F(t, x(t))$ *is measurable, then there exists a continuous function* $g\colon\ M \mapsto L_p(T,X)$ *such that, for every* $x(\cdot) \in M$ *a.e. on* T,

$$g(x)(t) \in F(t, x(t)).$$

Moreover,

$$g(x)(t) \in \mathrm{ext}\,\overline{\mathrm{co}}\, F(t, x(t)). \tag{8.4}$$

If $F\colon\ T \times X \mapsto ck(X)$, *then the requirement of the measurability of* $F(t, x(t))$ *can be omitted.*

Proof. Set

$$\mathcal{F}(x(\cdot)) = \{f(\cdot) \in L_p(T,X);\ f(t) \in F(t,x(t)) \text{ a.e.}\}, \quad x(\cdot) \in M,$$

and

$$\Gamma(x(\cdot)) = \{f(\cdot) \in L_p(T,X);\ f(t) \in \overline{\mathrm{co}}\, F(t,x) \text{ a.e.}\}, \quad x(\cdot) \in M.$$

By Theorem 1.5 in [27], $\Gamma(x(\cdot)) = \overline{\mathrm{co}}\,\mathcal{F}(x(\cdot))$, $x(\cdot) \in M$, and $\mathcal{F}(x(\cdot)) \subset \mathrm{d}cbL_p(T,X)$. As was established above, $\Gamma(x(\cdot))$ is a continuous in the Hausdorff metric $h_p(\cdot,\cdot)$ multifunction from M to $\mathrm{d}cwkL_p(T,X)$. Then, by Theorem 7.1, there exists a L_p-continuous selector $g(\cdot)$ of the multifunction $\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma(\cdot)$ such that inclusion (8.3) holds. By Proposition 6.2, we have that

$$\overline{\mathrm{dec}}\,\mathrm{st}\,\Gamma(\cdot) \subset \mathcal{F}(x(\cdot)), \quad x(\cdot) \in M.$$

From this and Corollary 5.2, we obtain the conclusion of Proposition 8.2.

If $F\colon\ T \times X \mapsto ck(X)$, then, according to Proposition 8.1, there exists a continuous function $g\colon\ M \mapsto L_p(T,X)$ such that inclusion (8.4) holds for every $x(\cdot) \in M$ a.e. on T.

Since $\mathrm{ext}\,\overline{\mathrm{co}}\, F(t,x(t)) \subset F(t,x)$, then the result is also true in this case.

Now consider an example of a multifunction satisfying all the hypotheses of Proposition 8.2.

Set $X = R^2$, $T = [0,1]$. For all $x = (x_1, x_2)$, $\|x\| \neq 0$ let the set $F_1(x)$ on the plane (v_1, v_2) be the ellipse arc

$$v_1 = \cos\phi, \qquad v_2 = \|x\|/(1 + \|x\|)\cdot \sin\phi,$$

$$\|x\|^{-1} \leqslant \phi \leqslant \|x\|^{-1} + 2\pi - \|x\|/(1 + \|x\|),$$

and at the point $x = (0,0)$ the set $F_1(0)$ consists of two points: $(-1,0)$, $(1,0)$.

Denote by $\partial\,\overline{\mathrm{co}}\, F_1(x)$ the boundary of the set $\overline{\mathrm{co}}\, F_1(x)$, and by Q the set of all points $x = (x_1, x_2)$ with rational coordinates. Consider the mapping $F\colon\ T \times X \mapsto$

X defined by the rule:

$F(t,x) = \lambda(t) \cdot F_1(x)$ at $x \in Q$,
$F(t,x) = \lambda(t) \cdot \partial \overline{\text{co}} F_1(x)$ at $x \in X \setminus Q$, where $\lambda(t) > 0$ is defined for every $t \in T$ and p-integrable on T.

For every fixed t, the multifunction F is lower semicontinuous at every point $x \in Q$, upper semicontinuous at every $x \in X \setminus Q$, and

$$\|F(t,x)\| \leqslant \sqrt{2}\lambda(t), \ t \in T, \quad x \in X.$$

Consequently, the multifunction $F(t,x)$ does not possess the property of lower semicontinuity at every point $x \in X$. However, $\overline{\text{co}} F(t,x)$ is a Caratheodory-type mapping.

Take as fixed $x(\cdot) \in C(T,X)$. Denote by T_0 the set of all $t \in T$ such that $x(t)$ is a point with rational coordinates. Then T_0 is an F_σ set and the restriction of $F_1(t,x(t))$ to T_0 is a lower semicontinuous multifunction. Hence, $F(t,x(t))$ is measurable on T_0. Analogously, the restriction of $F(t,x(t))$ to $T \setminus T_0$ is a measurable multifunction. Therefore, for every $x(\cdot) \in C(T,X)$ the multifunction $F(t,x(t))$ is measurable.

Then the multifunction $F(t,x)$ satisfies the hypotheses of Proposition 8.2.

9. Comments

In the present paper, the main results were obtained by the development of the method based on the Baire category theorem.

It should be mentioned that the idea of applying the Baire category theorem to differential inclusions in R has appeared in [10]. Subsequently, this method has been developed in [14, 16, 17] for proving the existence of solutions to the Cauchy problem for nonconvex-valued differential inclusions in Banach spaces. Further contributions concerning the existence of extreme solutions and relaxation theorems can be found in [4, 6, 7, 8, 18, 19, 33, 32, 34].

The existence of extreme continuous selectors and relaxation theorems were realized for the first time by the Baire category method in [34] for some class of multivalued maps. The present paper is devoted to the development of the results [34] for more general classes of multifunctions. It should be mentioned that the technique based on the Baire category theorem has been used in [4] in order to prove the existence of so-called directionally continuous selectors for a special class of multifunctions with nonclosed, nonconvex values.

(1) Proposition 2.1 was proved in [26] in order to obtain joint continuous selectors for a finite set of multifunctions and can be used in different fields.

(2) The functions under consideration in Section 3 were first introduced in [32] for proving the existence of extremal solutions and the relaxation theorems for differential inclusions in R^n. The functions from Section 3 have similar properties

to the Choquet function [9, 11]. It should be mentioned that the Choquet function has a restricted field of applicability and can be used only in separable, reflexive Banach spaces.

(3) Theorem 6.6 is proved by using some ideas of [26].

(4) The proof of Theorem 7.1 is obtained in the standard way [14, 16, 17], if we apply the method based on the Baire category theorem.

(5) In Section 8, we used some ideas from [35] for the construction of the example.

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