BASIC FEATURES OF THE TIME FINITE ELEMENT APPROACH FOR DYNAMICS*

MARCO BORRI, CARLO BOTTASSO and PAOLO MANTEGAZZA
Politecnico di Milano, Dipartimento di Ingegneria Aerospaziale,
Piazza Leonardo da Vinci 32; 20133 Milano, Italy

(Received: 20 June 1991; accepted in revised form: 18 November 1991)

ABSTRACT. Very general weak forms may be developed for dynamic systems, the most general being analogous to a Hu-Washizu three-field formulation, thus paralleling well-established weak methods of solid mechanics. In this work two different formulations are developed: a pure displacement formulation and a two-field mixed formulation. With the objective of developing a thorough understanding of the peculiar features of finite elements in time, the relevant methodologies associated with this approach for dynamics are extensively discussed. After having laid the theoretical bases, the finite element approximation and the linearization of the resulting forms are developed, together with a method for the treatment of holonomic and nonholonomic constraints, thus widening the horizons of applicability over the vast world of multibody system dynamics. With the purpose of enlightening on the peculiar numerical behavior of the different approaches, simple but meaningful examples are illustrated. To this aim, significant parallels with elastostatics are emphasized.

SOMMARIO. Due differenti formulazioni agli elementi finiti nel tempo sono presentate in questo lavoro quali casi particolari di una formulazione generale a tre campi: la prima è una formulazione agli spostamenti, mentre la seconda è una formulazione mista dove i campi indipendenti sono costituiti da spostamenti e momenti cinetici. Dopo aver sviluppato la linearizzazione e l'approssimazione agli elementi finiti delle forme, viene discussa una tecnica per il trattamento di vincoli di olonomia ed anolonomia. Le principali caratteristiche numeriche dei due metodi vengono infine evidenziate facendo anche ricorso ad esempi semplici ma significativi.

KEY WORDS. Finite elements, Time integration, Dynamics.

INTRODUCTION

At the beginning of the 1970s a renewed interest in finite elements in time as numerical applications of Hamilton's principle gave origin to several interesting works, pioneered by Fried [1] and independently by Argyris and Scharpf [2], and continued by Bailey [3]. The resort to Hamilton's Principle as the starting point for the development of methods for numerical solution of dynamic problems is well motivated, since it seems natural to think that everything that works from a theoretical point of view should also work numerically. However, the different implications associated with this approach have led to vigorous discussions, lasting several years [4–11].

One of the most crucial problems arises as a consequence of the different treatments reserved to the boundary terms. There has been much discussion in the literature concerning this fundamental topic. Nevertheless, it has been shown in [12] that the boundary terms must neither be dropped nor approximated, since they must be retained to allow a correct and fully consistent numerical solution. More recently, Peters and Izadpanah [13] offered a lucid analysis of the different treatments of the boundary terms, giving a proof of convergence, independently developed by Quarteroni [14], and showing how this proof cannot be established when the boundary momenta are approximated in terms of the time derivatives of their associated generalized coordinates, a common procedure often adopted in the literature [1–6, 9–10, 15–19].

Nowadays, the work of a number of authors has proved the time finite element method a mature and well-understood approach for dynamics, so that up to date a really broad class of dynamic problems has been successfully solved; they include rigid body dynamics [3], [20–24], wave propagation [6, 25] and optimal control [26], the treatment of complex phenomena such as nonlinear instabilities of airfoils [12], stability and trim analysis of helicopter rotors, together with the study of the behavior of periodic systems and the perturbation analysis about periodic solutions [27–32].

Finite elements in time for dynamics offer some sound advantages which make them competitive with the classical differential approach:

1. From a theoretical point of view, thanks to the significant analogies which may be drawn with finite elements in space, the general understanding of the numerical behavior and implications associated with the different formulations and with the choice of the shape functions is deeper.
2. Most of the numerical methods and methodologies developed for finite elements in space can be easily shared and are readily applicable to this method for dynamics, thus achieving a substantial unification.

*Paper presented at the '*International Technical Specialists' Meeting on Rotorcraft Basic Research*', March 25–27, 1991, Georgia Institute of Technology, Atlanta, Georgia, USA.

Meccanica **27**: 119–130, 1992.

3. The invariant represented by the energy of the system is preserved by particular finite element formulations. Even if this does not guarantee that other significant invariants possibly present in the system are actually preserved, it is a remarkable property worth of attention.
4. The class of problems which may be solved with a single general-purpose program is broader. This possibly represents the most crucial practical advantage. The additional effort required by the development of the tangent matrix of the dynamic system, upon which the time finite element method relies for the solution process, is greatly rewarded. Besides the classical initial value problems, linearized stability analyses are straightforwardly accomplished, since the transition matrix is readily available. This is definitely interesting, as linearized stability analyses are important in assessing the characteristics of a dynamic system. Moreover, unlike conventional differential approaches, periodic problems can be tackled in a natural way. The solution is attained assembling a suitable number of elements and imposing the appropriate periodic constraint relations, thus implying virtually no practical difference with the solution process required by the initial value problems. These unique characteristics justify the interest which this method has raised in the field of rotor dynamics.

From this preliminary discussion, the soundness of the time finite element approach is definitively assessed. In this work we face the problem of attempting a thorough analysis of this method when applied to general dynamic problems and of all the relevant associated methodologies. The discussion is presented in the context of a very general weak formulation which significantly parallels the weak formulations which are familiar to elastostatics; in fact the same numerical weaknesses, namely the locking problem, are often found to negatively affect even the forms for dynamics in certain instances, so that a bridge between the space and the time finite elements is made.

Specifically, two major forms are treated: a primal pure displacement form and a mixed one, closely analogous to a two-field Hellinger–Reissner form. Both the primal and the mixed forms are shown to arise naturally as special cases of a three-field formulation of the problem of motion: the development of the former is accomplished by enforcing the compatibility conditions and displacement boundary conditions *a priori*, while the latter may be obtained when the constitutive relations are satisfied *a priori*. A similar presentation has been carried out in [23], but is reported here again since it seems particularly attractive from a theoretical point of view.

Since the ability to treat constraints both of the holonomic and of the nonholonomic type is a prerequisite for the solution of a broad class of problems, a weak formulation of the constraint equations is extensively discussed. If it is examined through its equivalent differential formulation, this method may be understood as a modification of the Lagrangian multipliers technique, while it appears as a consistent weak formulation in its integral version which is intimately connected with the weak forms for dynamics addressed here.

GENERAL WEAK FORMS FOR DYNAMICS

For the sake of simplicity, let us consider a material system formed by n particles. With minor alterations it is possible to treat continuous systems or the special case of rigid bodies [23, 24]. Very well known fundamental dynamic equations for the system under consideration are here reported, with the purpose of casting them in a weak form. To this aim, let $\mathbf{x}$ be the vector of all the positions of the particles in the system and let a tilde ($\tilde{}$) denote a quantity referred to coordinates defined by $\mathbf{x}$. Then, the following relations are known to hold:

1. Momentum balance condition:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{m}\tilde{\mathbf{v}}) = \tilde{\mathbf{f}} \tag{1}$$

where $\tilde{m}$ is the generalized mass, $\tilde{\mathbf{v}}$ the velocity and $\tilde{\mathbf{f}}$ the force acting upon the particle system.

2. Velocity definition:

$$\tilde{\mathbf{v}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} \tag{2}$$

3. Constitutive relation:

$$\tilde{\mathbf{p}} = \tilde{m}\tilde{\mathbf{v}} \tag{3}$$

where $\tilde{\mathbf{p}}$ denotes the momentum.

4. Displacement boundary conditions:

$$\begin{aligned} \mathbf{x}(t_i) &= \mathbf{x}_i^b \\ \mathbf{x}(t_{i+1}) &= \mathbf{x}_{i+1}^b \end{aligned} \tag{4}$$

5. Momentum boundary conditions:

$$\begin{aligned} \tilde{\mathbf{p}}(t_i) &= \tilde{\mathbf{p}}_i^b \\ \tilde{\mathbf{p}}(t_{i+1}) &= \tilde{\mathbf{p}}_{i+1}^b \end{aligned} \tag{5}$$

where $(\,)^b$ denotes boundary values and (t_i, t_{i+1}) are boundary times.

We wish here to recall a first analogy which may be drawn with elastostatics, noting that the velocity definition parallels the strain–displacement relation and the constitutive relation parallels the constitutive stress–strain relation in elasticity.

Each of these equations may be expressed in weak form introducing a weight or test function and integrating over the time interval of interest. If one uses test functions that introduce consistent units of work or energy – which is, however, by no means mandatory – the following is

obtained:

$$\int_{t_i}^{t_{i+1}} \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{p}} - \tilde{\mathbf{f}} \right) \cdot \delta\mathbf{x} + \left(\tilde{\mathbf{v}} - \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} \right) \cdot \delta\tilde{\mathbf{p}} + \right.$$
$$\left. + (\tilde{\mathbf{p}} - \tilde{m}\tilde{\mathbf{v}}) \cdot \delta\tilde{\mathbf{v}} \right\} \mathrm{d}t$$
$$= \{(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}^b) \cdot \delta\mathbf{x} - (\mathbf{x} - \mathbf{x}^b) \cdot \delta\tilde{\mathbf{p}}\}|_{t_i}^{t_{i+1}} \tag{6}$$

which is the most general weak form, since none of the relations (1)–(5) is required to be satisfied *a priori*.

The independent fields are the position $\mathbf{x}$, the velocity $\tilde{\mathbf{v}}$ and the momentum $\tilde{\mathbf{p}}$. Integrating by parts the term $\mathrm{d}(\tilde{\mathbf{p}})/\mathrm{d}t$ to accomplish a simplification of the right-hand side term $\tilde{\mathbf{p}} \cdot \delta\mathbf{x}|_{t_i}^{t_{i+1}}$, a three-field formulation equivalent to the Hu-Washizu [33] formulation for elastostatics is obtained, so that in the following we will refer to this form as a 'Hu-Washizu Form'.

Primal Form

A displacement formulation may be devised by resorting to the principle of virtual work, thus requiring a displacement field compatible with the deformations. This implies that the equation

$$\tilde{\mathbf{v}} - \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = 0 \tag{7}$$

is identically satisfied, and that the displacement boundary conditions are identically satisfied as well, so that the second boundary term in (6) is dropped, leading to the following displacement form

$$\int_{t_i}^{t_{i+1}} (m\tilde{\mathbf{v}} \cdot \delta\tilde{\mathbf{v}} + \tilde{\mathbf{f}} \cdot \delta\mathbf{x})\,\mathrm{d}t = \tilde{\mathbf{p}}^b \cdot \delta\mathbf{x}|_{t_i}^{t_{i+1}} \tag{8}$$

where (7) is understood, $\mathbf{x}$ being the only independent field. If the position vector $\mathbf{x}$ is stated as a function of a suitable number of generalized coordinates $\mathbf{q}$ and of time t, Equation (8) becomes

$$\int_{t_i}^{t_{i+1}} \{\delta\mathscr{L}(\dot{\mathbf{q}}, \mathbf{q}, t) + \mathbf{Q}(\dot{\mathbf{q}}, \mathbf{q}, t) \cdot \delta\mathbf{q}\}\,\mathrm{d}t = \mathbf{p}^b \cdot \delta\mathbf{q}|_{t_i}^{t_{i+1}} \tag{9}$$

where the Lagrangian function is denoted by $\mathscr{L}$ and $\mathbf{Q} = \mathbf{f} \cdot \partial\mathbf{x}/\partial\mathbf{q}$ are in general nonconservative generalized forces.

Equation (8) is well known as 'Hamilton's law of varying action', which becomes Hamilton's principle if the test functions are chosen so as to vanish at the boundaries [34]. In the following we will refer to this form with the name of 'primal form', since it deals only with one independent and thus primal field.

It will be shown in the last section that the analogies between these weak forms and the well-known weak forms of elastostatics, are not restricted to a slight resemblance. Particularly, the locking phenomenon which may be observed in solid mechanics in pure displacement formulations, has a corresponding analog even in pure displacement formulations for dynamics, namely the primal form.

This remark sets forth the need to develop an alternative weak form where the independent fields are represented by generalized coordinates and momenta, thus establishing what will be referred to as a 'mixed form'. This second approach seems to be much more alluring even from the point of view of Hamiltonian mechanics: the phase space of a system is represented giving the same dignity and the same order of approximation to its two components, the generalized coordinates $\mathbf{q}$ and momenta $\mathbf{p}$. A single field formulation has not this kind of parallelism in the treatment of $\mathbf{q}$ and $\mathbf{p}$ since the momenta are introduced by means of the time derivatives of the generalized coordinates, thus negatively affecting its numerical behavior.

Mixed Form

A Legendre transformation can be applied to the Lagrangian function $\mathscr{L}$, transforming the velocities into momenta and the Lagrangian function into the Hamiltonian function, thus leading to the sought-for mixed form.

Let us turn back to the previous three-field Hu-Washizu formulation; then, it is possible to enforce *a priori* the constitutive relation, thus implying that the term $\tilde{\mathbf{p}} - \tilde{m}\tilde{\mathbf{v}}$ is null. Now $\tilde{\mathbf{v}}$ can be understood as a function of $\tilde{\mathbf{p}}$, i.e. $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}(\tilde{\mathbf{p}})$. Again we like to parallel this approach to the Hellinger–Reissner formulation in solid mechanics. Let us introduce the Hamiltonian function, defined as

$$\mathscr{H}(\tilde{\mathbf{p}}, \mathbf{x}) = \tilde{\mathbf{p}} \cdot \tilde{\mathbf{v}}(\tilde{\mathbf{p}}) - \mathscr{L}(\tilde{\mathbf{v}}(\tilde{\mathbf{p}}), \mathbf{x})$$

We note explicitly that the Lagrangian function $\mathscr{L}$ is now to be understood as a function of $\tilde{\mathbf{p}}$ and $\mathbf{x}$, and not of $\dot{\mathbf{x}}$, $\mathbf{x}$ as for the single field formulation. Integrating by parts the terms $\mathrm{d}\mathbf{x}/\mathrm{d}t$ and $\mathrm{d}\tilde{\mathbf{p}}/\mathrm{d}t$ and transforming to generalized coordinates, the following mixed form, symmetric in the test functions $(\delta\mathbf{p}, \delta\mathbf{q})$, is obtained

$$\int_{t_i}^{t_{i+1}} \left\{ \mathbf{p} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{q} - \mathbf{q} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{p} - \delta\mathscr{H}(\mathbf{p}, \mathbf{q}) + \mathbf{Q} \cdot \delta\mathbf{q} \right\} \mathrm{d}t$$
$$= (\mathbf{p}^b \cdot \delta\mathbf{q} - \mathbf{q}^b \cdot \delta\mathbf{p})|_{t_i}^{t_{i+1}}. \tag{10}$$

The independent fields are in this case $\mathbf{p}$ and $\mathbf{q}$. This is the weakest possible form, as all boundary conditions are of the natural type and the trial functions $(\mathbf{p}, \mathbf{q})$ have the same continuity requirements. Even the test functions have the same continuity requirements, but an order greater than the trial functions, since $\mathbf{p}$ and $\mathbf{q}$ should be piecewise continuous and $\delta\mathbf{p}$ and $\delta\mathbf{q}$ should be piecewise differentiable.

LINEARIZATION

The linearization of the forms previously presented is necessary in order to resort to a Newton–Raphson like numerical solution.

A general linearized primal form is written as

$$\int_{t_i}^{t_{i+1}} \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{q},\, \delta\mathbf{q} \right) \cdot \mathbf{T}_p \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\mathbf{q},\, \mathrm{d}\mathbf{q} \right) \right\} \mathrm{d}t$$
$$= \delta\mathbf{q} \cdot \mathbf{p}^b \Big|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{q},\, \delta\mathbf{q} \right) \cdot \mathbf{R}_p \right\} \mathrm{d}t \tag{11}$$

where $\mathbf{T}_p$ and $\mathbf{R}_p$ are, respectively, the tangent matrix and the residual vector. At the given state $(\dot{\mathbf{q}}, \mathbf{q})_g$, denoted by the subscript $(\,)_g$, the tangent matrix for the primal form is given by

$$\mathbf{T}_p = \begin{bmatrix} \dfrac{\partial^2 \mathscr{T}}{\partial \dot{\mathbf{q}}^2} & \dfrac{\partial^2 \mathscr{T}}{\partial \mathbf{q} \partial \dot{\mathbf{q}}} \\ \dfrac{\partial^2 \mathscr{T}}{\partial \dot{\mathbf{q}} \partial \mathbf{q}} + \dfrac{\partial \mathbf{Q}}{\partial \dot{\mathbf{q}}} & \dfrac{\partial^2 \mathscr{T}}{\partial \mathbf{q}^2} + \dfrac{\partial \mathbf{Q}}{\partial \mathbf{q}} \end{bmatrix}_g \tag{12}$$

and the residual vector by

$$\mathbf{R}_p = \left(\frac{\partial \mathscr{T}}{\partial \dot{\mathbf{q}}}, \frac{\partial \mathscr{T}}{\partial \mathbf{q}} + \mathbf{Q} \right)_g. \tag{13}$$

Analogously, a general linearized mixed form may be expressed as

$$\int_{t_i}^{t_{i+1}} \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{q} \cdot \mathrm{d}\mathbf{p} - \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{p} \cdot \mathrm{d}\mathbf{q} + \right.$$
$$\left. - (\delta\mathbf{p},\, \delta\mathbf{q}) \cdot \mathbf{T}_m \cdot (\mathrm{d}\mathbf{p},\, \mathrm{d}\mathbf{q}) \right\} \mathrm{d}t$$
$$= (\delta\mathbf{p} \cdot \mathbf{q}^b - \delta\mathbf{q} \cdot \mathbf{p}^b)\Big|_{t_i}^{t_{i+1}} +$$
$$- \int_{t_i}^{t_{i+1}} \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{q} \cdot \mathbf{p}_g - \frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{p} \cdot \mathbf{q}_g + \right.$$
$$\left. + (\delta\mathbf{p},\, \delta\mathbf{q}) \cdot \mathbf{R}_m \right\} \mathrm{d}t \tag{14}$$

where, at the given state $(\mathbf{p}, \mathbf{q})_g$, the tangent matrix is

$$\mathbf{T}_m = \begin{bmatrix} -\dfrac{\partial^2 \mathscr{H}}{\partial \mathbf{p}^2} & -\dfrac{\partial^2 \mathscr{H}}{\partial \mathbf{p} \partial \mathbf{q}} \\ -\dfrac{\partial^2 \mathscr{H}}{\partial \mathbf{q} \partial \mathbf{p}} + \dfrac{\partial \mathbf{Q}}{\partial \mathbf{p}} & -\dfrac{\partial^2 \mathscr{H}}{\partial \mathbf{q}^2} + \dfrac{\partial \mathbf{Q}}{\partial \mathbf{q}} \end{bmatrix}_g \tag{15}$$

and the residual vector is

$$\mathbf{R}_m = \left(-\frac{\partial \mathscr{H}}{\partial \mathbf{p}}, -\frac{\partial \mathscr{H}}{\partial \mathbf{p}} + \mathbf{Q} \right)_g. \tag{16}$$

FINITE ELEMENT APPROXIMATION

In order to develop a finite element approximation, let us subdivide the time interval $(t_{\text{initial}}, t_{\text{final}})$ in a certain number of time nodal points. The time interval $(t_{\text{initial}}, t_{\text{final}})$ is so covered with an appropriate number of consecutive nonoverlapping time elements, each one made of two or more time nodes. The generic time element is defined by the time instants (t_i, t_{i+1}).

This procedure gives rise to two different possibilities: or an implicit step-by-step self-starting integration formula is obtained, or an assembly process may be developed to obtain a solution over a time of interest, giving rise to a triangular system which resembles the step-by-step procedure.

Primal Form

Let us consider an n node time element. Let

$$\bar{\mathbf{q}} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$$
$$\delta\bar{\mathbf{q}} = (\delta\mathbf{q}_1, \delta\mathbf{q}_2, \ldots, \delta\mathbf{q}_n)$$

be vectors of trial and test function nodal values respectively. The parametric approximations are

$$\text{Trial functions} \begin{cases} \mathbf{q} = \mathbf{N} \cdot \bar{\mathbf{q}} \\ \dot{\mathbf{q}} = \dot{\mathbf{N}} \cdot \bar{\mathbf{q}} \end{cases}$$

$$\text{Test functions} \begin{cases} \delta\mathbf{q} = \mathbf{N} \cdot \delta\bar{\mathbf{q}} \\ \delta\dot{\mathbf{q}} = \dot{\mathbf{N}} \cdot \delta\bar{\mathbf{q}} \end{cases}$$

where $\mathbf{N}$ are piecewise Lagrangian shape functions, with the property that

$$N_r(t_s) = \delta_{rs}$$

where δ_{rs} is the Kronecker symbol.

Performing the integrations in Equation (11), resorting to Newton's method, and considering $\delta\bar{\mathbf{q}}$ as free variations, which is definitely the case for an initial value problem, a time-marching procedure is obtained at the jth step

$$\mathbf{K}_{p_j} \cdot \Delta\bar{\mathbf{q}}_j = \mathbf{B}_p \cdot (\mathbf{p}_i^b,\, \mathbf{p}_{i+1}^b) - \mathbf{F}_{p_j} \tag{17}$$

where $\Delta\bar{\mathbf{q}}_j$ are increments to the generalized coordinates at the time nodal points, and

$$\mathbf{K}_{p_j} = \int_{t_i}^{t_{i+1}} \{(\dot{\mathbf{N}},\, \mathbf{N})^T \cdot \mathbf{T}_p \cdot (\dot{\mathbf{N}},\, \mathbf{N})\}\, \mathrm{d}t \tag{18}$$

is the integrated tangent matrix, while

$$\mathbf{F}_{p_j} = \int_{t_i}^{t_{i+1}} \{(\dot{\mathbf{N}},\, \mathbf{N})^T \cdot \mathbf{R}_p\}\, \mathrm{d}t \tag{19}$$

is the integrated residual vector. The matrix $\mathbf{B}_p$ is given by

$$\mathbf{B}_p = \begin{bmatrix} -\mathbf{I} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \mathbf{I} \end{bmatrix}^T. \tag{20}$$

Since $\mathbf{B}_p$ has the form given by (20), it is always possible to eliminate the variables pertinent to the interior nodes in elements with more than two nodes.

By making use of the equations previously developed, periodic boundary value problems may be solved with no additional effort. The solution is obtained enforcing the periodicity constraints $\mathbf{q}_{\text{initial}} = \mathbf{q}_{\text{final}}$ and $\mathbf{p}_{\text{initial}} = \mathbf{p}_{\text{final}}$, assuming a period $T = t_{\text{final}} - t_{\text{initial}}$ and assembling a suitable number of elements. The number of elements which are necessary to obtain an accurate solution of the

problem of motion may be however different from the number of elements suitable for obtaining an accurate stability analysis.

As explicitly noted in [35], the power of this time finite element formulation stems from the fact that the shape functions have to ensure continuity of the generalized displacements $\mathbf{q}$ only, and not of their derivatives, even if higher order approximations can be profitably used. The key to this property is given by the boundary terms: if no forcing function is present, the velocity is continuous over a node instant, while if a forcing function is present it is weighted by the shape functions in (19) exactly as in elastostatics, and it is transformed in weighted impulses at the nodes, thus providing a correct variation of momentum.

Mixed Form

For a mixed formulation, different orders of approximation are requested for the trial functions $(\mathbf{p}, \mathbf{q})$ and the test functions $(\delta\mathbf{p}, \delta\mathbf{q})$, since only the derivatives of the test functions with respect to time appear. Moreover it should be pointed out that the values of $(\mathbf{p}, \mathbf{q})$ evaluated at the boundaries of a time element are not requested to be equal to $(\mathbf{p}^b, \mathbf{q}^b)$. The only restriction we are making is that $(\mathbf{p}^b, \mathbf{q}^b)_i$ depend solely on the boundary node i, and not on the subintervals that share the node i as a vertex. In other words, we are assuming that

$$(\mathbf{p}^b, \mathbf{q}^b)_i^- = (\mathbf{p}^b, \mathbf{q}^b)_i^+ \tag{21}$$

for all i. Let

$$\bar{\mathbf{q}} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{n-1})$$

$$\bar{\mathbf{p}} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-1})$$

be vectors of element internal variables, while

$$\delta\bar{\mathbf{q}} = (\delta\mathbf{q}_1, \delta\mathbf{q}_2, \ldots, \delta\mathbf{q}_n)$$

$$\delta\bar{\mathbf{p}} = (\delta\mathbf{p}_1, \delta\mathbf{p}_2, \ldots, \delta\mathbf{p}_n)$$

are vectors of nodal variables. The parametric approximations for $(\mathbf{p}, \mathbf{q})$ and $(\delta\mathbf{p}, \delta\mathbf{q})$ are thus

$$\text{Trial functions} \begin{cases} \mathbf{q} = \mathbf{M} \cdot \bar{\mathbf{q}} \\ \mathbf{p} = \mathbf{M} \cdot \bar{\mathbf{p}} \end{cases}$$

$$\text{Test functions} \begin{cases} \delta\mathbf{q} = \mathbf{N} \cdot \delta\bar{\mathbf{q}} \\ \delta\mathbf{p} = \mathbf{N} \cdot \delta\bar{\mathbf{p}} \\ \delta\dot{\mathbf{q}} = \dot{\mathbf{N}} \cdot \delta\bar{\mathbf{q}} \\ \delta\dot{\mathbf{p}} = \dot{\mathbf{N}} \cdot \delta\bar{\mathbf{p}} \end{cases}$$

where $\mathbf{M}$ and $\mathbf{N}$ are suitable shape functions.

The following iterative procedure is obtained

$$\mathbf{K}_{m_j} \cdot (\Delta\bar{\mathbf{p}}, \Delta\bar{\mathbf{q}})_j = \mathbf{B}_m \cdot (\mathbf{p}_i^b, \mathbf{q}_i^b, \mathbf{p}_{i+1}^b, \mathbf{q}_{i+1}^b) - \mathbf{F}_{m_j} \tag{22}$$

where once again the integrated tangent matrix is given by

$$\mathbf{K}_{m_j} = \int_{t_i}^{t_{i+1}} \{\dot{\mathbf{N}}^T \cdot \mathbf{I}_m \cdot \mathbf{M} + \mathbf{N}^T \cdot \mathbf{T}_m \cdot \mathbf{M}\}\, dt \tag{23}$$

while the integrated residual vector is given by

$$\mathbf{F}_{m_j} = \int_{t_i}^{t_{i+1}} \{\dot{\mathbf{N}}^T \cdot \mathbf{I}_m \cdot \mathbf{M} \cdot (\bar{\mathbf{p}}, \bar{\mathbf{q}}) + \mathbf{N}^T \cdot \mathbf{R}_m\}\, dt \tag{24}$$

where

$$\mathbf{I}_m = \begin{bmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix} \tag{25}$$

and the matrix $\mathbf{B}_m$ is given by

$$\mathbf{B}_m = \begin{bmatrix} -\mathbf{I}_m & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \mathbf{I}_m \end{bmatrix}^T. \tag{26}$$

Even for this form, since $\mathbf{B}_m$ has the expression given by (26), it is always possible to eliminate the internal variables.

STABILITY ANALYSIS

One of the key features of the finite elements in time for dynamics is represented by the ability to perform linearized stability analyses with no significant effort. This may be extremely useful in assessing the characteristics of a mechanical system.

The stability analysis is performed by making use of the transition matrix $\mathbf{A}$ which maps the initial perturbed state vector $(d\mathbf{p}_i, d\mathbf{q}_i)$ at time t_i into the perturbed state vector $(d\mathbf{p}_{i+1}, d\mathbf{q}_{i+1})$ at time t_{i+1} by means of the following equation

$$(d\mathbf{p}_{i+1}, d\mathbf{q}_{i+1}) = \mathbf{A} \cdot (d\mathbf{p}_i, d\mathbf{q}_i). \tag{27}$$

The stability limits are reached when the spectral radius of the transition matrix $\mathbf{A}$ attains unitary modulus.

We recall here the expression of the time-marching procedure which states Newton's method for a primal approach:

$$\mathbf{K}_{p_j} \cdot \Delta\bar{\mathbf{q}}_j = \mathbf{B}_p \cdot (\mathbf{p}_i^b, \mathbf{p}_{i+1}^b) - \mathbf{F}_{p_j}.$$

Let us split the vector of unknowns $\Delta\bar{\mathbf{q}}$ as

$$\Delta\bar{\mathbf{q}} = \begin{Bmatrix} \Delta\bar{\mathbf{q}}^b \\ \Delta\bar{\mathbf{q}}^m \end{Bmatrix}$$

where the subscript $(\,)_j$ has been dropped for simplicity of notation, and the superscripts $(\,)^b$ and $(\,)^m$ refer to boundary and middle nodes, respectively. It we apply the same partitioning to the matrices $\mathbf{K}_p$, $\mathbf{B}_p$ and to the vector $\mathbf{F}_p$, we thus get

$$\begin{aligned} &\mathbf{K}_p^{bb} \cdot \Delta\bar{\mathbf{q}}^b + \mathbf{K}_p^{bm} \cdot \Delta\bar{\mathbf{q}}^m = \mathbf{B}_p^b \cdot (\mathbf{p}_i^b, \mathbf{p}_{i+1}^b) - \mathbf{F}_p^b \\ &\mathbf{K}_p^{mb} \cdot \Delta\bar{\mathbf{q}}^b + \mathbf{K}_p^{mm} \cdot \Delta\bar{\mathbf{q}}^m = \mathbf{B}_p^m \cdot (\mathbf{p}_i^b, \mathbf{p}_{i+1}^b) - \mathbf{F}_p^m. \end{aligned} \tag{28}$$

Since $\mathbf{B}_p$ has the expression given by Equation (20), then $\mathbf{B}_p^m \equiv 0$; Equation (28) thus becomes

$$\hat{\mathbf{K}}_p \cdot \Delta\bar{\mathbf{q}}^b = \mathbf{B}_p^b \cdot (\mathbf{p}_i^b, \mathbf{p}_{i+1}^b) - \hat{\mathbf{F}}_p \tag{29}$$

where

$$\hat{\mathbf{K}}_p = \mathbf{K}_p^{bb} - \mathbf{K}_p^{bm} \cdot \mathbf{K}_p^{mm^{-1}} \cdot \mathbf{K}_p^{mb} \tag{30}$$

and

$$\hat{\mathbf{F}}_p = \mathbf{F}_p^b - \mathbf{K}_p^{bm} \cdot \mathbf{K}_p^{mm^{-1}} \cdot \mathbf{F}_p^m. \tag{31}$$

Partitioning $\hat{\mathbf{K}}_p$ as

$$\hat{\mathbf{K}}_p = \left[\begin{array}{c|c} \hat{\mathbf{K}}_{11} & \hat{\mathbf{K}}_{12} \\ \hline \hat{\mathbf{K}}_{21} & \hat{\mathbf{K}}_{22} \end{array}\right] \tag{32}$$

the transition matrix $\mathbf{A}_p$ which maps the perturbation $(\mathrm{d}\mathbf{p}_i, \mathrm{d}\mathbf{q}_i)$ into $(\mathrm{d}\mathbf{p}_{i+1}, \mathrm{d}\mathbf{q}_{i+1})$ is easily obtained as

$$\mathbf{A}_p = \begin{bmatrix} -\hat{\mathbf{K}}_{22} \cdot \hat{\mathbf{K}}_{12}^{-1} & \hat{\mathbf{K}}_{21} - \hat{\mathbf{K}}_{22} \cdot \hat{\mathbf{K}}_{12}^{-1} \cdot \hat{\mathbf{K}}_{11} \\ -\hat{\mathbf{K}}_{12}^{-1} & -\hat{\mathbf{K}}_{12}^{-1} \cdot \hat{\mathbf{K}}_{11} \end{bmatrix}. \tag{33}$$

Let us now turn to a mixed approach, recalling the expression of the iterative procedure obtained by resorting to Newton's method

$$\mathbf{K}_{m_j} \cdot (\Delta\bar{\mathbf{p}}, \Delta\bar{\mathbf{q}})_j = \mathbf{B}_m \cdot (\mathbf{p}_i^b, \mathbf{q}_i^b, \mathbf{p}_{i+1}^b, \mathbf{q}_{i+1}^b) - \mathbf{F}_{m_j}$$

and assuming a partition of $\mathbf{K}_m$ given by

$$\mathbf{K}_m = \left[\begin{array}{c} \mathbf{K}_m^i \\ \hline \mathbf{K}_m^m \\ \hline \mathbf{K}_m^f \end{array}\right] \tag{34}$$

where the superscripts $(\,)^i$, $(\,)^m$ and $(\,)^f$ refer respectively to initial, middle and final nodes of the test functions. Recalling the expression of $\mathbf{B}_m$ given by Equation (26), the following relations are found:

$$\begin{aligned} \mathbf{K}_m^i \cdot (\Delta\bar{\mathbf{p}}, \Delta\bar{\mathbf{q}}) &= -\mathbf{I}_m \cdot (\mathbf{p}_i^b, \mathbf{q}_i^b) - \mathbf{F}_m^i \\ \mathbf{K}_m^m \cdot (\Delta\bar{\mathbf{p}}, \Delta\bar{\mathbf{q}}) &= -\mathbf{F}_m^m \\ \mathbf{K}_m^f \cdot (\Delta\bar{\mathbf{p}}, \Delta\bar{\mathbf{q}}) &= \mathbf{I}_m \cdot (\mathbf{p}_{i+1}^b, \mathbf{q}_{i+1}^b) - \mathbf{F}_m^f. \end{aligned} \tag{35}$$

The transition matrix may be found eliminating $(\Delta\bar{\mathbf{p}}, \Delta\bar{\mathbf{q}})$ from Equations (35). For example, solving the first two groups of equations in (35) for $(\Delta\bar{\mathbf{p}}, \Delta\bar{\mathbf{q}})$ and substituting into the last, gives

$$\begin{aligned} &-\mathbf{K}_m^f \cdot \begin{bmatrix} \mathbf{K}_m^i \\ \mathbf{K}_m^m \end{bmatrix}^{-1} \cdot \begin{pmatrix} \mathbf{I}_m \cdot (\mathbf{p}_i^b, \mathbf{q}_i^b) + \mathbf{F}_m^i \\ \mathbf{F}_m^m \end{pmatrix} \\ &\quad = \mathbf{I}_m \cdot (\mathbf{p}_{i+1}^b, \mathbf{q}_{i+1}^b) - \mathbf{F}_m^f. \end{aligned} \tag{36}$$

For a two-node element, which thus has no middle nodes, Equation (36) is significantly simplified to

$$\begin{aligned} &-\mathbf{K}_m^f \cdot \mathbf{K}_m^{i^{-1}} \cdot (\mathbf{I}_m \cdot (\mathbf{p}_i^b, \mathbf{q}_i^b) + \mathbf{F}_m^i) \\ &\quad = \mathbf{I}_m \cdot (\mathbf{p}_{i+1}^b, \mathbf{q}_{i+1}^b) - \mathbf{F}_m^f. \end{aligned} \tag{37}$$

Noticing that $\mathbf{I}_m^{-1} = -\mathbf{I}_m$, the transition matrix is then given by

$$\mathbf{A}_m = \mathbf{I}_m \cdot \mathbf{K}_m^f \cdot \mathbf{K}_m^{i^{-1}} \cdot \mathbf{I}_m. \tag{38}$$

WEAK FORM OF THE CONSTRAINTS

Since constraints naturally arise in many fields of mechanics, it is important to develop the ability to treat equations of constraint.

If one employs a classical differential approach, it is straightforward to resort to the Lagrangian multipliers technique. So many examples are available in the literature, that a review is not applicable here [36].

On the contrary, although many works have been published on finite elements in time, no consistent way of introducing constraint relations was known until 1981, when a weak form of the constraints has been presented in [28]. This topic has been discussed more extensively in [37].

Let us consider the class of constraints expressed by

$$\boldsymbol{\psi}(\dot{\mathbf{q}}, \mathbf{q}, t) = \mathbf{A}(\mathbf{q}, t) \cdot \dot{\mathbf{q}} + \mathbf{a}(\mathbf{q}, t) = 0. \tag{39}$$

Equations (39) are generic functions of the generalized coordinates and of time, but are only linear functions of the generalized velocities. This class groups the vast majority of the constraints which may be practically met in mechanics. Equations $\boldsymbol{\psi}(\dot{\mathbf{q}}, \mathbf{q}, t) = 0$ represent holonomic constraints if a function $\boldsymbol{\phi}$ exists such that $\boldsymbol{\phi}(\mathbf{q}, t) = 0$ and $\dot{\boldsymbol{\phi}}(\mathbf{q}, t) = \boldsymbol{\psi}(\dot{\mathbf{q}}, \mathbf{q}, t)$, so that the differential equations $\boldsymbol{\psi}(\dot{\mathbf{q}}, \mathbf{q}, t)$ are integrable. Otherwise they represent nonholonomic constraints, which thus are given as nonintegrable relations between the differentials of the variables, and not as relations between the variables themselves.

Equations (39) entail the following equations in the virtual displacements

$$\mathbf{A} \cdot \delta\mathbf{q} = 0. \tag{40}$$

In the context of weak formulations, Equations (39) and (40) should be enforced with a suitable choice of test functions, which may be chosen as $\delta\boldsymbol{\mu}$ for the functions $\boldsymbol{\psi}$, and as $\dot{\boldsymbol{\mu}}$ for the functions $\mathbf{A} \cdot \delta\mathbf{q}$. This approach represents a strategy in accordance with that previously employed, since it allows an integration by parts which reduces the continuity requirements. The weak form of the constraints is thus given by

$$\int_{t_i}^{t_{i+1}} \left(\delta\boldsymbol{\mu} \cdot \boldsymbol{\psi} - \boldsymbol{\mu} \cdot \frac{\partial \boldsymbol{\psi}}{\partial \dot{\mathbf{q}}} \cdot \delta\mathbf{q} \right) \mathrm{d}t = 0 \tag{41}$$

which, integrated by parts, leads to

$$\begin{aligned} &\int_{t_i}^{t_{i+1}} \left\{ \delta(\boldsymbol{\mu} \cdot \boldsymbol{\psi}) - \boldsymbol{\mu} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \boldsymbol{\psi}}{\partial \dot{\mathbf{q}}} - \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{q}} \right) \cdot \delta\mathbf{q} \right\} \mathrm{d}t \\ &\quad = \boldsymbol{\mu} \cdot \frac{\partial \boldsymbol{\psi}}{\partial \dot{\mathbf{q}}} \cdot \delta\mathbf{q} \Big|_{t_i}^{t_{i+1}} \end{aligned} \tag{42}$$

which is consistent with the weak forms developed for dynamics. Here the only assumption made is that the differential operator d and the variational operator δ can commute even for nonholonomic constraints, so that the transposition relation $\mathrm{d}(\delta q) = \delta(\mathrm{d}q)$ holds for each coordinate.

Constrained Primal and Mixed Forms

The three terms of Equation (42) may be directly combined with the three terms of the primal form expressed by (9) to obtain a constrained primal form, which may be written as

$$\int_{t_i}^{t_{i+1}} (\delta \mathscr{L}^* + \mathbf{Q}^* \cdot \delta \mathbf{q})\, \mathrm{d}t = \mathbf{p}^{*^b} \cdot \delta \mathbf{q}|_{t_i}^{t_{i+1}} \tag{43}$$

where the following definitions hold

$$\mathbf{p}^* \hat{=} \mathbf{p} + \boldsymbol{\mu} \cdot \frac{\partial \boldsymbol{\psi}}{\partial \dot{\mathbf{q}}}$$

$$\mathscr{L}^* \hat{=} \mathscr{L} + \boldsymbol{\mu} \cdot \boldsymbol{\psi}$$

$$\mathbf{Q}^* \hat{=} \mathbf{Q} + \boldsymbol{\mu} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \boldsymbol{\psi}}{\delta \dot{\mathbf{q}}} - \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{q}} \right).$$

If $\mathscr{L}^*$ is understood as a modified Lagrangian function, $\mathbf{p}^*$ as modified momenta and $\mathbf{Q}^*$ as modified forces, one can interpret the modifying force term originated by the constraints as a weighted form of integrability conditions of the constraint equations. As a matter of fact, in the case of holonomic constraints, it is easy to show that the modifying term of the generalized forces is identically zero. In fact, if the constraints are holonomic the integrability of Equations (39) implies the existence of a function ϕ such that $\dot{\phi}(\mathbf{q}, t) = \boldsymbol{\psi}(\dot{\mathbf{q}}, \mathbf{q}, t)$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \boldsymbol{\psi}}{\partial \dot{\mathbf{q}}} - \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{q}} = 0$$

which proves that the modifying terms are zero.

As the true momenta $\mathbf{p}$ are related to the Lagrangian function $\mathscr{L}$ by the relation

$$\mathbf{p} = \frac{\partial \mathscr{L}}{\partial \dot{\mathbf{q}}}$$

the same relation is easily recognized between the modified momenta $\mathbf{p}^*$ and the modified Lagrangian function $\mathscr{L}^*$

$$\mathbf{p}^* = \frac{\partial \mathscr{L}^*}{\partial \dot{\mathbf{q}}}$$

which is due to the boundary terms in (42).

Analogously, it is straightforward to develop a constrained mixed form, by introducing the concept of a modified Hamiltonian function defined by

$$\mathscr{H}^* = \mathbf{p}^* \cdot \dot{\mathbf{q}} - \mathscr{L}^*.$$

If the modified momenta $\mathbf{p}^*$, the generalized coordinates $\mathbf{q}$ and the multipliers $\boldsymbol{\mu}$ are understood to be independent fields, the constrained mixed form is written

$$\int_{t_i}^{t_{i+1}} \left\{ \mathbf{p}^* \cdot \frac{\mathrm{d}}{\mathrm{d}t} \delta \mathbf{q} - \mathbf{q} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \delta \mathbf{p}^* - \delta \mathscr{H}^* + \mathbf{Q}^* \cdot \delta \mathbf{q} \right\} \mathrm{d}t$$
$$= (\mathbf{p}^{*^b} \cdot \delta \mathbf{q} - \mathbf{q}^b \cdot \delta \mathbf{p}^*)|_{t_i}^{t_{i+1}}. \tag{44}$$

Equation (44) is again completely similar to the unconstrained formulation, since the effect of the constraints has been assimilated in the definitions of $\mathbf{p}^*$, $\mathscr{L}^*$ and $\mathbf{Q}^*$. Moreover, while the generalized momenta $\mathbf{p}$ must satisfy the conditions expressed by Equations (39), the modified momenta $\mathbf{p}^*$ must not. This is definitely interesting from a numerical point of view and it has been observed that the corresponding differential equivalent of these weak forms, which is expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{q}} = \mathbf{Q} - \mathbf{A} \cdot \dot{\boldsymbol{\mu}}$$
$$\mathbf{A} \cdot \dot{\mathbf{q}} + \mathbf{a} = 0 \tag{45}$$

has a better numerical behavior in certain instances compared to a classical Lagrangian multipliers approach. It can be recognized that the Lagrangian multipliers $\boldsymbol{\mu}$ here employed are nothing but the opposite in sign of the time derivatives of those used in the classical literature on the subject ([36], [38], [39]). This difference is of great conceptual significance: as a matter of fact, Equations (45) may be understood as a constraint stabilization technique which does not suffer from the problems of other well-known stabilization techniques, e.g. Baumgarte's method [40], where problem dependent values for some stabilization parameters have to be chosen, and moreover it seems physically and mathematically more motivated.

The primal and mixed constrained forms have been successfully used in the context of rigid body dynamics to enforce the presence of the unitary quaternion relation [21], and in the context of multibody systems to enforce different kinds of mechanical joints [22], [23].

NUMERICAL STUDIES

Let us consider as a very simple but really enlightening example, a single-degree-of-freedom (SDoF) oscillator. Let q be the position of the mass and p its momentum.

In primal form, the SDoF problem becomes

$$\int_{t_i}^{t_{i+1}} \{ m\dot{q} \cdot \delta \dot{q} - (c\dot{q} + kq) \cdot \delta q \}\, \mathrm{d}t = p^b \cdot \delta q|_{t_i}^{t_{i+1}} \tag{46}$$

which produces the following transition matrix, if a two-node time element is adopted:

$$\mathbf{A}_p = \frac{1}{\rho^2 + 6\rho\xi + 6} \cdot \begin{bmatrix} -2\rho^2 - 6\rho\xi + 6 & .5(\rho^2 - 12)\rho^2 \dfrac{m}{\Delta t} \\ 6\dfrac{\Delta t}{m} & -2\rho^2 + 6\rho\xi + 6 \end{bmatrix} \tag{47}$$

where q is linear in the time interval $(t_i, t_{i+1}) = \Delta t$. In (47), $\xi = c/c_{\mathrm{cr}}$, where c_{cr} is the critical damping defined as $c_{\mathrm{cr}} = 2\sqrt{km}$, and for the sake of conciseness the position $\rho = \omega \Delta t$ has been assumed, where $\omega = \sqrt{k/m}$.

The careful reader will not miss the fact that the evolution of (p_{i+1}, q_{i+1}) as Δt increases is governed by

terms arising from the potential energy, thus leading to a conditionally stable integration scheme. The same behavior may be observed even if time elements with more than two nodes are adopted: in Figure 1 all the integration schemes present a stability boundary, where the spectral radius emerges from the unitary limit. Moreover, the disturbing presence of instability 'bubbles' may be noticed. The growth of potential energy is indeed the locking phenomenon we were previously referring to. It may be observed in all those situations where different energy contributions, in this case the potential and kinetic energies, are described with polynomial approximations. The analogy with elastostatics is even deeper: it is easy to prove, at least experimentally [12], that a reduced element quadrature is able to eliminate the problem, and this is exactly the same method employed in elastostatics as an ad hoc procedure to remove this kind of problem. If we now turn to the mixed approach, thus adopting constant shape functions for the trial functions $(\mathbf{p}, \mathbf{q})$ and linear shape functions for the test functions $(\delta\mathbf{p}, \delta\mathbf{q})$, we are able to develop an integration scheme characterized by the following transition matrix:

$$\mathbf{A}_m = \frac{1}{\rho^2+4\rho\xi+4} \cdot \begin{bmatrix} -\rho^2-4\rho\xi+4 & -4\rho^2\dfrac{m}{\Delta t} \\ 4\dfrac{\Delta t}{m} & -\rho^2+4\rho\xi+4 \end{bmatrix}. \tag{48}$$

Here it is easily recognizable that the same order of approximation employed for the generalized coordinates and the momenta inhibits the locking phenomenon, since the potential energy and the kinetic energy are completely unconstrained and thus able to exchange completely with each other in one period of oscillation. This means that the procedure is unconditionally stable, and this characteristic has been obtained without any ad hoc procedure necessary in pure displacement formulations.

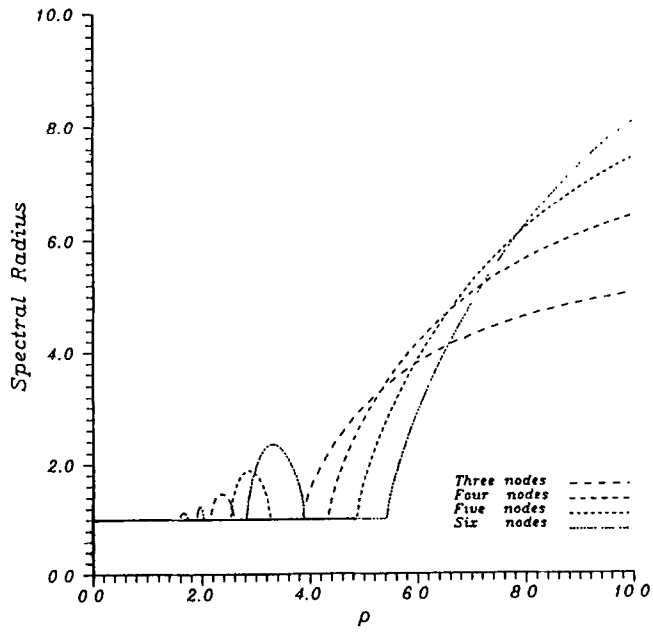

Fig. 1. Spectral radius for the SDoF problem in primal form. Time finite elements with different number of modes are considered.

Moreover when there is no physical damping, it is remarkable that the total energy of the system is preserved since it is immediate to prove from (48) that

$$\frac{1}{2}\frac{p_{i+1}^2}{m} + \frac{k}{2}q_{i+1}^2 = \frac{1}{2}\frac{p_i^2}{m} + \frac{k}{2}q_i^2. \tag{49}$$

This is not the case for the primal approach, in which we have, by making use of matrix (47), that

$$\frac{1}{2}\frac{p_{i+1}^2}{m} + \frac{k}{2}q_{i+1}^2 = \frac{1}{2}\mathscr{A}\frac{p_i^2}{m} + \frac{k}{2}\mathscr{B}q_i^2 + \mathscr{C}p_iq_i \tag{50}$$

where

$$\mathscr{A} = \frac{144+48\rho^2+16\rho^4}{(12+2\rho^2)^2}$$

$$\mathscr{B} = \frac{144+48\rho^2-8\rho^4+\rho^6}{(12+2\rho^2)^2}$$

$$\mathscr{C} = \omega\frac{12\rho^3-4\rho^5}{(12+2\rho^2)^2}.$$

Only at the limit case $\Delta s = 0$, do $\mathscr{A}$ and $\mathscr{B}$ attain a unitary value, while $\mathscr{C}$ become zero. The same basic behavior may be observed if elements with more than two nodes are employed.

Although the energy is thus not preserved by the primal formulation, it is however possible to identify a quantity which is actually preserved. To this aim, let us note that from Equation (46), adopting a two-node time element, one gets

$$\begin{aligned} \mathscr{D}q_i - \mathscr{E}q_{i+1} &= -p_i \\ -\mathscr{E}q_i + \mathscr{D}q_{i+1} &= -p_{i+1} \end{aligned} \tag{51}$$

where

$$\mathscr{D} = \frac{m}{\Delta t} - \frac{k}{3}\Delta t$$

and

$$\mathscr{E} = \frac{m}{\Delta t} + \frac{k}{6}\Delta t.$$

From (51), the following expression is obtained:

$$\frac{1}{2}\frac{p_{i+1}^2}{m} + \frac{\mathscr{E}^2-\mathscr{D}^2}{2m}q_{i+1}^2 = \frac{1}{2}\frac{p_i^2}{m} + \frac{\mathscr{E}^2-\mathscr{D}^2}{2m}q_i^2 \tag{52}$$

where

$$\frac{\mathscr{E}^2-\mathscr{D}^2}{2m} = \frac{k}{2}\left(1-\frac{1}{12}\rho^2\right). \tag{53}$$

From Equations (52) and (53) it may be argued that the motion of the system is confined over an ellipse in the

phase space when the algorithm is stable. On the contrary, when the stability limit is exceeded, the motion degenerates and becomes a hyperbola.

Adopting a reduced order integration, the coefficients $\mathscr{D}$ and $\mathscr{E}$ which appear in (51) become

$$\mathscr{D} = \frac{m}{\Delta t} - \frac{k}{4}\Delta t$$

$$\mathscr{E} = \frac{m}{\Delta t} + \frac{k}{4}\Delta t$$

so that

$$\frac{\mathscr{E}^2 - \mathscr{D}^2}{2m} = \frac{k}{2} \tag{54}$$

and the algorithm is unconditionally stable and preserves the energy as the mixed form does.

It is interesting to investigate the order of convergence of the primal and of the mixed form as the number of nodes per element is varied. To this aim, we introduce the theoretical solution of the SDoF problem subject to the initial conditions $q(0)=q_0$, $\dot{q}(0)=\dot{q}_0$, which is given by the following expressions of the evolution operator $\mathbf{E}$ which maps the initial state vector $(\dot{q}_0, q_0)$ into the state vector $(\dot{q}(t), q(t))$ at the generic time instant t.

Underdamped system $(0 \leqslant \xi < 1)$

Let $\bar{\omega} = \sqrt{1-\xi^2}$ and $\alpha = \omega\xi$. Then

$$\mathbf{E}_u(t) = \mathrm{e}^{-\alpha t} \sin \bar{\omega} t \cdot \begin{bmatrix} \tan^{-1} \bar{\omega} t - \alpha \bar{\omega}^{-1} & -\bar{\omega} - \alpha^2 \bar{\omega}^{-1} \\ \bar{\omega}^{-1} & \alpha \bar{\omega}^{-1} + \tan^{-1} \bar{\omega} t \end{bmatrix}. \tag{55}$$

Critically damped system $(\xi = 1)$

$$\mathbf{E}_c(t) = \mathrm{e}^{-\omega t} \begin{bmatrix} 1-\omega t & -\omega^2 t \\ t & 1+\omega t \end{bmatrix}. \tag{56}$$

Overdamped system $(\xi > 1)$

Let $\hat{\omega} = \omega\sqrt{\xi^2 - 1}$ and $\alpha = \omega\xi$. Then

$$\mathbf{E}_0(t) = \mathrm{e}^{-\alpha t} \sinh \hat{\omega} t \cdot \begin{bmatrix} \tanh^{-1} \hat{\omega} t - \alpha \hat{\omega}^{-1} & -\hat{\omega} - \alpha^2 \hat{\omega}^{-1} \\ \hat{\omega}^{-1} & \alpha \hat{\omega}^{-1} + \tanh^{-1} \hat{\omega} t \end{bmatrix}. \tag{57}$$

From the analytical expressions of the transition matrices, the order of convergence may be obtained expanding in Taylor series the same transition matrices and the evolution operators.

Tables 1, 2 and 3 present the orders of approximation of $\dot{q}$ and q, which are denoted by $\kappa_{\dot{q}_p}$ and κ_{q_p} for the primal form, and of p and q, denoted by κ_{p_m} and κ_{q_m} for the mixed form. In the third column the order of convergence κ is reported. The growth of the order of convergence is remarkably of two for each node added to the time element. Note how for the primal form, in absence of physical damping, at a given number of nodes, $\kappa_{\dot{q}_p}$ is not equal to κ_{q_p}.

TABLE I
Order of approximation and of convergence for the primal form. Undamped case.

Primal form ($\xi = 0$)			
Number of nodes	$\kappa_{\dot{q}_p}$	κ_{q_p}	κ
2	2	3	2
3	5	4	4
4	6	7	6

TABLE II
Order of approximation and of convergence for the primal form. Damped case.

Primal form ($\xi \neq 0$)			
Number of nodes	$\kappa_{\dot{q}_p}$	κ_{q_p}	κ
2	2	2	2
3	4	4	4
4	6	6	6

TABLE III
Order of approximation and of convergence for the mixed form. Damped and undamped cases.

Mixed form			
Number of nodes	κ_{p_m}	κ_{q_m}	κ
2	2	2	2
3	4	4	4
4	6	6	6

It is worth pointing out another interesting property of the mixed form. Let us consider again the SDoF problem, which in first-order form may be written as

$$\dot{\mathbf{y}}(t_i) = \mathbf{S} \cdot \mathbf{y}(t_i) \tag{58}$$

where $\mathbf{y}(t_i) = (p(t_i), q(t_i))$ and

$$\mathbf{S} = \begin{bmatrix} -c/m & -k \\ 1/m & 0 \end{bmatrix}. \tag{59}$$

The algorithmic counterpart of (58) is expressed by means of the transition matrix as

$$\mathbf{y}_i = \mathbf{A} \cdot \mathbf{y}_{i-1}. \tag{60}$$

Let us find out an algorithmic analog of (58) by means of (60). One gets

$$\mathbf{y}_{i-1} = \mathbf{A}^{-1} \cdot \mathbf{y}_i \tag{61}$$

$$\dot{\mathbf{y}}_i = \dot{\mathbf{A}} \cdot \mathbf{y}_{i-1} \tag{62}$$

and thus the matrix we are looking for is given by $\tilde{\mathbf{S}} = \dot{\mathbf{A}} \cdot \mathbf{A}^{-1}$. The difference $\tilde{\mathbf{S}} - \mathbf{S}$ may be understood as a global measure of accuracy, since it accounts for the differences existing between the original system representing the SDoF problem and the equivalent discrete system

implied by a certain algorithm. Moreover, while the eigenvalues of the transition matrix are the algorithmic counterparts of the eigenvalues of the evolution matrix, the eigenvalues of $\tilde{\mathbf{S}}$ are the algorithmic counterparts of the eigenvalues of the real system.

When no physical damping is present, the matrix $\mathbf{S}$ possesses the symplectic structure

$$\mathbf{S} = \begin{bmatrix} 0 & -k \\ 1/m & 0 \end{bmatrix}.$$

Generally speaking, an algorithm modifies the symplectic structure of the matrix S, introducing terms $\tilde{S}_{1,1}$ and $\tilde{S}_{2,2}$ different from zero. Considering the expression of S given by (59) in the presence of damping, it is possible to argue that the condition $\tilde{S}_{1,1} \neq 0$ implies the introduction by means of the algorithm of a certain amount of damping that is not really present in the physical system, while the condition $\tilde{S}_{2,2} \neq 0$ may be explained as a violation of the constitutive relation $p = m\dot{q}$.

The symplectic structure is preserved by the algorithms originated by the mixed form, while it is not preserved by those derived by the primal form. For example, for a two-node time element one gets

$$\tilde{\mathbf{S}}_p = \frac{1}{36+12\rho^2+\rho^4} \cdot \begin{bmatrix} -3\rho^2 \cdot \Delta t & -(36+3\rho^2+\rho^4) \cdot k \\ (36+\rho^2) \cdot \dfrac{1}{m} & 3\rho^2 \cdot \Delta t \end{bmatrix} \tag{63}$$

and

$$\tilde{\mathbf{S}}_m = \frac{1}{4+\rho^2} \cdot \begin{bmatrix} 0 & -4 \cdot k \\ 4 \cdot \dfrac{1}{m} & 0 \end{bmatrix}. \tag{64}$$

The symmetry of the multiplicative terms that enter into the extra-diagonal elements for the mixed form, is immediately noticed. The same elements are not symmetrical for the primal form. These remarks remain valid even in the presence of physical damping. This is a further consequence of the same order of approximation required by the mixed form for the generalized coordinates $\mathbf{q}$ and the generalized momenta $\mathbf{p}$. The same kind of behavior addressed here has been noticed when the number of nodes of the time element is increased.

Let us now address a different problem originated in the context of rigid body dynamics, namely that of a spinning top. The same example has been presented in [23], [24], but it is given again here since, in the authors' opinion, it is very effective in proving the capability of the mixed form to correctly evaluate the stability limits of a physical system.

We assume that the top is rotating at constant speed about a vertical axis and is acted upon by gravity. Let Ω denote the actual rotation speed of the top and Ω_c be the critical speed, which may be expressed as

$$\Omega_c = \frac{2}{J_a}\sqrt{mgdJ_t} \tag{65}$$

where J_a and J_t are, respectively, the axial and transverse moments of inertia referred to the point of contact, m is the mass, d is the distance from the contact point to the center of gravity and g is the acceleration due to gravity.

The tangent matrix for the spinning top problem in primal form may be written as

$$\mathbf{T}_p = \begin{bmatrix} J_t & -\frac{1}{2}iJ_a\Omega \\ \frac{1}{2}iJ_a\Omega & mgd \end{bmatrix} \tag{66}$$

if we resort to the complex notation $\varphi = \varphi_1 + i\varphi_2$ for the sake of simplicity, with $i = \sqrt{-1}$, and by making use of the fact that the rotation about the vertical axis is decoupled from the others.

The integrated tangent matrix thus becomes

$$\mathbf{K}_p = \begin{bmatrix} \mathscr{F} & -\mathscr{G} + i\mathscr{H} \\ -\mathscr{G} - i\mathscr{H} & \mathscr{F} \end{bmatrix} \tag{67}$$

where the following positions are understood

$$\mathscr{F} = \frac{J_t}{\Delta t} + \frac{mgd\,\Delta t}{3}$$

$$\mathscr{G} = \frac{J_t}{\Delta t} + \frac{mgd\,\Delta t}{6}$$

$$\mathscr{H} = \frac{J_a\Omega}{2}.$$

The eigenvalues of the associated transition matrix $\mathbf{A}_p$ are then

$$\lambda_p^{1,2} = \frac{-2mgd\,\Delta t^2 - 6J_t}{3i\Omega J_a\,\Delta t - 6J_t + mgd\,\Delta t^2} \pm \pm \frac{\sqrt{-3J_a^2\Omega^2 + m^2g^2d^2\,\Delta t^2 + 12J_tmgd}}{3i\Omega J_a\Delta t - 6J_t + mgd\,\Delta t^2}. \tag{68}$$

The stability limit, attained when the spectral radius is equal to 1, is reached when

$$3J_a^2\Omega^2 - m^2g^2d^2\,\Delta t^2 - 12J_tmgd = 0.$$

This means, in terms of Ω, that

$$\Omega^2 = \Omega_c^2\left(1 + \frac{mgd\,\Delta t^2}{12}\right)$$

and the boundary stability speed is a function of Δt, which is thus correctly evaluated only as the limiting case of $\Delta t = 0$.

Let us turn to the mixed approach. In complex notation, the tangent matrix is

$$\mathbf{T}_m = \begin{bmatrix} -J_t^{-1} & i\dfrac{J_a\Omega}{2J_t} \\ -i\dfrac{J_a\Omega}{2J_t} & mgd \end{bmatrix} \tag{69}$$

thus leading to the following form of the integrated tangent matrix:

$$\mathbf{K}_m = \begin{bmatrix} -\frac{\Delta t}{2J_t} & 1+\frac{iJ_a\Omega\Delta t}{4J_t} \\ -1-\frac{iJ_a\Omega\Delta t}{4J_t} & \frac{mgd\Delta t}{2} \\ -\frac{\Delta t}{2J_t} & -1+\frac{iJ_a\Omega\Delta t}{4J_t} \\ 1-\frac{iJ_a\Omega\Delta t}{4J_t} & \frac{mgd\Delta t}{2} \end{bmatrix}. \tag{70}$$

The eigenvalues of the associated transition matrix $\mathbf{A}_m$ attain unitary modulus for a value of Ω which does not depend on Δt and is exactly Ω_c, so that the stability limit is not a function of the time step as it happens to be with the primal form. This behavior is strictly analogous to the unconditional stability found for the SDoF problem.

At this point, for the sake of completeness, it would be interesting to give a comparison of the computational cost of the finite element in time approach with respect to other standard time-marching procedures. This, however, is beyond the scope of the present work, since rough *a priori* estimates are difficult to give and possibly unreliable for realistic problems. Note in fact that these methods are extremely accurate, especially when higher order elements are employed. But while in such cases the single elements retain a high sparsity level, the bandwidth of the tangent matrix is drastically increased. This indicates the need to resort to special techniques such as full sparse solvers (not skyline) in order to efficiently take advantage of the potential numerical superiority of these methods. It is thus clear that meaningful comparisons can be done only with highly optimized codes capable of dealing with realistic examples, such as flexible multibody systems with large numbers of degrees of freedom.

CONCLUDING REMARKS

A comprehensive review of the basic theory concerned with finite elements in time for dynamics has been given. The different formulations arising in this context have been presented in the frame of a very general weak formulation, since in the authors' opinion this gives a consistent picture of the situation and permits a deeper understanding of the numerical implications. To this purpose, significant analogies with the well-known weak forms for elastostatics and their behavior have been emphasized. Moreover, it has been shown how to formally develop the finite element approximation, the linearization of the resulting forms and how initial value problems, periodic problems and linearized stability analyses may be carried out.

Since no method for dynamics may be really useful without the ability to take into account the presence of constraint relations, a methodology which achieves this object has been addressed, improving further on the capabilities of this approach and extending its horizons over the world of multibody systems. It has been shown that in this way the constrained problem is consistently and elegantly expressed by means of modified forms which are not formally different from their unconstrained counterparts.

Eventually, two simple examples have been discussed with the purpose of illustrating the basic numerical features of the primal and mixed forms. Both the examples lead to assess the superior performance of the mixed approach over the primal. The remarkable property of the mixed form to preserve the total energy of the system in the problem of the SDoF system has been emphasized. This behavior is a consequence of the same treatment and order of approximation adopted for the generalized coordinates and their associated momenta which is allowed by the resort to a mixed, two-field approach.

REFERENCES

1. Fried, I., 'Finite element analysis of time dependent phenomena', *AIAA Jl*, **7** (1969) 1170–1173.
2. Argyris, J. H. and Scharpf, D. W., 'Finite elements in time and space', *Aeronaut. J. Roy. Aeronaut. Soc.*, **73** (1969) 1041–1044.
3. Bailey, C. D., 'Application of Hamilton's law of varying action', *AIAA Jl*, **13**(9) (1975) 1154–1157.
4. Smith, C. V. and Smith, D. R., 'Comment on "Application of Hamilton's law of varying action"', *AIAA Jl*, **15**(2) (1977) 284–286.
5. Bailey, C. D., 'Reply by Author to C. V. Smith Jr and D. R. Smith', *AIAA Jl*, **15**(2) (1977) 286–287.
6. Simkins, T. E., 'Unconstrained variational statements for initial and boundary-value problems', *AIAA Jl*, **16**(6) (1978) 559–563.
7. Smith, C. V., 'Comment on "Unconstrained variational statements for initial and boundary-value problems"', *AIAA Jl*, **17**(1) (1979) 126–127.
8. Simkins, T. E., 'Reply by Author to C. V. Smith', *AIAA Jl*, **17**(1) (1979) 127–128.
9. Bailey, C. D., 'Hamilton's law and the stability of nonconservative continuous systems', *AIAA Jl*, **18**(3) (1980) 347–349.
10. Smith, C. V., 'Comment on "Hamilton's law and the stability of nonconservative continuous systems"', *AIAA Jl*, **19**(3) (1981) 415.
11. Bailey, C. D., 'Reply by Author to C. V. Smith Jr, *AIAA Jl*, **19**(3) (1981) 416.
12. Borri, M., Ghiringhelli, G. L., Lanz, M., Mantegazza, P. and Merlini, T., 'Dynamic response of mechanical systems by a weak Hamiltonian formulation', *Comput. Struct.*, **20** (1985) 495–508.
13. Peters, D. A. and Izadpanah, A., 'hp-Version finite elements for the space time domain', *Comput. Mech.*, **3** (1988) 73–88.
14. Quarteroni, A., 'Some remarks on finite elements for second order initial value problems', Private communication, 1990.
15. Smith, D. R. and Smith, C. V., 'When is Hamilton's principle an extremum principle?', *AIAA Jl*, **0** (1974) 1573–1576.
16. Bailey, C. D., 'Hamilton, Ritz, and elastodynamics', *J. Appl. Mech.*, **43** (1976) 684–688.
17. Simkins, T. E., 'Finite elements for initial value problems in dynamics', *AIAA Jl*, **19**(10) (1981) 1357–1362.
18. Baruch, M. and Riff, R., 'Hamilton's principle, Hamilton's law – 6" correct formulations', *AIAA Jl*, **20**(5) (1982) 687–691.
19. Riff, R. and Baruch, M., 'Stability of time finite elements', *AIAA Jl*, **22**(8) (1984) 1171–1173.
20. Borri, M. and Mantegazza, P., 'Finite time element approximation of dynamics of non-holonomic systems', in *AMSE Congress*, Williamsburg, Virginia, August 1986.

21. Borri, M., Mello, F., Iura, M. and Atluri, N., 'Primal and mixed forms of Hamilton's principle for constrained rigid and flexible dynamical systems: Numerical studies', in *Non-Linear Vibrations, Stability, and Dynamics of Structures and Mechanisms*, Donaldson Brown Continuing Education Center, Virginia Politechnic Institute and State University, June 1988.
22. Borri, M. and Atluri, N., 'Time finite element method for the constrained dynamics of a rigid body', in *ICES-88*, Atlanta, Georgia, April 1988.
23. Mello, F., 'Weak formulations in analytical dynamics, with applications to multi-rigid-body systems, using time finite elements', PhD Thesis, Georgia Institute of Technology, Atlanta, Georgia, December 1989.
24. Borri, M., Mello, F. and Atluri, S. N., 'Variational approaches for dynamics and time-finite-elements: Numerical studies', *Comput. Mech.*, **7**(1) (1990) 49–76.
25. Iura, M., Borri, M. and Atluri, S. N., 'Analysis of travelling wave responses of structures', in *ICES-88*, Atlanta, Georgia, April 1988.
26. Hodges, D. H. and Bless, R. R., 'A weak Hamiltonian finite element method for optimal control problems', *J. Guidance Control Dynam.* (1990). To appear.
27. Borri, M., Lanz, M. and Mantegazza, P., 'Helicopter rotor dynamics by finite element time discretization', *L'Aerotec. Missili Spazio*, **60** (1981) 193–200.
28. Borri, M., Lanz, M. and Mantegazza, P., 'A general purpose program for rotor blade dynamics', in *Seventh European Rotorcraft and Powered Lift Aircraft Forum*, Deutsche Gesellschaft für Luft und Raumfahrt, Garmish-Partenkirchen, September 1981.
29. Borri, M., Lanz, M., Mantegazza, P., Orlandi, D. and Russo, A., 'STAHR: a Program for stability and trim analysis of helicopter rotors', in *Eighth European Rotorcraft Forum*, Association Aeronatique et Astronatique de France, Aix-en-Provence, September 1982.
30. Borri, M. and Mantegazza, P., 'Some contributions on structural and dynamic modelling of helicopter rotor blades', *L'Aerotec. Missili Spazio*, **00** (1985) 143–154.
31. Izadpanah, A., 'Calculation of Floquet stability by generalization of Hamilton's law to a bilinear formulation', in *40th Ann. Nat'l Forum of the American Helicopter Society*, Arlington, Texas, 1985.
32. Borri, M., 'Helicopter rotor dynamics by finite element time approximation', *Comput. Math. Applic.*, **12A**(1) (1986) 149–160.
33. Washizu, K., *Variational Methods in Elasticity and Plasticity*, Pergamon Press, Oxford, 1968.
34. Hamilton, W. R., 'On a general method in dynamics', *Trans. Roy. Soc. London* (1834) 247–308.
35. Borri, M., Lanz, M. and Mantegazza, P., 'Comment on "Time finite element discretization of Hamilton's law of varying action"', *AIAA Jl*, **23**(9) (1985) 1457–1458.
36. Neimark, J. I. and Fufaev, N. A., *Dynamics of Nonholonomic Systems*. Vol. 33 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, Rhode Island, 1972.
37. Borri, M. and Mantegazza, P., 'Finite time elements approximation of the dynamics of nonholonomic systems', *Modelling, Simulation Control*, **12**(1) (1987) 7–18.
38. Lanczos, C., *The Variational Principles of Mechanics* (2nd edn), University of Toronto Press, Toronto, 1964.
39. Levi-Civita, T. and Amaldi, U., *Lezioni di Meccanica Razionale*, Zanichelli Editore, Bologna, 1974 (in Italian).
40. Baumgarte, J., 'Stabilization of constraints and integrals of motion in dynamical systems', *Comput. Meth. Appl. Mech. Engng*, **1** (1972) 1–16.