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Lattice Operators Underlying Dynamic Systems

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Abstract. This paper investigates algebraic and continuity properties of increasing set operators underlying dynamic systems. We recall algebraic properties of increasing operators on complete lattices and some topologies used for the study of continuity properties of lattice operators. We apply these notions to several operators induced by a differential equation or differential inclusion. We especially focus on the operators associating with any closed subset its reachable set, its exit tube, its viability kernel or its invariance kernel. Finally, we show that morphological operators used in image processing are particular cases of operators induced by constant differential inclusion.

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1. Introduction

A *complete lattice* [15, 16, 6] is a partial ordered set such that every subset $\mathcal{H}$ has a supremum and an infimum denoted by $\vee\mathcal{H}$ and $\wedge\mathcal{H}$. As an important example for the following, we mention, that the power space $\mathcal{F}(X)$, the space of all closed subsets of X supplied with the inclusion order, is a complete lattice and the supremum and the infimum are given by

$$\forall\mathcal{H} \subset \mathcal{F}(X), \quad \vee\mathcal{H} = \overline{\bigcup_{K\in\mathcal{H}} K}, \qquad \wedge\mathcal{H} = \bigcap_{K\in\mathcal{H}} K.$$

By an operator, we mean a mapping of a complete lattice into itself.

In this paper, we focus on set operators defined on a complete lattice induced by a differential inclusion (in particular, on the power space $\mathcal{P}(X)$, or on the space $\mathcal{F}(X)$ of closed subsets of X).

An *algebraic dilation* δ distributes over suprema, and dually an *algebraic erosion* ε distributes over infima. The pair (ε, δ) is called an *adjunction* if for all x, y, we have $\delta(y) \leqslant x \Leftrightarrow y \leqslant \varepsilon(x)$. It is well known [32, 12] that if the operators ε and δ constitute an adjunction, then ε is an erosion, and δ a dilation. Furthermore, to every dilation there corresponds a unique erosion such that the pair forms an adjunction; dually, to every erosion can be associated a unique dilation that both operators constitute an adjunction. The composite operators have such algebraic properties as increasing, extensive (or anti-extensive) and idempotent. An operator which is increasing extensive (resp. anti-extensive) and idempotent is an *algebraic closing* (resp. an *algebraic opening*). From a theoretic point of view, the dilation, the erosion, the adjunction, the opening and the closing are the most important algebraic notions in mathematical morphology.

A differential inclusion $x'(t) \in F(x(t))$ where F: $X \rightsquigarrow X$ is a set-valued map (i.e. $F(x) \subset X$), corresponds to a generalisation [4, 5] of the notion of the differential equation $x'(t) = f(x(t))$ where the dynamic is multivalued and nondeterministic. The main example is given by the control system $F(x) = \{f(x,u)\}_{u \in U(x)}$.

Many set operators [2] can be deduced from the framework of differential inclusions. For example, we associate with any set-valued map F the set $\mathcal{S}(x_0)$, called the *reachable set*, of solutions to the differential inclusion $x'(t) \in F(x(t))$ starting from x_0 and the reachable map ϑ_F defined by

$$\vartheta_F(h, x_0) \;:=\; \{x(h)\}_{x(\cdot) \in \mathcal{S}(x_0)},$$

where $x(\cdot)$ range over $\mathcal{S}(x_0)$. Then, we show that under adequate assumptions, the operator $K \mapsto \vartheta_F(h, K) = \bigcup_{x_0 \in K} \vartheta_F(h, x_0)$ commutes with the supremum, and is an algebraic dilation. The set $\mathrm{Exit}_F(K,t)$ (resp. $\mathrm{EXIT}_F(K,t)$) is the subset of initial states $x \in K$ such that one solution (resp. all solutions) $x(\cdot)$ to differential inclusion $x'(t) \in F(x(t))$ starting at x_0 remains in K for all time in $[0,t]$. The operator Viab_F which, with any closed subset K associated with its 'viability kernel', corresponds to the $\mathrm{Exit}_F(.,\infty)$ operator. Furthermore, it is *increasing anti-extensive and idempotent*. We deduce that it is an *algebraic opening*. In the same way, the operator Inv_F which, with any closed subset K associated with its 'invariance kernel', corresponds to the $\mathrm{EXIT}_F(.,\infty)$ operator and it is an algebraic opening which commutes with the infimum.

We deduce from these previous results some continuity properties of these operators.

In the last part, we will study a particular example: the differential constant inclusion $x'(t) \in B$, where B is a symmetrical compact convex set. As a particular case of the previous results, we provide algebraic and continuity properties of the induced operators which appear to be usual morphological operators where B is usually called *the structuring element.*

2. Lattice Framework

In this section, we briefly recall some basic notions and some results that we shall use later on. For more details on lattice theory, consult [15, 16, 6].

2.1. ALGEBRAIC DEFINITIONS AND PROPERTIES ON A COMPLETE LATTICE

By an *operator*, we shall mean a mapping of a complete lattice $\mathcal{L}$ into a complete lattice $\mathcal{M}$.

DEFINITION 2.1. An operator ψ: $\mathcal{L} \mapsto \mathcal{M}$ is: *increasing* if $A \leqslant B$ implies $\psi(A) \leqslant \psi(B)$, *extensive* (resp. *antiextensive*) if $\mathcal{L} = \mathcal{M}$ and $\psi(A) \geqslant A$ (resp. $\psi(A) \leqslant A$), and *idempotent*, if $\mathcal{L} = \mathcal{M}$ and $\psi^2 = \psi$.

It is obvious that if ψ is an increasing operator, then for any family $K_i \in \mathcal{L}$, we have

$$\psi(\bigvee K_i) \geqslant \bigvee \psi(K_i) \quad \text{and} \quad \psi(\bigwedge K_i) \leqslant \bigwedge \psi(K_i).$$

If $\mathcal{L}$ and $\mathcal{M}$ are lattices included in Boolean lattices* (for example, $\mathcal{F}(X)$ in $\mathcal{P}(X)$), then every element K in $\mathcal{L}$ (resp. in $\mathcal{M}$) has a unique complement in the Boolean lattice which we denote by K^c (in $\mathcal{P}(X)$, we have $K^c = X \setminus K$). The *dual operator*** of an operator ψ: $\mathcal{L} \mapsto \mathcal{M}$ is given by ψ^*: $\mathcal{L}^* \mapsto \mathcal{M}^*$, where $\psi^*(K) = (\psi(K^c))^c$ and $\mathcal{L}^* = \{K^c \mid K \in \mathcal{L}\}$. It is clear that ψ^* is increasing if and only if ψ is, ψ^* is idempotent if and only if ψ is, and ψ^* is extensive if and only if ψ is anti-extensive.

For example, the space $\mathcal{F}(X)$ of all closed subsets of X and the space $\mathcal{F}^*(X)$ of all open subsets of X are duals, and we have

$$\forall \mathcal{H} \subset \mathcal{F}(X), \quad \vee \mathcal{H} = \overline{\bigcup_{K \in \mathcal{H}} K}, \quad \wedge \mathcal{H} = \bigcap_{K \in \mathcal{H}} K,$$

$$\forall \mathcal{H} \subset \mathcal{F}^*(X), \quad \vee \mathcal{H} = \bigcup_{K \in \mathcal{H}} K, \quad \wedge \mathcal{H} = \operatorname{Int}\left(\bigcap_{K \in \mathcal{H}} K\right),$$

where $\operatorname{Int}(K)$ denotes the interior of K.

2.1.1. *Algebraic Dilation and Erosion*

DEFINITION 2.2. We say [32, 18, 13] that ψ is an *algebraic dilation* (resp. an *algebraic erosion*) if ψ distributes over the suprema (resp. over the infima), i.e. $\psi(\bigvee K_i) = \bigvee \psi(K_i)$ (resp. $\psi(\bigwedge K_i) = \bigwedge \psi(K_i)$).

* In a complete lattice $\mathcal{L}$, there exists a smallest element denoted by $\emptyset$ and a greatest element denoted by E. If $x, y \in \mathcal{L}$ are such that $x \wedge y = \emptyset$ and $x \vee y = E$, the y is called a *complement* of x. A lattice $\mathcal{L}$ is called *complemented* if all elements in $\mathcal{L}$ have a complement. A Boolean lattice is a complemented distributive lattice, i.e. every element has a unique complement and such that the supremum distributes over the infimum and conversely the infimum distributes over the supremum.

** The lattice structure of the dual lattice $\mathcal{L}^* = \{K^c \mid K \in \mathcal{L}\}$ is induced by the Morgan's Laws.

DEFINITION 2.3 [13]. Let δ: $\mathcal{L} \mapsto \mathcal{M}$ and ε: $\mathcal{M} \mapsto \mathcal{L}$ be two operators on a complete lattice $\mathcal{L}$. Then we will say that (ε, δ) is an *adjunction* if for every $A \in \mathcal{L}$ and for every $B \in \mathcal{M}$, we have

$$\delta(A) \leqslant B \Leftrightarrow A \leqslant \varepsilon(B). \tag{1}$$

We will denote $\varepsilon^{\bowtie} = \delta$ and $\delta^{\bowtie} = \varepsilon$ when (ε, δ) is an adjunction.

If (ε, δ) is an adjunction, it follows automatically [17, 32, 13] that δ is an algebraic dilation and ε an algebraic erosion.

For example, in the case of Minkowski operations on $\mathcal{P}(\mathbb{R}^n)$, the subtraction and the addition by a set B form an adjunction.

An automorphism ψ of $\mathcal{L}$ is both an algebraic dilation and an algebraic erosion and (ψ^{-1}, ψ) and (ψ, ψ^{-1}) are adjunctions.

PROPOSITION 2.4 [32, 13].

1. *For any algebraic dilation* δ: $\mathcal{L} \mapsto \mathcal{M}$, *there exists a unique algebraic erosion* $\delta^{\bowtie}$: $\mathcal{M} \mapsto \mathcal{L}$ *such that* $(\delta^{\bowtie}, \delta)$ *is an adjunction.*
2. *For any algebraic erosion* ε: $\mathcal{M} \mapsto \mathcal{L}$, *there exists a unique algebraic dilation* $\varepsilon^{\bowtie}$: $\mathcal{L} \mapsto \mathcal{M}$ *such that* $(\varepsilon, \varepsilon^{\bowtie})$ *is an adjunction.*

2.1.2. *Algebraic Opening and Closing*

DEFINITION 2.5 [29]. We will say that ψ: $\mathcal{L} \mapsto \mathcal{L}$ is an *algebraic opening* (resp. an *algebraic closing*) if ψ is increasing, idempotent and anti-extensive (resp. extensive).

The topological opening $K \mapsto \mathrm{Int}(K)$ is an algebraic opening on $\mathcal{P}(X)$. The closed convex closure operator $K \mapsto \overline{\mathrm{co}}(K)$ is an algebraic closing on $\mathcal{F}(X)$.

PROPOSITION 2.6 [29]. *Given an adjunction* (ε, δ), *then we have* $\delta\varepsilon\delta = \delta$ *and* $\varepsilon\delta\varepsilon = \varepsilon$. *The operator* $\delta\varepsilon$ *is an algebraic opening and the operator* $\varepsilon\delta$ *is an algebraic closing.*

Remark 2.7. We observe that if ψ is an algebraic dilation (resp. an algebraic erosion), then its dual operator ψ^* is an algebraic erosion (resp. an algebraic dilation). Furthermore, if ψ is an algebraic opening (resp. an algebraic closing), then ψ^* is an algebraic closing (resp. an algebraic opening).

2.1.3. *Subset of Fixpoints of Algebraic Opening and Closing*

DEFINITION 2.8 (Subset of fixpoints). Let ψ be an operator on $\mathcal{L}$ and $K \in \mathcal{L}$. We say that K is a *fixpoint* of ψ if $\psi(K) = K$. The set of all fixpoints of ψ is called the subset of *fixpoints* of ψ and it is denoted by

$$\mathrm{Fix}(\psi) := \{K \in \mathcal{L} \mid \psi(K) = K\}.$$

It is obvious that $\text{Fix}(\psi)$ is ψ-closed, because $\forall K \in \text{Fix}(\psi)$ we have $\psi(K) = K$, and $\psi(\psi(K)) = \psi(K)$ implies that $\psi(K) \in \text{Fix}(\psi)$. Then, for every opening ψ: $\mathcal{L} \mapsto \mathcal{L}$, there is an associated subset of fix points. Since ψ is idempotent, $\text{Fix}(\psi)$ is nothing but the image of $\mathcal{L}$ under ψ, i.e. $\text{Fix}(\psi) = \psi(\mathcal{L})$.

THEOREM 2.9 (Tarski fixpoint theorem) [19]. *Let ψ be an increasing idempotent operator on $\mathcal{L}$. Then* $\text{Fix}(\psi)$ *is a nonempty complete lattice included in $\mathcal{L}$.*

Algebraic opening and closing are completely characterized by their subset of fixpoints.

PROPOSITION 2.10 [17]. *If ψ is an opening, then its subset of fixpoints is closed under suprema, i.e. if $K_i \in \text{Fix}(\psi)$ for $i \in I$, then $\bigvee_{i \in I} K_i \in \text{Fix}(\psi)$. Conversely, every subset $\mathcal{B}$ of $\mathcal{L}$ which is closed under suprema is the subset of fixpoints of a unique opening ψ given by*

$$\psi(K) = \bigvee \{B \in \mathcal{B} \mid B \subset K\}.$$

For example, in $\mathcal{P}(\mathbb{R}^n)$, the subset of fixpoints of the topological opening $K \mapsto \text{Int}(K)$ is the family of all open sets which is closed under union and invariant under translation. Moreover, the interior of a set K is the union of all open balls inside K, i.e. $\text{Int}(K) = \bigcup \{B \in \mathcal{B} \mid B \subset K\}$, *where $\mathcal{B}$ is the family of all open balls.*

PROPOSITION 2.11 [17]. *If ϕ is a closing, then its subset of fixpoints is closed under infima, that is if $K_i \in \text{Fix}(\psi)$ for $i \in I$, then $\bigwedge_{i \in I} K_i \in \text{Fix}(\psi)$. Conversely, every subset $\mathcal{B}$ of $\mathcal{L}$ which is closed under infima is the subset of fixpoints of unique closing ϕ given by*

$$\phi(K) = \bigwedge \{B \in \mathcal{B} \mid B \supset K\}.$$

For example, in $\mathcal{P}(\mathbb{R}^n)$, the subset of fixpoints of the closed convex closure $K \mapsto \overline{\text{co}}(K)$ is the family of all closed convex sets which is closed under intersection and invariant under translation. Moreover, the closed convex hull of a set K is the intersection of all closed half hyper-planes which contain K, i.e. $\overline{\text{co}}(K) = \bigcap \{B \in \mathcal{B} \mid K \subset B\}$, *where $\mathcal{B}$ is the family of all closed half hyper-planes.*

PROPOSITION 2.12 [29]. *Let ψ be an algebraic opening and θ be an increasing anti-extensive operator. Then the following four statements are equivalent:*

1. $\psi \leqslant \theta$, (i.e. $\forall K \in \mathcal{L},\ \psi(K) \leqslant \theta(K)$),
2. $\psi\theta = \psi$,
3. $\theta\psi = \psi$,
4. $\text{Fix}(\psi) \subset \text{Fix}(\theta)$.

From this proposition, it follows that an algebraic opening is uniquely determined by its subset of fixpoints.

COROLLARY 2.13 [29]. *Let* ψ_1 *and* ψ_2 *be two algebraic openings, then* $\psi_1 = \psi_2$ *if and only if* $\mathrm{Fix}(\psi_1) = \mathrm{Fix}(\psi_2)$.

2.2. ORDER CONTINUITY OF LATTICE OPERATORS

Throughout this section, we assume that $\mathcal{L}$ is a complete lattice.

DEFINITION 2.14 [14]. For a sequence K_n in $\mathcal{L}$ we define $\limsup K_n = \bigwedge_{N\geqslant 1} \bigvee_{n\geqslant N} K_n$.

It is clear that

- $\limsup(K_n \wedge L_n) \leqslant (\limsup K_n) \wedge (\limsup L_n)$,
- $\limsup(K_n \vee L_n) \geqslant (\limsup K_n) \vee (\limsup L_n)$.

For example, on the lattice $\mathcal{F}(X)$ of the closed subsets of X, the lim sup is given by

$$\limsup K_n := \bigcap_{N\geqslant 1} \left(\overline{\bigcup_{n\geqslant N} K_n} \right)$$

If we now consider the complete lattice $\mathcal{F}(X)$, where X is a topological space which is Hausdorff, locally compact and admits a countable base. On $\mathcal{F}(X)$, we can first define limits of sets introduced by Painlevé in 1902, and called the *Kuratowski upper limits* of sequences of sets:

DEFINITION 2.15 [5]. Let $(K_n)_{n\in\mathbf{N}}$ be a sequence of subsets of a metric space X. We say that the subset

$$\mathrm{Lim\,sup}_{n\to\infty} K_n := \left\{ x \in X \mid \liminf_{n\to\infty} d(x, K_n) = 0 \right\}$$

is the *upper limit* of the sequence K_n.

Upper limits are obviously *closed* and we have on $\mathcal{F}(X)$, $\mathrm{Lim\,sup} K_n = \limsup K_n$

It is easy to check that:

PROPOSITION 2.16 [5]. *If* $(K_n)_{n\in\mathbf{N}}$ *is a sequence of subsets of a metric space, then* $\mathrm{Lim\,sup}_{n\to\infty} K_n$ *is the set* of cluster points of sequences $x_n \in K_n$, i.e., *of limits of subsequences* $x_{n'} \in K_{n'}$.

A weak notion of continuity can be defined using the limits of sets as follows:

DEFINITION 2.17 [5]. Let ψ: $\mathcal{F}(X) \mapsto \mathcal{F}(X)$ be an operator. We say that ψ is *upper-semi-continuous* (u.s.c.) if $\mathrm{Lim\,sup}_{n\to\infty}\psi(K_n) \subseteq \psi(\mathrm{Lim\,sup}_{n\to\infty}K_n)$.

We can show that:

PROPOSITION 2.18 [17, 32, 12]. *Every increasing erosion is u.s.c. on $\mathcal{F}(X)$.*

3. Operators Induced by Differential Inclusion

In this section, we recall some operators induced by differential inclusions as the reachable map, the exit tube, the viability kernel map and the invariance kernel map. For more details on the differential inclusion theory and viability theory, see [4, 2] or [11].

3.1. DIFFERENTIAL INCLUSION, REACHABLE SET AND ACCESSIBILITY SET

Indeed, control systems provide the main example of differential inclusion, and are governed by a family differential equation

$$x'(t) = f(x(t), u(t)), \quad \text{where } u(t) \in U(x(t)).$$

The single-valued map f describes the dynamics of the system: it associates with state x of the system and the control u with the velocity $f(x, u)$ of the system. The set-valued map U describes a feedback map assigning to the state x the subset $U(x)$ of admissible controls. If we put $F(x) := f(x, U(x)) = \{f(x, u)\}_{u\in U(x)}$, then the control system is governed by the differential inclusion

$$x'(t) \in F(x(t)).$$

More generally, let F: $X \rightsquigarrow X$ be a set-valued map from the vector space X to itself. We define the notion of solution of differential inclusion $x'(t) \in F(x(t))$ as follows:

DEFINITION 3.1. We denote by $\mathcal{S}_F(x_0)$ the set of solutions $x(\cdot)$ to the differential inclusion:

$$\forall t \in I, \quad x'(t) \in F(x(t)), \qquad x(0) = x_0 \tag{2}$$

starting at the initial state x_0, where $x(\cdot)$: $I \mapsto X$ is an absolutely continuous function (i.e. $x(\cdot) \in W^{1,1}(0, \infty, X)$).

We also denote by $\vartheta_F(h, x_0)$ the set of the values $x(h)$ at time h of the solutions x of (2).

DEFINITION 3.2. For all subsets $K \subset X$, $\vartheta_F(h, K) = \bigcup_{y\in K} \vartheta_F(h, y)$ is the *reachable set* from K at time h of F.

The *reachable map* $t \rightsquigarrow \vartheta_F(t, x_0)$ enjoys the semi-group property: $\forall t$, $s \geqslant 0$, $\vartheta_F(t+s, x_0) = \vartheta_F(t, \vartheta_F(s, x_0))$.

DEFINITION 3.3. The set-valued map $\mathrm{Acc}_F(t,\cdot)$: $y \rightsquigarrow \mathrm{Acc}_F(t,y) = \bigcup_{s\leqslant t} \vartheta_F(s,y)$ is called the *accessibility map for F at t* [2]. The *accessibility tube* of K is the set-valued map $t \to \mathrm{Acc}_F(t,K) = \bigcup_{y\in K} \mathrm{Acc}_F(t,y)$.

3.2. EXIT TUBE

Let K be a closed subset of X and $x(\cdot)$: $[0,+\infty[\mapsto X$ be a continuous function. We denote by τ_K the *exit functional* associating with $x(\cdot)$ its *exit time* $\tau_K(x(\cdot))$ defined in [2] (See Figure 1) as

$$\tau_K(x(\cdot)) := \bigwedge \{t \in [0,+\infty[\mid x(t) \notin K\}.$$

It is obvious that $\forall t \in [0, \tau_K(x(\cdot))[$, $x(t) \in K$, and if $\tau_K(x(\cdot))$ is finite, then $x(\tau_K(x(\cdot))) \in \partial K$, where ∂K denotes the boundary of K.

Then we can associate the function $\tau_K^\sharp$: $K \mapsto \mathbb{R}_+ \cup \{+\infty\}$ defined by $\tau_K^\sharp$: $K \mapsto \mathbb{R}_+ \cup \{+\infty\}$ (resp. $\tau_K^\flat$: $K \mapsto \mathbb{R}_+ \cup \{+\infty\}$) which is defined by

$$\tau_K^\sharp(x_0) := \bigvee_{x(\cdot)\in\mathcal{S}_F(x_0)} \tau_K(x(\cdot)),$$

$$\left(\text{resp. } \tau_K^\flat(x_0) := \bigwedge_{x(\cdot)\in\mathcal{S}_F(x_0)} \tau_K(x(\cdot))\right)$$

and is called the *exit function* (resp. the *global exit function*).

When the set-valued dynamic is sufficiently regular (Marchaud), in a sense we precise below, the exit functions satisfy some continuity properties.

We denote by $\|F(x)\| := \bigvee_{y\in F(x)} \|y\|$ and say that F has *linear growth* if there exists a positive constant c such that: $\forall x \in \mathrm{Dom}(F)$, $\|F(x)\| \leqslant c(\|x\|+1)$.

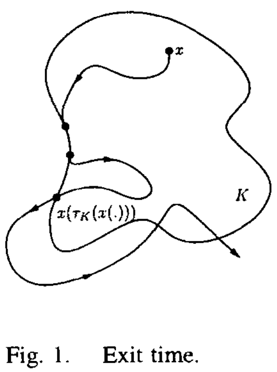

Fig. 1. Exit time.

We shall say that F is a *Marchaud map* if it is upper semicontinuous, has compact convex images and linear growth. A set-valued map F: $X \rightsquigarrow X$ is *Lipschitz around* $x \in \mathrm{Dom}(F)$ if there exist a positive constant λ and a neighborhood $\mathcal{U}$ of x such that

$$\forall x_1 x_2 \in \mathcal{U}, \quad F(x_1) \subset F(x_2) + \lambda \|x_1 - x_2\| B,$$

where B is the unit ball of X. In this case, F is also called Lipschitz (or λ-Lipschitz) on $\mathcal{U}$.

PROPOSITION 3.4 [2]. *Let* F: $X \rightsquigarrow X$ *be a Marchaud map, and* $K \subset X$ *be a closed subset. Then the exit function* $\tau_K^\sharp$ *is upper semicontinuous. Furthermore, if* F *is Lipschitz, then the global exit function* $\tau_K^\flat$ *is upper semicontinuous.*

DEFINITION 3.5. If F is a Marchaud (resp. Lipschitz) set-valued map, we can respectively associate the two closed subsets

$$\mathrm{Exit}_F(K, t) := \{x_0 \in K \mid \tau_K^\sharp(x_0) \geqslant t\},$$
$$\mathrm{EXIT}_F(K, t) := \{x_0 \in K \mid \tau_K^\flat(x_0) \geqslant t\},$$

with any $t \geqslant 0$.

We shall say that the set-valued map $t \mapsto \mathrm{Exit}_F(K, t)$ (resp. $t \mapsto \mathrm{EXIT}_F(K, t)$) is the *exit tube* (resp. the *global exit tube*).

When $t_1 \leqslant t_2$, then

$$\mathrm{Exit}_F(K, t_2) \subseteq \mathrm{Exit}_F(K, t_1) \subseteq \cdots \subseteq \mathrm{Exit}_F(K, 0) = K.$$

3.3. VIABILITY KERNEL AND INVARIANCE KERNEL

Let K be a subset of the domain of F. A function $x(\cdot)$: $I \mapsto X$ is said to be *viable* in K on the *interval* $I \subset \mathbb{R}^+$ if and only if

$$\forall t \in I, \quad x(t) \in K.$$

DEFINITION 3.6 (Viability Kernel) [2]. Let F be a Marchaud set-valued map and K be a closed subset. The subset $\mathrm{Exit}_F(K, +\infty) = \bigcap_{t \geqslant 0} \mathrm{Exit}_F(K, t)$ is called the *viability kernel* of K for F, denoted by $\mathrm{Viab}_F(K)$. It is the subset of initial states such that at least one solution of the differential inclusion (2) starting from them is viable in K, i.e.

$$\mathrm{Viab}_F(K) = \{x_0 \in K \mid \exists x(\cdot) \text{ solution of (2) such that } x(t) \in K\}.$$

We also can introduce the concept of invariance kernels:

DEFINITION 3.7 (Invariance Kernel). Let F be a Lipschitz Marchaud set-valued map and K be a closed subset. We shall say that

$$\mathrm{EXIT}_F(K,+\infty)=\bigcap_{t\geqslant 0}\mathrm{EXIT}_F(K,t)$$

is called the *invariance kernel* of K for F, denoted by $\mathrm{Inv}_F(X)$. It is the subset of the initial states such that any solution of the differential inclusion (2) starting from them is viable in K, i.e.

$$\mathrm{Inv}_F(X)=\{x_0\in K\mid \forall x(\cdot)\text{ solution of (2) such that } x(t)\in K\}.$$

4. Algebraic and Continuity Properties of Operators Induced by Differential Inclusion

4.1. REACHABLE MAP

PROPOSITION 4.1. *The operator* $K\mapsto\vartheta_F(h,K)$ *is an algebraic dilation on* $\mathcal{F}(X)$.

Proof.

$$\vartheta_F(h,\bigcup_i K_i)=\bigcup_{x\in\bigcup_i K_i}\vartheta_F(h,x)=\bigcup_i\bigcup_{x\in K_i}\vartheta_F(h,x)=\bigcup_i\vartheta_F(h,K_i).$$

4.2. EXIT TUBES

Let $F\colon X\rightsquigarrow X$ be a set-valued map. The operator $K\mapsto\mathrm{Exit}_F(K,t)$ is an increasing anti-extensive operator.

PROPOSITION 4.2. *Let* $F\colon X\rightsquigarrow X$ *be a set-valued map. The operator* $\mathrm{EXIT}_F(\cdot,t)\colon K\mapsto\mathrm{EXIT}_F(K,t)$ *is an algebraic erosion on* $\mathcal{F}(X)$.

Proof. Since

$$\mathrm{EXIT}_F(\cdot,t)(K)=\{x_0\in K\mid\forall s\leqslant t,\vartheta_F(s,x_0)\subset K\},$$

it is obvious that

$$\begin{aligned}\mathrm{EXIT}_F(\cdot,t)(\bigcap K_i)&=\mathrm{EXIT}_F(\bigcap K_i,t)=\bigcap\mathrm{EXIT}_F(K_i,t)\\&=\bigcap\mathrm{EXIT}_F(\cdot,t)(K_i).\end{aligned}$$

Let $F\colon X\rightsquigarrow X$ be a Marchaud map, then $K\mapsto\vartheta_F(h,K)$ is semi-continuous in the sense of $\liminf$, since this operator is an algebraic dilation on $\mathcal{P}(X)$, and $K\mapsto\mathrm{EXIT}_F(\cdot,t)(K)=\mathrm{EXIT}_F(K,t)$ is semi-continuous in the sense of $\limsup$ as an algebraic erosion on $\mathcal{F}(X)$. Since $K\mapsto\mathrm{EXIT}_F(\cdot,t)(K)$ is increasing and semi-continuous in the sense of $\liminf$, we deduce that

PROPOSITION 4.3. *Let* F: $X \rightsquigarrow X$ *be a Marchaud map. The operator* $\mathrm{EXIT}_F(\cdot, t)$: $K \mapsto \mathrm{EXIT}_F(K, t)$ *is u.s.c. on* $\mathcal{F}(X)$.

4.3. ACCESSIBILITY TUBE

PROPOSITION 4.4. *Let* F *be a Marchaud set-valued map in* X. *The set-valued map* $\mathrm{Acc}_F(t, \cdot)$: $K \mapsto \mathrm{Acc}_F(t, K) = \bigcup_{y \in K} \mathrm{Acc}_F(t, y)$ *is a dilation on* $\mathcal{F}(X)$.

Proof. It is obvious from Proposition 4.1.

THEOREM 4.5. *Let* F *be a Marchaud set-valued map in* X. *Let us consider the two operators on* $\mathcal{F}(X)$: $\mathrm{EXIT}_F(\cdot, t)$: $K \mapsto \mathrm{EXIT}_F(K, t)$ *and* $\mathrm{Acc}_F(\cdot, t)$: $K \mapsto \mathrm{Acc}_F(K, t)$. *Then* $(\mathrm{EXIT}_F(\cdot, t), \mathrm{Acc}_F(\cdot, t))$ *constitutes an adjunction,* i.e.

$$\mathrm{EXIT}_F(\cdot, t)^{\bowtie} = \mathrm{Acc}_F(\cdot, t).$$

Proof. We observe that $\mathrm{EXIT}_F(t, K) = \{x \mid \bigcup \vartheta(t, x) \subset K\}$, then we can deduce that

$$\begin{aligned} A \subset \mathrm{EXIT}_F(t, B) &\Leftrightarrow \forall x \in A, \quad x \in \mathrm{EXIT}_F(t, B) \\ &\Leftrightarrow \forall x \in A, \quad \bigcup_{h \leqslant t} \vartheta_F(h, x) \subset B \\ &\Leftrightarrow \bigcup_{x \in A} \bigcup_{h \leqslant t} \vartheta_F(h, x) \subset B \\ &\Leftrightarrow \mathrm{Acc}_F(t, A) \subset B. \end{aligned}$$

From Proposition 2.6, we can deduce the following corollary.

COROLLARY 4.6. *The map* $K \mapsto \mathrm{Acc}_F(\mathrm{EXIT}_F(K, t), t)$ *is an algebraic opening and the operator* $K \mapsto \mathrm{EXIT}_F(\mathrm{Acc}_F(K, t), t)$ *is an algebraic closing on* $\mathcal{F}(X)$.

4.4. PROPERTIES OF THE VIABILITY KERNEL

Let us consider a Marchaud map F: $X \rightsquigarrow X$. Let Viab_F be the following operator on $\mathcal{F}(X)$ defined by Viab_F: $K \mapsto \mathrm{Viab}_F(K)$. In this section, we will study some properties of this operator.

4.4.1. *Algebraic Properties*

PROPOSITION 4.7. *The operator* Viab_F: $K \mapsto \mathrm{Viab}_F(K)$ *is an algebraic opening on* $\mathcal{F}(X)$.

Proof.

- $\text{Viab}_F(K) \subset K \Rightarrow \text{Viab}_F(\text{Viab}_F(K)) \subset \text{Viab}_F(K)$. Let $x_0 \in \text{Viab}_F(K)$ then there exists $x(\cdot) \in \mathcal{S}_F(x_0)$ such that $\forall t,\ x(t) \in K$. If there exists t_0 such that $x(t_0) \notin \text{Viab}_F(K)$, then $\exists t_1 < t_0$ such that $x(t_0 + t_1) \notin K$, and that is impossible. Thus, Viab_F is idempotent.
- It is obvious that Viab_F is anti-extensive because $\forall K \in \mathcal{F}(X)$, $\text{Viab}_F(K) \subset K$.
- Viab_F is increasing: Suppose $K \subset L$, $\forall x_0 \in \text{Viab}_F(K), \exists x(\cdot)$ such that $\forall t \in [0,T]$, $x(t) = \vartheta_F(t, x_0)$ and $x(t) \in K \subset L$. This implies $x_0 \in \text{Viab}_F(L)$ then we can deduce that $\text{Viab}_F(K) \subset \text{Viab}_F(L)$.

Corollary 4.8 follows from the remark of Section 2.7.

COROLLARY 4.8. *The operator* $\text{Abs}_F(\cdot)$ *defined by*

$$\text{Abs}_F(\Omega) = X \backslash \text{Viab}_F\left(X \backslash \Omega\right)$$

is an algebraic closing on the space $\mathcal{F}^*(X)$ *of all open subsets of* X.

4.4.2. *Continuity Property*

PROPOSITION 4.9 [2]. *Let us consider a set-valued map* $F\colon X \rightsquigarrow X$ *satisfying uniform linear growth and an arbitrary sequence of closed sets* (K_n). *Then* $\text{Lim sup}_{n\to\infty}\text{Viab}_F(K_n) \subset \text{Viab}_F(\text{Lim sup}_{n\to\infty} K_n)$.

From this proposition, it is obvious that Viab_F is u.s.c.

From Proposition 4.7, we deduce that

$$\text{Viab}_F(K) = \bigcup\{L \in \mathcal{F}(X) \mid L \in \text{Fix}(\text{Viab}_F), L \subset K\}$$

where $\text{Fix}(\text{Viab}_F)$ is the set of all viability domains for F. This set is a complete lattice. The domain of the solution map ϑ_F is the largest closed viability domain contained in the domain of F, and

$$\bigwedge \text{Fix}(\text{Viab}_F) = \emptyset, \qquad \bigvee \text{Fix}(\text{Viab}_F) = \text{Dom}(\mathcal{S}_F).$$

4.5. PROPERTIES OF THE INVARIANCE KERNEL AND INVARIANCE ENVELOPE

Let us consider a Marchaud map $F\colon X \rightsquigarrow X$. Let Inv_F be the following operator on $\mathcal{F}(X)$ defined by $K \mapsto \text{Inv}_F(K)$. In this section, we will study some algebraic properties of this operator.

PROPOSITION 4.10. *The operator* $\text{Inv}_F\colon K \mapsto \text{Inv}_F(K)$ *is an increasing algebraic erosion and an algebraic opening on* $\mathcal{F}(X)$.

It is clear that $\mathrm{Inv}_F(K_1 \cap K_2) = \mathrm{Inv}_F(K_1) \cap \mathrm{Inv}_F(K_2)$ and, more generally, that the invariance kernel of any intersection of closed subsets K_i $(i \in I)$ is the intersection of invariance kernels of the K_i.

It follows that:

PROPOSITION 4.11. *The operator* Inv_F: $K \mapsto \mathrm{Inv}_F(K)$ *is u.s.c. on* $\mathcal{F}(X)$.

We have also [2] a proposition* on the lower semi-continuity of $K \mapsto \mathrm{Inv}_F(K)$.

It follows that if the solution map $\mathcal{S}_F$ is lower semi-continuous, then the operator Inv_F: $K \mapsto \mathrm{Inv}_F(K)$ is semi-continuous in the sense of lim inf. Since it is an algebraic erosion, it is semi-continuous in the sense of lim sup. We deduce that, under the previous assumptions, the operator Inv_F: $K \mapsto \mathrm{Inv}_F(K)$ is continuous.

DEFINITION 4.12 [28]. Assume that F: $X \rightsquigarrow X$ is a Lipschitz Marchaud set-valued map. The invariance envelope is defined by

$$\mathrm{Env}_F(K) = \overline{\mathrm{Acc}_F(K, +\infty)} = \overline{\bigcup_{t \geqslant 0} \mathrm{Acc}_F(K, t)}.$$

Let F: $X \rightsquigarrow X$ be Lipschitz Marchaud set-valued map. Let K, H be two non-empty closed subsets of X, then

- $K \subset \mathrm{Env}_F(K)$,
- if $H \subset K$, then $\mathrm{Env}_F(H) \subset \mathrm{Env}_F(K)$,
- the subset K is invariant if and only if $K = \mathrm{Env}_F(K)$.

It follows that

PROPOSITION 4.13. *The operator* Env_F: $K \mapsto \mathrm{Env}_F(K)$ *is an algebraic dilation and closing on* $\mathcal{F}(X)$.

Proof.

$$\begin{aligned} &\mathrm{Env}_F(K_1 \cup K_2) \\ &\quad = \overline{\mathrm{Acc}_F(K_1, +\infty) \cup \mathrm{Acc}_F(K_2, +\infty)} \\ &\quad \subseteq \overline{\mathrm{Acc}_F(K_1, +\infty)} \cup \overline{\mathrm{Acc}_F(K_2, +\infty)} = \mathrm{Env}_F(K_1) \cup \mathrm{Env}_F(K_2), \end{aligned}$$

$$\left.\begin{array}{l} \mathrm{Env}_F(K_1) \subset \mathrm{Env}_F(K_1 \cup K_2) \\ \mathrm{Env}_F(K_2) \subset \mathrm{Env}_F(K_1 \cup K_2) \end{array}\right\} \Rightarrow \mathrm{Env}_F(K_1) \cup \mathrm{Env}_F(K_2) \subseteq \mathrm{Env}_F(K_1 \cup K_2).$$

---

* Let us assume that F is Lipschitz. Then the lower limit of closed subsets $K_n \subset \Omega$ invariant under F is also invariant under F. In particular, the lower limit of the invariance kernels of a sequences of closed subsets $K_n \subset \Omega$ contains the invariance kernels of the lower limit of the sequence K_n:

$$\mathrm{Lim\,inf}_{n\to\infty}\left(\mathrm{Inv}_F(K_n)\right) \supset \mathrm{Inv}_F\left(\mathrm{Lim\,inf}_{n\to\infty}(K_n)\right).$$

Then we have $\text{Env}_F(K_1 \cup K_2) = \text{Env}_F(K_1) \cup \text{Env}_F(K_2)$.

Furthermore, we can show [28] that $\text{Env}_F(K) = X \setminus \overline{\text{Inv}_{-F}(X \setminus K)}$. Since Inv_F is an opening, by complementation duality, we deduce that Env_F is a closing.

Since the intersection of two invariance domains is still an invariance domain, the invariance envelope $\text{Env}_F(K)$ is defined as the intersection of all closed invariant subsets containing K. Then we obtain $\text{Env}_F(K) = \bigcap\{B \in \mathcal{B} \mid K \subset B\}$, where $\mathcal{B}$ is the family of all invariant domains under F.

5. Application to Morphological Operators

It appears that morphological operators are a particular case of operators induced by differential inclusions.

Let B be a subset of a topological vector space X. We consider the multivalued map T_B defined by $T_B(x) = B_x = \{x + b \text{ where } b \in B\} = B \oplus \{x\}$. We put $\check{B} = -B = \{-b \mid b \in B\}$ the symetrical set of B.

We recall [17] that if K and B are two subsets of X,

- The *morphological dilation* of K by B is defined by $K \oplus \check{B} = \{x \mid B_x \cap K \neq \emptyset\}$.
- The *morphological erosion* of K by B is defined by $K \ominus \check{B} = \{x \mid B_x \subset K\}$.
- The *morphological opening* of K by B is defined by $K_B = (K \ominus \check{B}) \oplus B = \bigcup_x\{B_x \mid B_x \subset K\}$.
- The *morphological closing* of K by B is defined by $K^B = (K \oplus \check{B}) \ominus B$.

We say that B is a *barrel* set of the finite-dimensional vector space X if B is a convex, symmetric (i.e. $\check{B} = B$), compact set with a nonempty interior. We can equip the space X with a metric derived from a norm associated with a barrel set B defined by $\|\cdot\|_B$: $\|x_0\|_B = \min\{\lambda,\ \lambda \geqslant 0;\ x_0 \in \lambda B\}$ and the induced distance d^B is defined by [30]:

$$\begin{aligned} d^B(x_0, y) &= \|x_0 - y\|_B \Leftrightarrow d^B(x_0, y) \\ &= \min\{\lambda,\ \lambda \geqslant 0;\ y \in \{x_0\} \oplus \lambda B\}, \end{aligned} \tag{3}$$

i.e., $d^B(x_0, y)$ is the size of the biggest homothetic set of B centered on x_0 and containing y. It is obvious that in the plane $\mathbb{R}^2$, the Euclidean norm, the L_1-norm, i.e. $\|x_0\|_1 = |\alpha_1| + |\alpha_2|$, and the L_∞-norm, i.e. $\|x_0\|_\infty = \bigvee(|\alpha_1|, |\alpha_2|)$ where $x_0 = (\alpha_1, \alpha_2)$, are $\|\cdot\|_B$ for B, respectively, be a disk, a diamond and a square.

LEMMA 5.1. *Let B be a barrel and K be a compact subset of X such that $\overline{\text{Int}(K)} = K$, then we have*

$$\forall K \in X, \quad \tau_K^{\flat}(x_0) = d^B(x_0, K^c),$$

where $d^B(x_0, y) = \|x_0 - y\|_B$, *and* $d^B(x_0, Y) = \bigwedge_{y \in Y} d^B(x_0, y)$ *is the distance to* K^c *function associated with* B.

From Lemma 5.1, we deduce the following proposition.

PROPOSITION 5.2. *Let* B *be a barrel and* K *be a closed subset of* X, *then we have*

$$\begin{aligned}
&\mathrm{Exit}_B(K, h) = K, \\
&\vartheta_B(h, K) = K \oplus hB, \\
&\mathrm{Acc}_F(h, K) = K \oplus hB, \\
&\mathrm{EXIT}_B(K, h) = K \ominus hB, \\
&\mathrm{Acc}_F(\mathrm{EXIT}_F(K, t), t) = (K \ominus hB) \oplus hB = K_B, \\
&\mathrm{EXIT}_F(\mathrm{Acc}_F(K, t), t) = (K \oplus hB) \ominus hB = K^B.
\end{aligned}$$

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