THE (3+1)-DIMENSIONAL MONGE-AMPÈRE EQUATION IN DISCONTINUITY WAVE THEORY: APPLICATION OF A RECIPROCAL TRANSFORMATION

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ABSTRACT. It is shown that the complete exceptionality condition for discontinuity waves associated with a second-order non-linear hyperbolic equation of the form

$u_{tt} + f(x_i, t, u, u_i, u_{it}, u_{jk}) = 0, \quad i = 1, 2, 3; \quad j \leqslant k$

leads to a Monge–Ampère-type equation in 3 + 1 dimensions. Application of a novel reciprocal transformation shows that an important subclass may be reduced to linear canonical form. Specialization to 1 + 1 dimensions yields linearization of a Boillat-type equation satisfying the complete exceptionality criterion. In this last case the transformation allowing the linearization coincide with the one introduced by Hoskins and Bretherton in the theory of atmospheric frontogenesis and so-called *geostrophic transformation*. Finally, always in 1 + 1 dimensions, we show that the Monge–Ampère equation is also strictly exceptional, i.e. the only possible shocks are characteristic.

SOMMARIO. Si dimostra che la condizione di completa eccezionalita' per le onde di discontinuita' associate con una equazione non-lineare iperbolica della forma:

$u_{tt} + f(x_i, t, u, u_i, u_{it}, u_{jk}) = 0, \quad i = 1, 2, 3; \quad j \leqslant k$

e' soddisfatta se l'equazione e' di tipo Monge-Ampère in 3 + 1 dimensioni. Inoltra, esiste una transformazione reciproca che riduce una sottoclasse di tali equazioni a forma canonica lineare. La particolarizzazione di tale transformazione al caso di 1 + 1 dimensioni linearizza l'equazione non-lineare del secondo ordine, che gode della proprieta' di essere completamente eccezionale, ottenuta da Boillat. Tale trasformazione coincide con quella detta *geostrofica* e introdotta da Hoskins e Bretherton nella teoria della frontogenesi atmosferica. Infine, sempre nel caso a 1 + 1 dimensioni, si dimostra che l'equazione di Monge–Ampère presenta anche il carattere di stretta eccezionalita' cioe' i soli urti possibili si propagano con velocita' caratteristiche.

KEY WORDS. Discontinuity, Waves Reciprocal transformation, Exceptionality

1. THE COMPLETE EXCEPTIONALITY CONDITIONS: A MONGE-AMPÈRE EQUATION IN 3+1 DIMENSIONS

Herein, we consider discontinuity waves Σ given by $\phi(x_i, t) = 0$ associated with second-order non-linear hyperbolic equations of the form

$$u_{tt} + f(x_i, t, u, u_i, u_{it}, u_{jk}) = 0, \quad i = 1, 2, 3; \quad j \leqslant k \quad (1.1)$$

As is well known, the Cauchy problem associated with (1.1) is defined in terms of the third-order quasi-linear equation obtained by taking the derivative with respect to one independent variable, say t.

Application of the standard transposition

$$\partial_t \to -\lambda\delta; \qquad \partial_i \to n_i\delta; \qquad \delta = \left(\frac{\partial}{\partial\phi}\right)_{\phi=0^-} - \left(\frac{\partial}{\partial\phi}\right)_{\phi=0^+} \quad (1.2)$$

where

$$\lambda = -\phi_t/|\nabla\phi|, \qquad n_i = \phi_i/|\nabla\phi| \quad (1.3)$$

produce the characteristic root $\lambda = 0$, which we discard because it is artificially introduced by the time derivative, and the characteristic polynomial condition:

$$P(\lambda) := \lambda^2 - \lambda f_{u_{0i}} n_i + f_{u_{ik}} n_i n_k = 0 \quad \text{on } \Sigma \quad (1.4)$$

$(u_{0i} := \partial^2 u/\partial t\, \partial x_i)$

in analogy with the (2+1)-dimensional case [1]. The relation (1.4) determines the possible speeds of propagation λ of the discontinuities in the third-order derivatives of u, through Σ, in terms of n_i and the derivatives of f with respect to the second-order derivatives of u.

The complete exceptionality condition in the case under consideration is

$$-\lambda\lambda_{u_{0i}} n_i + \lambda_{u_{jk}} n_j n_k = 0 \quad \text{on } \Sigma \quad (1.5)$$

to be taken together with (1.4). The derivative of $P(\lambda)$ with respect to u_{0j} multiplied by n_j and the derivatives of $P(\lambda)$ with respect to u_{rs} multiplied by $n_r n_s$ now give, in turn,

$$P'\lambda_{u_{0j}} n_j - \lambda f_{u_{0i}} n_i n_j + f_{u_{jk} u_{0j}} n_j n_i n_k = 0, \quad (1.6)$$

$$P'\lambda_{u_{rs}} n_r n_s - f_{u_{0i} u_{rs}} n_i n_r n_s + f_{u_{jk} u_{rs}} n_i n_k n_r n_s = 0. \quad (1.7)$$

$(P' := \partial_\lambda P)$

Subtraction of $\lambda \times (1.6)$ and (1.7) yields, on use of (1.4) and the exceptionality condition,

$$(f_{u_{0i}}f_{u_{0k}u_{0j}} - 2f_{u_{0i}u_{kj}})n_i n_k n_j = 0, \tag{1.8}$$

$$(f_{u_{ik}u_{rs}} - f_{u_{ik}}f_{u_{0r}u_{0s}})n_j n_k n_r n_s = 0. \tag{1.9}$$

If we now set

$$u_{0i} = s_i;\ u_{ii} = r_i;\ u_{23} = p_1;\ u_{13} = p_2;\ u_{12} = p_3 \tag{1.10}$$

taking into account the constraint $j < k$ in (1.1), then 'in extenso', (1.8), (1.9) (requiring to be satisfied $\forall n_1, n_2, n_3$) provide a system of simultaneous non-linear equations which determine $f(x_i, t, u, u_i, s_i, r_i, p_i)$, namely:

$$\begin{aligned}
&f_{s_1}f_{s_1s_1} - 2f_{s_1r_1} = 0,\\
&f_{r_1r_1} - f_{r_1}f_{s_1s_1} = 0,\\
&f_{s_2}f_{s_2s_2} - 2f_{s_2r_2} = 0,\\
&f_{r_2r_2} - f_{r_2}f_{s_2s_2} = 0,\\
&f_{s_1}f_{s_2s_2} + 2(f_{s_2}f_{s_1s_2} - f_{s_1r_2} - f_{s_2p_3}) = 0, \qquad S_{123}\\
&f_{s_2}f_{s_1s_2} + 2(f_{s_1}f_{s_1s_2} - f_{s_2r_1} - f_{s_1p_3}) = 0,\\
&2[f_{p_3r_1} - f_{r_1}f_{s_1s_2}] - f_{p_3}f_{s_1s_1} = 0,\\
&2[f_{p_3r_2} - f_{r_2}f_{s_1s_2}] - f_{p_3}f_{s_2s_2} = 0,\\
&f_{p_3p_3} + 2(f_{r_1r_2} - f_{p_3}f_{s_1s_2}) - f_{r_1}f_{s_2s_2} - f_{r_2}f_{s_1s_1} = 0,
\end{aligned}$$

together with

$$\begin{aligned}
&2[f_{p_2p_3} + f_{r_1p_1} - f_{p_2}f_{s_1s_2} - f_{p_3}f_{s_1s_3} - f_{r_1}f_{s_2s_3}]\\
&\quad - f_{p_1}f_{s_1s_1} = 0 \qquad \mathscr{E}_{123}
\end{aligned}$$

augmented by their cyclic interchanges S_{231}, S_{312}, $\mathscr{E}_{231}$, $\mathscr{E}_{312}$ and the single constraint

$$\sum_{1,2,3} [f_{s_1}f_{s_2s_3} - f_{s_1p_1}] = 0. \tag{1.11}$$

It is observed that x_i, t, u, u_i are so-called *inessential* variables in the above since they do not occur explicitly.

In the $(2+1)$-dimensional case, it has been shown in [1] that solution of the system S_{123} yields

$$f = \square_1/\Delta_1 \tag{1.12}$$

where

$$\begin{aligned}
\Delta_1 &:= \alpha_1(r_1r_2 - p_3^2) + \alpha_2 r_1 + \alpha_3 r_2 + \alpha_4 p_3 + \alpha_5,\\
\square_1 &:= -(\alpha_2 + \alpha_1 r_2)s_1^2 - (\alpha_3 + \alpha_1 r_1)s_2^2 + (2\alpha_1 p_3 - \alpha_4)s_1 s_2\\
&\quad + (\alpha_6 r_2 + \alpha_7 p_3 + \alpha_8)s_1 + (-\alpha_7 r_1 - \alpha_6 p_3 + \alpha_9)s_2\\
&\quad + \alpha_{10}(r_1r_2 - p_3^2) + \alpha_{11} r_1 + \alpha_{12} r_2 + \alpha_{13} p_3 + \alpha_{14}
\end{aligned} \tag{1.13}$$

with the α_i, $i = 1, \ldots, 14$, dependent on the variables x_i, t, u, u_i $(i = 1, 2)$. Insertion of (1.12) into (1.1) delivers the general $(2+1)$-dimensional Monge–Ampère equation [1]:

$$\begin{aligned}
&\bar{\alpha}_1(u_{00}(u_{11}u_{22} - u_{12}^2) - u_{01}^2 u_{22} - u_{02}^2 u_{11} + 2u_{01}u_{02}u_{12}]\\
&\quad + 2[\bar{\alpha}_2(u_{01}u_{02} - u_{00}u_{12}) + \bar{\alpha}_3(u_{12}u_{02} - u_{01}u_{22})\\
&\quad + \bar{\alpha}_4(u_{12}u_{01} - u_{02}u_{11})]\\
&\quad + \bar{\alpha}_5(u_{00}u_{11} - u_{01}^2) + \bar{\alpha}_6(u_{00}u_{22} - u_{02}^2) + \bar{\alpha}_7(u_{11}u_{22} - u_{12}^2)\\
&\quad + \bar{\alpha}_8 u_{00} + \bar{\alpha}_9 u_{01} + \bar{\alpha}_{10} u_{02} + \bar{\alpha}_{11} u_{11} + \bar{\alpha}_{12} u_{22}\\
&\quad + \bar{\alpha}_{13} u_{12} + \bar{\alpha}_{14} = 0
\end{aligned} \tag{1.14}$$

where the $\bar{\alpha}_i$, $i = 1, \ldots, 14$, have arbitrary dependence on x_i, t, u, u_i, $i = 1, 2$.

If we set

$$\Delta^{(2+1)} := \det u_{ij}, \quad i, j = 0, 1, 2 \quad \text{with } u_{ij} = u_{ji} \tag{1.15}$$

then it is observed that the Monge–Ampère equation (1.14) admits the more compact representation

$$\begin{aligned}
&\bar{\alpha}_1 \Delta^{(2+1)} + \bar{\alpha}_2 \Delta^{(2+1)}_{u_{12}} + \bar{\alpha}_3 \Delta^{(2+1)}_{u_{01}} + \bar{\alpha}_4 \Delta^{(2+1)}_{u_{02}}\\
&\bar{\alpha}_5 \Delta^{(2+1)}_{u_{22}} + \bar{\alpha}_6 \Delta^{(2+1)}_{u_{11}} + \bar{\alpha}_7 \Delta^{(2+1)}_{u_{00}}\\
&\quad + \bar{\alpha}_8 u_{00} + \bar{\alpha}_9 u_{01} + \bar{\alpha}_{10} u_{02} + \bar{\alpha}_{11} u_{11} + \bar{\alpha}_{12} u_{22}\\
&\quad + \bar{\alpha}_{13} u_{13} + \bar{\alpha}_{14} = 0.
\end{aligned} \tag{1.16}$$

In the $(3+1)$-dimensional case under consideration, the system S_{123} again delivers the relation (1.12) with Δ_1, $\square_1$ given by (1.13) but where the α_i now depend on t, x_i, u, u_i, $i = 1, 2, 3$ together with $u_{03} = s_3$, $u_{33} = r_3$, $u_{23} = p_1$, $u_{13} = p_2$. On cyclic interchange, it is seen that the systems S_{231} and S_{312} admit solutions

$$f = \square_2/\Delta_2 \tag{1.17}$$

and

$$f = \square_3/\Delta_3 \tag{1.18}$$

where

$$\begin{aligned}
\Delta_2 &:= \beta_1(r_2r_3 - p_1^2) + \beta_2 r_2 + \beta_3 r_3 + \beta_4 p_1 + \beta_5,\\
\square_2 &:= -(\beta_2 + \beta_1 r_3)s_2^2 - (\beta_3 + \beta_1 r_2)s_3^2 + (2\beta_1 p_1 - \beta_4)s_2 s_3\\
&\quad + (\beta_6 r_3 + \beta_7 p_1 + \beta_8)s_2 + (-\beta_7 r_2 - \beta_6 p_1 + \beta_9)s_3\\
&\quad + \beta_{10}(r_2r_3 - p_1^2) + \beta_{11} r_2 + \beta_{12} r_3 + \beta_{13} p_1 + \beta_{14},
\end{aligned} \tag{1.19}$$

and

$$\begin{aligned}
\Delta_3 &:= \gamma_1(r_3r_1 - p_2^2) + \gamma_2 r_3 + \gamma_3 r_1 + \gamma_4 p_2 + \gamma_5,\\
\square_3 &:= -(\gamma_2 + \gamma_1 r_1)s_3^2 - (\gamma_3 + \gamma_1 r_3)s_1^2 + (2\gamma_1 p_2 - \gamma_4)s_3 s_1\\
&\quad + (\gamma_6 r_1 + \gamma_7 p_2 + \gamma_8)s_3 + (-\gamma_7 r_3 - \gamma_6 p_2 + \gamma_9)s_1\\
&\quad + \gamma_{10}(r_3r_1 - p_2^2) + \gamma_{11} r_3 + \gamma_{12} r_1 + \gamma_{13} p_2 + \gamma_{14},
\end{aligned} \tag{1.20}$$

in turn. In the above, the β_i depend on t, x_i, u, u_i, $i = 1, 2, 3$ together with $u_{01} = s_1$, $u_{11} = r_1$, $u_{13} = p_2$, $u_{12} = p_3$ while the γ_i depend on t, x_i, u, u_i, $i = 1, 2, 3$ together with $u_{02} = s_2$, $u_{22} = r_2$, $u_{12} = p_3$, $u_{23} = p_1$. Thus, we have the compatibility conditions

$$\square_1/\Delta_1 = \square_2/\Delta_2 = \square_3/\Delta_3. \tag{1.21}$$

In particular, if it is required that

$$\square_1 = \square_2 = \square_3 \tag{1.22}$$

and

$$\Delta_1 = \Delta_2 = \Delta_3 \tag{1.23}$$

then the relations (1.21) hold and an extensive calculation shows that

$$f = \Box_i/\Delta_i \tag{1.24}$$

where

$$\begin{aligned}\Delta_i = {}& \delta_1[r_1r_2r_3 - r_1p_1^2 - r_2p_2^2 - r_3p_3^2 + 2p_1p_2p_3] \\ & + \delta_2[r_1r_2 - p_3^2] + \delta_3[r_2r_3 - p_1^2] + \delta_4[r_3r_1 - p_2^2] \\ & + \delta_5[r_3p_3 - p_1p_2] + \delta_6[r_1p_1 - p_2p_3] + \delta_7[r_2p_2 - p_3p_1] \\ & + \delta_8r_1 + \delta_9r_2 + \delta_{10}r_3 + \delta_{11}p_1 + \delta_{12}p_2 + \delta_{13}p_3 + \delta_{14},\end{aligned} \tag{1.25}$$

$$\begin{aligned}\Box_i = {}& -s_1^2\,\Delta_{i,r_1} - s_2^2\,\Delta_{i,r_2} - s_3^2\,\Delta_{i,r_3} \\ & - s_1s_2\,\Delta_{ip_3} - s_2s_3\,\Delta_{ip_1} - s_3s_1\,\Delta_{ip_2} \\ & + \psi_1s_1 + \psi_2s_2 + \psi_3s_3 + \omega,\end{aligned} \tag{1.26}$$

with

$$\begin{aligned}\psi_1 = {}& \varepsilon_1(r_2r_3 - p_1^2) + \varepsilon_2(r_2p_2 - p_3p_1) + \varepsilon_3(r_3p_3 - p_1p_2) \\ & + \varepsilon_4r_2 + \varepsilon_5r_3 + \varepsilon_6p_1 + \varepsilon_7p_2 + \varepsilon_8p_3 + \varepsilon_9, \\ \psi_2 = {}& -\varepsilon_3(r_3r_1 - p_2^2) + \varepsilon_2(r_1p_1 - p_2p_3) - \varepsilon_1(r_3p_3 - p_1p_2) \\ & - \varepsilon_8r_1 + \varepsilon_{10}r_3 + \varepsilon_{11}p_1 + \varepsilon_{12}p_2 - \varepsilon_4p_3 + \varepsilon_{13}, \\ \psi_3 = {}& -\varepsilon_2(r_2r_1 - p_3^2) - \varepsilon_1(r_2p_2 - p_3p_1) + \varepsilon_3(r_1p_1 - p_2p_3) \\ & - \varepsilon_7r_1 - \varepsilon_{11}r_2 - \varepsilon_{10}p_1 - \varepsilon_5p_2 - (\varepsilon_6 + \varepsilon_{12})p_3 + \varepsilon_{14},\end{aligned} \tag{1.27}$$

and

$$\begin{aligned}\omega = {}& \zeta_1[r_1r_2r_3 - r_1p_1^2 - r_2p_2^2 - r_3p_3^2 + 2p_1p_2p_3] \\ & + \zeta_2[r_1r_2 - p_3^2] + \zeta_3[r_2r_3 - p_1^2] + \zeta_4[r_3r_1 - p_2^2] \\ & + \zeta_5[r_3p_3 - p_1p_2] + \zeta_6[r_1p_1 - p_2p_3] + \zeta_7[r_2p_2 - p_3p_1] \\ & + \zeta_8r_1 + \zeta_9r_2 + \zeta_{10}r_3 + \zeta_{11}p_1 + \zeta_{12}p_2 + \zeta_{13}p_3 + \zeta_{14}.\end{aligned} \tag{1.28}$$

In the above, δ_j, ε_j, ζ_j, $j = 1, \ldots, 14$, have arbitrary dependence on x_i, t, u, u_i, $i = 1, 2, 3$.

If, in analogy with (1.15), we set

$$\Delta^{(3+1)} := \det u_{ij}, \quad i, j = 0, 1, 2, 3 \quad \text{with } u_{ij} = u_{ji} \tag{1.29}$$

where the variables t, x_l, $l = 1, 2, 3$, are represented by $0, 1, 2, 3$ then insertion of (1.24) into (1.1) produces a generalized Monge–Ampère-type equation in $3+1$ dimensions, namely

$$\begin{aligned}& \lambda\Delta^{(3+1)} + \sum \mu_{ij}\,\Delta^{(3+1)}_{u_{ij}} + \sum \nu_{kl,mn}\,\Delta^{(3+1)}_{u_{kl},u_{mn}} \\ & + \sum \zeta_{ij}u_{ij} + \eta = 0\end{aligned} \tag{1.30}$$

where

$i, j = 0, 1, \ldots, 3, \quad i \leqslant j$

$k, l, m, n = 0, 1, \ldots, 3; \quad k < l, \quad k \leqslant m < n.$

It is noted that, in view of the relationship

$$\Delta^{(3+1)}_{u_{01},u_{23}} + \Delta^{(3+1)}_{u_{02},u_{13}} + \Delta^{(3+1)}_{u_{03},u_{13}} = 0 \tag{1.31}$$

there are a total of 42 independent coefficients λ, μ_{ij}, $\nu_{kl,mn}$, ξ_{ij}, η each with arbitrary dependence on the variables t, x_i, u, u_i.

2. REDUCTION VIA A RECIPROCAL TRANSFORMATION

The application of reciprocal-type transformations to the linearization of certain non-linear boundary value problems in $1+1$ dimensions, notably in heat conduction and soil mechanics, is well established [3–9].

An involutory, so-called *geostrophic transformation* has been employed to linearize a two-dimensional Monge–Ampère equation which arises in the theory of atmospheric frontogenesis [10]. An important boundary value problem was solved by this method. In the present context of discontinuity waves, a cognate transformation may be used to linearize a $(1+1)$-dimensional Monge–Ampère equation of the Boillat type [11].

Thus, in $1+1$ dimensions, the complete exceptionality condition leads to the Monge–Ampère equation obtained by Boillat, namely

$$u_{tt} + \Box/\Delta = 0 \tag{2.1}$$

where, in the notation of (1.13),

$$\left.\begin{aligned}\Delta &:= \alpha_2r_1 + \alpha_5, \\ \Box &:= -\alpha_2s_1^2 + \alpha_8s_1 + \alpha_{11}r_1 + \alpha_{14}\end{aligned}\right\} \tag{2.2}$$

with the α_i, in general, dependent on the variables x, t, u, u_x. In the sequel, attention is restricted to the case when the α_i are constant.

The transformation

$$\left.\begin{aligned}& x^* = x - k_1u_x, \qquad t^* = t - k_0u_t \\ & u^* = -u + \tfrac{1}{2}(k_0u_t^2 + k_1u_x^2) \\ & 0 < |J(x^*, t^*;\, x, t)| < \infty\end{aligned}\right\} R \tag{2.3}$$

is now introduced.

Under R,

$$\begin{aligned}\partial_{x^*} &= [(1 - k_0u_{tt})\partial_x + k_0u_{xt}\partial_t]/J \\ \partial_{t^*} &= [k_1u_{xt}\partial_x + (1 - k_1u_{xx})\partial_t]/J\end{aligned} \tag{2.4}$$

where

$$J = J(x^*, t^*;\, x, t) = (1 - k_1u_{xx})(1 - k_0u_{tt}) - k_0k_1u_{xt}^2. \tag{2.5}$$

Use of (2.3) and (2.4) produces the relations

$$u^*_{x^*} = -u_x, \tag{2.6}$$

$$u^*_{t^*} = -u_t \tag{2.7}$$

whence

$$\begin{aligned}& x^{**} = x^* - k_1u^*_{x^*} = x - k_1(u_x + u^*_{x^*}) = x \\ & t^{**} = t^* - k_0u^*_{t^*} = t - k_0(u_t + u^*_{t^*}) = t \\ & u^{**} = -u^* + \tfrac{1}{2}(k_0u^{*2}_{t^*} + k_1u^{*2}_{x^*}) = u\end{aligned} \tag{2.8}$$

so that $R^2 = I$, that is, R is reciprocal.

Application of the reciprocal relations

$$x = x^* - k_1 u^*_{x^*}, \qquad t = t^* - k_0 u^*_{t^*}\} \tag{2.9}$$

shows that

$$u_{xx} = (J^* - J^*_{\rho_1^*})/k_1 J^*, \qquad u_{tt} = (J^* - J^*_{\rho_0^*})/k_0 J^* \qquad u_{xt} = J^*_{u^*_{x^*t^*}}/2k_0 k_1 J^* \tag{2.10}$$

where

$$J^* - J^*(x, t; x^* t^*) = \rho_0 \rho_1 - k_0 k_1 u^{*2}_{x^*t^*} \tag{2.11}$$

with

$$\rho_0^* = 1 - k_0 u^*_{t^*t^*}, \qquad \rho_1^* = 1 - k_1 u^*_{x^*x^*}. \tag{2.12}$$

Now, 'in extenso' the Monge–Ampère equation (2.1) with Δ_1, $\Box_1$ given by (2.2) becomes

$$\alpha_2(u_{xx}u_{tt} - u_{xt}^2) + \alpha_5 u_{tt} + \alpha_8 u_{xt} + \alpha_{11} u_{xx} + \alpha_{14} = 0. \tag{2.13}$$

On application of R to (2.13) one obtains

$$(k_0\alpha_{11} + k_1\alpha_5 + \alpha_2 + k_0 k_1 \alpha_{14})(u^*_{x^*x^*}u^*_{t^*t^*} - u^{*2}_{x^*t^*}) - (\alpha_5 + k_0\alpha_{14})u^*_{t^*t^*} - \alpha_8 u^*_{x^*t^*} - (\alpha_{11} + k_1\alpha_{14})u^*_{x^*x^*} + \alpha_{14} = 0. \tag{2.14}$$

It is noted that the conditions

$$4(\alpha_2\alpha_{14} - \alpha_{11}\alpha_5) + \alpha_8^2 \begin{cases} > 0 & \text{hyperbolicity} \\ < 0 & \text{ellipticity} \end{cases}$$

are preserved under R. Moreover, if k_0, k_1 are chosen such that

$$k_0\alpha_{11} + k_1\alpha_5 + \alpha_2 + k_0 k_1 \alpha_{14} = 0, \tag{2.15}$$

then the Monge–Ampère equation (2.13) is reduced by R to the *linear* canonical form

$$(\alpha_5 + k_0\alpha_{14})u^*_{t^*t^*} + \alpha_8 u^*_{x^*t^*} + (\alpha_{11} + k_1\alpha_{14})u^*_{x^*x^*} - \alpha_{14} = 0. \tag{2.16}$$

If $u(x, t)$ is the solution of (2.16) then the solution of the *non-linear* equation (2.13) is given parametrically via the relations (2.3).

In the case of the (3+1)-dimensional Monge–Ampère equations (1.30) introduced in the previous section, linearization is sought by an extension of (2.3), namely by transformations of the type

$$\left.\begin{aligned} &x^* = x - k_1 u_x, \qquad y^* = y - k_2 u_y, \\ &z^* = z - k_3 u_z, \qquad t^* = t - k_0 u_t, \\ &u^* = -u + \tfrac{1}{2}(k_0 u_t^2 + k_1 u_x^2 + k_2 u_y^2 + k_3 u_z^2), \\ &0 < |J(x^*, y^*, z^*, t^*; x, y, z, t)| < \infty. \end{aligned}\right\} R \tag{2.17}$$

It is readily shown that

$$u^*_{x^*} = -u_x, \quad u^*_{y^*} = -u_y, \quad u^*_{z^*} = -u_z, \quad u^*_{t^*} = -u_t, \tag{2.18}$$

whence

$$\left.\begin{aligned} &x^{**} = x, \quad y^{**} = y, \quad z^{**} = z, \quad t^{**} = t \\ &u^{**} = u \end{aligned}\right\} \tag{2.19}$$

and, accordingly, $R^2 = I$.

The reciprocal nature of the transformation (2.17) is now exploited in an inverse approach. Thus, the reciprocal associate under R is sought of the *linear* (3+1)-dimensional equation

$$\sum_{\substack{i,j=0,1,2,3 \\ i \leq j}} \gamma_{ij} u^*_{ij} + \delta = 0, \quad \gamma_{ij} \in \mathbb{R}. \tag{2.20}$$

Here, the u^*_{ij} denote second partial derivatives with respect to the appropriate starred variables with, in this context, 0, 1, 2, 3 corresponding, in turn, to t^*, x^*, y^*, z^*.

Under R, in the notation of Section 1,

$$\begin{aligned} &u^*_{x^*x^*} = (J - J_{\rho_1})/(k_1 J), u^*_{y^*y^*} = (J - J_{\rho_2})/(k_2 J), \\ &u^*_{z^*z^*} = (J - J_{\rho_3})/(k_3 J),\ u^*_{t^*x^*} = J_{s_1}/(2k_0 k_1 J), \\ &u^*_{t^*y^*} = J_{s_2}/(2k_0 k_2 J),\ u^*_{t^*z^*} = J_{s_3}/(2k_0 k_3 J), \\ &u^*_{y^*z^*} = J_{p_1}/(2k_2 k_3 J),\ u^*_{x^*z^*} = J_{p_2}/(2k_3 k_1 J), \\ &u^*_{x^*y^*} = J_{p_3}/(2k_1 k_2 J) \end{aligned} \tag{2.21}$$

where, on expansion

$$\begin{aligned} J &= J(x^*, y^*, z^*, t^*; x, y, z, t) \\ &= k_0 k_1 k_2 k_3 [s_1^2 p_1^2 + s_2^2 p_2^2 + s_3^2 p_3^2 \\ &\quad - 2(s_2 s_3 p_2 p_3 + s_3 s_1 p_3 p_1 + s_1 s_2 p_1 p_2)] \\ &\quad - 2[k_2 k_3 k_0 s_2 s_3 p_1 \rho_1 + k_3 k_0 k_1 s_3 s_1 p_2 \rho_2 \\ &\quad + k_0 k_1 k_2 s_1 s_2 p_3 \rho_3 + k_1 k_2 k_3 p_1 p_2 p_3 \rho_4] \\ &\quad - [k_1 k_0 s_1^2 \rho_2 \rho_3 + k_2 k_0 s_2^2 \rho_3 \rho_1 + k_3 k_0 s_3^2 \rho_1 \rho_2 \\ &\quad + k_2 k_3 p_1^2 \rho_1 \rho_4 + k_3 k_1 p_2^2 \rho_2 \rho_4 + k_1 k_2 p_3^2 \rho_3 \rho_4] \\ &\quad + \rho_1 \rho_2 \rho_4 \rho_4, \end{aligned} \tag{2.22}$$

and we have adopted the notation

$$\rho_1 = 1 - k_1 u_{xx},\ \rho_2 = 1 - k_2 u_{yy},\ \rho_3 = 1 - k_3 u_{zz},\ \rho_4 = 1 - k_0 u_{tt}.$$

On insertion of the expressions (2.21) into (2.20) we obtain the reciprocal associate of (2.20) under R, namely

$$u_{tt} + \Box_R/\Delta_R = 0, \tag{2.23}$$

where Δ_R, $\Box_R$ are given by

$$\begin{aligned} \Delta_R &= k_0 k_1 k_2 k_3 \varepsilon [r_1 r_2 r_3 - r_1 p_1^2 - r_2 p_2^2 - r_3 p_3^2 + 2 p_1 p_2 p_3] \\ &\quad + k_0 k_1 k_2 (\gamma_{33} k_0 k_1 k_2 - \varepsilon)[r_1 r_2 - p_3^2] \\ &\quad + k_0 k_2 k_3 (\gamma_{11} k_0 k_2 k_3 - \varepsilon)[r_2 r_3 - p_1^2] \\ &\quad + k_0 k_3 k_1 (\gamma_{22} k_0 k_3 k_1 - \varepsilon)[r_3 r_1 - p_2^2] \\ &\quad - \gamma_{12} k_0 k_1 k_2 k_3^2 [r_3 p_3 - p_1 p_2] - \gamma_{23} k_0 k_2 k_3 k_1^2 [r_1 p_1 - p_2 p_3] \\ &\quad - \gamma_{13} k_0 k_3 k_1 k_2^2 [r_2 p_2 - p_3 p_1] \\ &\quad + k_1 (\varepsilon - \gamma_{22} k_0 k_1 k_3 - \gamma_{33} k_0 k_1 k_2) r_1 \\ &\quad - k_2 (\varepsilon - \gamma_{33} k_0 k_2 k_1 - \gamma_{11} k_0 k_2 k_3) r_2 \end{aligned}$$

$$-k_3(\varepsilon-\gamma_{11}k_0k_3k_2-\gamma_{22}k_0k_3k_1)r_3$$
$$+k_0k_1k_2k_3[\gamma_{23}p_1+\gamma_{13}p_2+\gamma_{12}p_3]$$
$$+\gamma_{11}k_2k_3k_0+\gamma_{22}k_3k_1k_0+\gamma_{33}k_1k_2k_0-\varepsilon, \tag{2.24}$$
$$(\varepsilon:=\gamma_{11}k_0k_2k_3+\gamma_{22}k_0k_3k_1+\gamma_{33}k_0k_1k_2$$
$$+\gamma_{00}k_1k_2k_3+k_0k_1k_2k_3\delta),$$

$$\Box_{\mathbf{R}} = -s_1^2\,\Delta_{\mathbf{R},r_1}-s_2^2\,\Delta_{\mathbf{R},r_2}-s_3^2\,\Delta_{\mathbf{R},r_3}$$
$$-s_1s_2\,\Delta_{\mathbf{R},p_3}-s_2s_3\,\Delta_{\mathbf{R},p_1}-s_3s_1\,\Delta_{\mathbf{R},p_2}$$
$$+\phi_1s_1+\phi_2s_2+\phi_3s_3+\chi, \tag{2.25}$$

$$\phi_1=k_0k_1k_2k_3[-\gamma_{01}k_2k_3(r_2r_3-p_1^2)$$
$$+\gamma_{03}k_1k_2(r_2p_2-p_3p_1)+\gamma_{02}k_1k_3(r_3p_3-p_1p_2)$$
$$+\gamma_{01}(k_2r_2+k_3r_3)-\gamma_{03}k_1p_2-\gamma_{02}k_1p_3-\gamma_{01}]$$
$$\phi_2=k_0k_1k_2k_3[-\gamma_{02}k_1k_3(r_3r_1-p_2^2)$$
$$+\gamma_{03}k_1k_2(r_1p_1-p_2p_3)+\gamma_{01}k_2k_3(r_3p_3-p_1p_2)$$
$$+\gamma_{02}(k_1r_1+k_3r_3)-\gamma_{03}k_2p_1-\gamma_{01}k_2p_3-\gamma_{02}]$$
$$\phi_3=k_0k_1k_2k_3[-\gamma_{03}k_1k_2(r_2r_1-p_3^2)$$
$$+\gamma_{01}k_2k_3(r_2p_2-p_3p_1)+\gamma_{02}k_1k_3(r_1p_1-p_2p_3)$$
$$+\gamma_{03}k_1r_1+\gamma_{03}k_2r_2-\gamma_{02}k_3p_1-\gamma_{01}k_3p_2-\gamma_{03}],$$
$$\chi=k_1^2k_2^2k_3^2\gamma_{00}[r_1r_2r_3-r_1p_1^2-r_2p_2^2-r_3p_3^2+2_{p1}p_2p_3]$$
$$+k_1k_2k_3\gamma_{00}[-k_1k_2(r_1r_2-p_3^2)-k_2k_3(r_2r_3-p_1^2)$$
$$-k_3k_1(r_3r_1-p_2^2)+k_1r_1+k_2r_2+k_3r_3-1]-k_4^{-1}\,\Delta_{\mathbf{R}}. \tag{2.26}$$

It is noted that the linearizable class of non-linear equations given by (2.24) and (2.25) lie within the class of $(3+1)$-dimensional Monge-Ampère equations as introduced in Section 1. The reciprocal relations provide a parametric representation of their solution in terms of the solution of the canonical linear equation (2.20).

3. STRICTLY EXCEPTIONAL SYSTEMS

It has been seen that, in $1+1$ dimensions, the complete exceptionality condition leads to the Monge–Ampère equation (2.13). The more stringent requirement of strict exceptionality wherein the only possible shocks are characteristic was introduced in [12] and subsequently examined in [13]. Here it is noted that the Monge–Ampère equation (2.13) with constant α_i, which has shown to be linearizable via a reciprocal transformation, is also strictly exceptional. Thus, in this case, if $\alpha_2 \neq 0$ then (2.13) can be rewritten as

$$(u_{xx}u_{tt}-u_{xt}^2)+au_{tt}+2bu_{xt}+cu_{xx}+d=0, \quad a,\ b,\ c,\ d\in\mathbb{R}. \tag{3.1}$$

The spatial derivative of (3.1) yields

$$(a+u_{xx})u_{ttx}+2(b-u_{xt})u_{xxt}+(c+u_{tt})u_{xxx}=0 \tag{3.2}$$

with associated characteristic polynomial

$$P(\lambda)=(a+u_{xx})\lambda^2+2(u_{xt}-b)\lambda+c+u_{tt}=0. \tag{3.3}$$

The roots μ, ν of (3.3) are real and distinct in the hyperbolic case, $\Delta=b^2-ac+d>0$, and are given by:

$$\mathrm{k} = \frac{b-u_{xt}+\sqrt{\Delta}}{a+u_{xx}}, \tag{3.4}$$

$$\nu = \frac{b-u_{xt}-\sqrt{\Delta}}{a+u_{xx}}. \tag{3.5}$$

The complete exceptionality of (3.1) is readily confirmed, since

$$\delta\lambda = \frac{\partial\lambda}{\partial u_{xx}}\,\delta u_{xx} + \frac{\partial\lambda}{\partial u_{xt}}\,\delta u_{xt} = 0. \tag{3.6}$$

Now, (3.2) may be written in one of the two equivalent forms

$$\partial_t\mu + \nu\,\partial_x\mu = 0, \tag{3.7}$$

$$\partial_t\nu + \mu\,\partial_x\nu = 0. \tag{3.8}$$

These, in turn, mean that μ=constant along the characteristics $dx/dt=\nu$ and ν=constant along the characteristics $dx/dt=\mu$. Moreover, (3.2) can be written in conservation form as

$$\partial_t(\mu+\nu)+\partial_x(\mu\nu)=0. \tag{3.9}$$

Consider the situation now wherein the second-order partial derivatives of u are discontinuous across a *shock* surface $\psi(x,t)$ moving with shock velocity $\sigma=-\psi_t/\psi_x$. If we denote the jump across $\psi(x,t)=0$ by $[\ ]=(\ )_{\psi=0^+}-(\ )_{\psi=0^-}$ then (3.9) provides the Rankine–Hugoniot relation

$$-\sigma[\mu+\nu]+[\mu\nu]=0, \tag{3.10}$$

that is, in the original variables,

$$-\sigma\left[\frac{2(b-u_{xt})}{a+u_{xx}}\right]+\left[\frac{c+u_{tt}}{a+u_{xx}}\right]=0. \tag{3.11}$$

Since $[\sigma]=0$, we have

$$\left[\frac{(a+u_{xx})\sigma^2-2\sigma(b-u_{xt})+c+u_{tt}}{a+u_{xx}}\right]=0 \tag{3.12}$$

and taking into account the relations

$$\sigma[u_{xx}]+[u_{xt}]=0, \tag{3.13}$$

$$\sigma[u_{xt}]+[u_{tt}]=0, \tag{3.14}$$

it is readily shown that

$$[(a+u_{xx})\sigma^2-2\sigma(b-u_{xt})+cu_{tt}]=[P(\sigma)]=0. \tag{3.15}$$

Hence, (3.11) yields

$$P(\sigma)\left[\frac{1}{a+u_{xx}}\right]=0 \tag{3.16}$$

and it follows from (3.13), (3.14) and (3.16) that if $\sigma\neq 0$ and

$[u_{xx}]=0$ then we must have that $[u_{xt}]=0$ together with $[u_{tt}]=0$ and shocks are precluded. If $\sigma=0$ then $[u_{xt}]=0$ and $[u_{tt}]=0$ but, because of (3.12), it follows that $[u_{xx}]=0$ and shocks are again precluded. Thus, for jumps in the second derivatives to occur, it is required that $[u_{xx}]\neq 0$ whence (3.16) implies that

$$P(\sigma)=0. \tag{3.17}$$

Thus, the only possible shocks are characteristic and, consequently, the linearizable Monge–Ampère equation (3.1) is strictly exceptional.

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