ON EQUATIONS FOR WAVE INTERACTIONS

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ABSTRACT. A number of new nonlinear evolution equations for the interaction of waves on the x, y plane are found. It is shown that these equations may be investigated by the inverse scattering method.

Recently it has been shown [1] that a number of practically important nonlinear evolution equations can be represented in the form

$$\frac{\partial \mathcal{L}}{\partial t} + [\mathscr{A}, \mathcal{L}] = \mathscr{B} \cdot \mathcal{L}, \tag{1}$$

if one uses the operator $\mathcal{L}$ of the following matrix structure:

$$\mathcal{L} = \left| \begin{matrix} L & \lambda\partial_y & v_1 & \cdots & v_{\mu_1} \\ & w_1 & \partial_x & & 0 \\ & \cdot & & \cdot & \\ & \cdot & & & \cdot \\ & \cdot & & & \\ & w_{\mu_1} & 0 & & \partial_x \end{matrix} \right|, \quad \mu_1 \geqslant 1, \tag{2}$$

where

$$\partial_x = \frac{\partial}{\partial x}, \qquad \partial_y = \frac{\partial}{\partial y},$$

the operator L has the form

$$L = \partial_x^{m_0+2} + \sum_{m=0}^{m_0} u_m \partial_x^m, \quad m_0 \geqslant 0, \tag{3}$$

the functions $u_0, \ldots, u_{m_0}, v_1, \ldots, v_{\mu_1}, w_1, \ldots, w_{\mu_1}$ depend on the spatial variables x, y and the time variable t, λ is a properly chosen constant, and $\mathscr{A}$, $\mathscr{B}$ are differential operators with respect to x, y. An explicit form of $\mathscr{A}$ and $\mathscr{B}$ will be specified below.

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The relation (1) may be written as follows

$$\mathcal{L} \cdot T = \hat{T} \cdot \mathcal{L}$$

where

$$T = \partial_t + \mathscr{A}, \qquad \hat{T} = \partial_t + \hat{\mathscr{A}}, \qquad \hat{\mathscr{A}} = \mathscr{A} - \mathscr{B}, \qquad \partial_t = \frac{\partial}{\partial t}.$$

This means that the operator T transforms any solution φ of the equation $\mathcal{L}\varphi = 0$ into a solution of the same equation. This fact is of principal importance for the application of the inverse scattering method to the investigation of nonlinear evolution equations, as has been demonstrated first in [2]. An important role for understanding the essence of this phenomenon has been played by papers [3, 4]. Later an essential progress in the development of the inverse scattering method and its applications to the investigation of various nonlinear systems was made by different authors. In particular, by means of the inverse scattering method, many important results concerning nonlinear systems depending on two or more spatial variables have been obtained [5, 6].

In the present paper, we incorporate the relation (1) to obtain four new nonlinear evolution systems describing interactions of waves on the x, y plane. Interacting waves of such a type usually appear in hydrodynamics and plasma physics. In fact, the number of nonlinear evolution equations integrable by the inverse scattering method with the operator $\mathcal{L}$ of the kind (2) can be considerably enlarged by taking into consideration other possibilities for the operators $\mathscr{A}$ and $\mathscr{B}$ entering relation (1). The method used in the present paper is based on some earlier results of the author [7–9]. Further applications of the method will be given in a separate paper to be published elsewhere.

Let us specify now the operators $\mathscr{A}$ and $\mathscr{B}$ entering relation (1) and derive the explicit form of the corresponding nonlinear evolution equations for various operators L of type (3), i.e., for different values of the number m_0. We shall investigate the cases $m_0 = 0$ and $m_0 = 1$.

1. Let us consider first the simplest case, when the operators $\mathscr{A}$ and $\mathscr{B}$ can be represented in the following matrix form:

$$\mathscr{A} = c \begin{vmatrix} P & \\ q_1 & \\ \vdots & 0 \\ q_{\mu_1} & \end{vmatrix}, \qquad \hat{\mathscr{A}} = c \begin{vmatrix} P & p_1 \ldots p_{\mu_1} \\ & 0 \end{vmatrix}$$

where c is a constant. Putting $\mathscr{B} = \mathscr{A} - \hat{\mathscr{A}}$ and substituting this into relation (1), we obtain the following system of equations:

$$\frac{\partial L}{\partial t} + c[P, L] + c\lambda \frac{\partial P}{\partial y} + c \sum_{\mu=1}^{\mu_1} (p_\mu \cdot w_\mu - v_\mu q_\mu) = 0,$$

$$\dot{v}_\mu + c(P \cdot v_\mu + p_\mu \partial_x) = 0, \tag{4}$$

$$\dot{w}_\mu = c(\omega_\mu P + \partial_x \cdot q_\mu), \quad \mu = 1, \ldots, \mu_1,$$

where the differentiation with respect to t is denoted by the dot. Further, let P be an arbitrary differential (with respect to x) operator of the form

$$P = \partial_x^{n_0+2} + \sum_{n=0}^{n_0} P_n \partial_x^n, \quad n_0 \geqslant 0. \tag{5}$$

Define now the operators $p_1, \ldots, p_{\mu_1}$ and $q_1, \ldots, q_{\mu_1}$ from the condition that the operators

$$\hat{P}_\mu = P \cdot v_\mu + p_\mu \partial_x, \qquad \hat{q}_\mu = w_\mu P + \partial_x \cdot q_\mu, \quad \mu = 1, \ldots, \mu_1,$$

are of zero order, i.e., that they are reduced to the multiplication by functions. Clearly this is always possible and their choice is unique. Now, it is easy to realize that the operators $p_1, \ldots, p_{\mu_1}$ and $q_1, \ldots, q_{\mu_1}$ defined in this way can be written as follows:

$$p_\mu = -v_\mu \partial_x^{n_0+1} + \alpha_\mu, \qquad q_\mu = -w_\mu \partial_x^{n_0+1} + \beta_\mu$$

where the order of the operators α_μ and β_μ is not larger than n_0, $\mu = 1, \ldots, \mu_1$. Consequently, the order of the operator

$$\gamma = \sum_{\mu=1}^{\mu_1} (p_\mu \cdot w_\mu - v_\mu q_\mu)$$

does not exceed n_0. Thus, if $m_0 \geqslant n_0$ and the operator P defined by equality (5) is chosen in such a way that the order of the operator $\delta = [P, L]$ does not exceed m_0, then for the above definition of the operators $P, p_1, \ldots, p_{\mu_1}$ and $q_1, \ldots, q_{\mu_1}$, the equalities (4) correctly determine some nonlinear evolution equations.

To illustrate this let us consider several examples.

(1) For $m_0 = 0$ we take

$$P = L, \qquad p_\mu = -\partial_x \cdot v_\mu - v'_\mu, \qquad q_\mu = -w_\mu \partial_x + w'_\mu, \quad \mu = 1, \ldots, \mu_1,$$

where the differentiation with respect to x is denoted by the prime. In this case system (4) is reduced to

$$
\begin{aligned}
&\frac{\partial u_0}{\partial t} + c\lambda \frac{\partial u_0}{\partial y} = 2c \sum_{\mu=1}^{\mu_1} \frac{\partial}{\partial x}(v_\mu w_\mu), \\
&\dot{v}_\mu + c(u_0 v_\mu + v_\mu'') = 0, \\
&\dot{w}_\mu = c(u_0 w_\mu + w_\mu''), \quad \mu = 1, \dots, \mu_1 .
\end{aligned}
\tag{6}
$$

When $c = \lambda = -i$ the system (6) has 2^{μ_1} invariant manifolds defined by the conditions

$$
u_0 = u_0 = u, \qquad w_\mu = i\epsilon_\mu \overline{v}_\mu, \quad \epsilon_\mu = \pm 1, \mu = 1, \dots, \mu_1 ,
$$

where the complex conjugation is denoted by the bar. On these manifolds, system (6) can be written in the form

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial u}{\partial y} = 2 \sum_{\mu=1}^{\mu_1} \epsilon_\mu \frac{\partial}{\partial x} |v_\mu|^2, \\
&i\frac{\partial v_\mu}{\partial t} + \frac{\partial^2 v_\mu}{\partial x^2} + u v_\mu = 0, \quad \mu = 1, \dots, \mu_1 .
\end{aligned}
\tag{7}
$$

(2) Further, for $m_0 = 1$ we take

$$
P = \partial_x^2 + \tfrac{2}{3} u_1 ,
$$

$$
p_\mu = -\partial_x \cdot v_\mu - v_\mu', \qquad q_\mu = -w_\mu \partial_x + w_\mu', \quad \mu = 1, \dots, \mu_1 .
$$

We have now

$$
[P, L] = f_1' \partial_x + f_0'
$$

where

$$
f_1 = 2u_0 - u_1', \qquad f_0 = u_0' - \tfrac{2}{3} u_1'' - \tfrac{1}{3} u_1^2 .
$$

As a consequence, the first equation of system (4) is equivalent to the following pair of equations

$$
\dot{u}_1 + c f_1' = 0,
$$

$$
\dot{u}_0 + c f_0' + \tfrac{2}{3} c\lambda \frac{\partial u_1}{\partial y} = 2c \sum_{\mu=1}^{\mu_1} \frac{\partial}{\partial x}(v_\mu w_\mu).
$$

Let us put now

$$u_1 = \tfrac{3}{2}u, \qquad u_0 = \tfrac{3}{4}u' + ip. \tag{8}$$

As a result, we obtain the system

$$\dot{u} + \frac{4i}{3}cp' = 0,$$

$$\dot{p} + \frac{i}{4}c(3u^2 + u'')' - ic\lambda\frac{\partial u}{\partial y} + 2ic\sum_{\mu=1}^{\mu_1}\frac{\partial}{\partial x}(v_\mu w_\mu) = 0.$$

Eliminating p from these equations, we get

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} + \frac{c^2}{3}\frac{\partial^2}{\partial x^2}\left(3u^2 + \frac{\partial^2 u}{\partial x^2}\right) - \\ &\quad - \frac{4c^2}{3}\lambda\frac{\partial^2 u}{\partial x \partial y} + \frac{8}{3}c^2\sum_{\mu=1}^{\mu_1}\frac{\partial^2}{\partial x^2}(v_\mu w_\mu) = 0.\end{aligned} \tag{9}$$

By virtue of (8) two other equations of system (4) have the following form:

$$\dot{v}_\mu + c(uv_\mu + v_\mu'') = 0, \tag{10}$$

$$\dot{w}_\mu = c(uw_\mu + w_\mu''), \quad \mu = 1, \ldots, \mu_1.$$

For $c = -i$ and $\lambda = -\frac{1}{4}$ systems (9), (10) have 2^{μ_1} invariant manifolds defined by the equalities

$$u = \bar{u}, \qquad w_\mu = \epsilon_\mu \bar{v}_\mu, \quad \epsilon_\mu = \pm 1, \quad \mu = 1, \ldots, \mu_1.$$

Thus, for $m_0 = 1$ the following system arises

$$\begin{aligned}&3\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) \\ &\quad = 8\sum_{\mu=1}^{\mu_1}\epsilon_\mu\frac{\partial^2}{\partial x^2}|v_\mu|^2,\end{aligned} \tag{11}$$

$$i\frac{\partial v_\mu}{\partial t} + \frac{\partial^2 v_\mu}{\partial x^2} + uv_\mu = 0, \quad \mu = 1, \ldots, \mu_1.$$

Substituting now in these equations t by y and y by t, we get the following important system:

$$3\frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right)$$

$$= 8 \sum_{\mu=1}^{\mu_1} \epsilon_\mu \frac{\partial^2}{\partial x^2} |v_\mu|^2, \tag{12}$$

$$i \frac{\partial v_\mu}{\partial y} + \frac{\partial^2 v_\mu}{\partial x^2} + u v_\mu = 0, \quad \mu = 1, \ldots, \mu_1.$$

(3) Let us return to the case $m_0 = 1$. We now put

$$P = L, \qquad p_\mu = -\partial_x^2 \cdot v_\mu - \partial_x \cdot v'_\mu - v''_\mu - u_1 v_\mu,$$

$$q_\mu = -w_\mu \partial_x^2 + w'_\mu \partial_x - w''_\mu - u_1 w_\mu, \quad \mu = 1, \ldots, \mu_1.$$

In this case system (4) can be written as

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} + c\lambda \frac{\partial u_1}{\partial y} = 3c \sum_{\mu=1}^{\mu_1} \frac{\partial}{\partial x}(v_\mu w_\mu), \\
&\frac{\partial u_0}{\partial t} + c\lambda \frac{\partial u_0}{\partial y} = 3c \sum_{\mu=1}^{\mu_1} \frac{\partial}{\partial x}\left(\frac{\partial v_\mu}{\partial x} w_\mu\right), \\
&\dot{v}_\mu + c(u_0 v_\mu + u_1 v'_\mu + v''_\mu) = 0, \\
&\dot{\omega}_\mu + c(-u_0 w_\mu + (u_1 w_\mu)' + w'''_\mu) = 0,
\end{aligned} \tag{13}$$

where $\mu = 1, \ldots, \mu_1$.

When $c = \lambda = i$ the system (13) has 2^{μ_1} invariant manifolds defined by conditions

$$u_1 = 2u, \qquad u_0 = u', \qquad w_\mu = \epsilon_\mu v_\mu, \quad \mu = 1, \ldots, \mu_1.$$

On these manifolds system (13) is reduced to the following form:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} = 3 \sum_{\mu=1}^{\mu_1} \epsilon_\mu v_\mu \frac{\partial v_\mu}{\partial x}, \\
&\frac{\partial v_\mu}{\partial t} + \frac{\partial u}{\partial x} v_\mu + 2u \frac{\partial v_\mu}{\partial x} + \frac{\partial^3 v_\mu}{\partial x^3} = 0, \quad \mu = 1, \ldots, \mu_1.
\end{aligned} \tag{14}$$

2. Let us consider now a more complicated case arising for $m_0 < n_0$, i.e., when the order of the operator L in (3) is less than the order of the operator P defined by (5). In this case it is necessary to slightly modify the above definition of the operators $P, p_1, \ldots, p_{\mu_1}$ and $q_1, \ldots, q_{\mu_1}$.

Omitting the details we give here one example corresponding to the case $m_0 = 0$. Let

$$u = u_0, \qquad P = \partial_x^3 + \tfrac{3}{2} u \partial_x + \tfrac{3}{4} u' - \tfrac{3}{2} \sigma + f,$$

$$p_\mu = -\partial_x^2 \cdot v_\mu' - \partial_x \cdot v_\mu' - v_\mu'' - \tfrac{3}{2} u v_\mu, \tag{15}$$

$$q_\mu = -w_\mu \partial_x^2 + w_\mu' \partial_x - w_\mu'' - \tfrac{3}{2} u w_\mu, \quad \mu = 1, \ldots, \mu_1,$$

where f is so far an unknown function, and $\sigma = \sum_{\mu=1}^{\mu_1} v_\mu w_\mu$.

Putting $\rho = \sum_{\mu=1}^{\mu_1} (v_\mu w_\mu' - v_\mu' w_\mu)$, we obtain that in the case under consideration system (4) transforms to the following form:

$$\frac{\partial u}{\partial t} + \frac{c}{4} \frac{\partial}{\partial x}\left(3u^2 + \frac{\partial^2 u}{\partial x^2}\right) + \frac{3c}{2} \frac{\partial \rho}{\partial x} - \frac{3c}{2} \lambda \frac{\partial \sigma}{\partial y} + c\lambda \frac{\partial f}{\partial y} = 0,$$

$$\dot{v}_\mu + c(v_\mu''' + \tfrac{3}{2} u v_\mu' + \tfrac{3}{4} u' v_\mu - \tfrac{3}{2} \sigma v_\mu + f v_\mu) = 0, \tag{16}$$

$$\dot{w}_\mu + c(w_\mu''' + \tfrac{3}{2} u w_\mu' + \tfrac{3}{4} u' w_\mu + \tfrac{3}{2} \sigma w_\mu - f w_\mu) = 0, \quad \mu = 1, \ldots, \mu_1,$$

if the function f in (15) obeys the condition

$$\frac{\partial f}{\partial x} = \frac{3\lambda}{4} \frac{\partial u}{\partial y}. \tag{17}$$

For $c = 4$, $\lambda = i$ and $f = (3i/4)p$, system (16), (17) has 2^{μ_1} invariant manifolds defined by the relations

$$u = \overline{u}, \qquad p = \overline{p}, \qquad w_\mu = i\epsilon_\mu \overline{v}_\mu, \quad \epsilon_\mu = \pm 1, \mu = 1, \ldots, \mu_1.$$

On these manifolds, system (16), (17) is

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + 6 \frac{\partial R}{\partial x} + 6 \frac{\partial S}{\partial y} - 3 \frac{\partial p}{\partial y} = 0,$$

$$\frac{\partial v_\mu}{\partial t} + 3 \frac{\partial u}{\partial x} v_\mu + 6u \frac{\partial v_\mu}{\partial x} + 4 \frac{\partial^3 v_\mu}{\partial x^3}$$

$$= 6iSv_\mu - 3ipv_\mu, \quad \mu = 1, \ldots, \mu_1,$$

$$\frac{\partial p}{\partial x} = \frac{\partial u}{\partial y},$$

where

$$S = \sum_{\mu=1}^{\mu_1} \epsilon_\mu |v_\mu|^2, \qquad R = i \sum_{\mu=1}^{\mu_1} \epsilon_\mu \left(v_\mu \frac{\partial \overline{v}_\mu}{\partial x} - \frac{\partial v_\mu}{\partial x} \overline{v}_\mu \right).$$

Eliminating p from these equations, we get

$$
\begin{aligned}
&3\frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) \\
&\quad = 6\frac{\partial^2 R}{\partial x^2} + 6\frac{\partial^2 S}{\partial x \partial y}, \\
&\frac{\partial v_\mu}{\partial t} + 3\frac{\partial u}{\partial x} v_\mu + 6u\frac{\partial v_\mu}{\partial x} + 4\frac{\partial^3 v_\mu}{\partial x^3} \\
&\quad = 6iSv_\mu - 3iv_\mu \int_{x_0}^{x} \frac{\partial u}{\partial y}\,\mathrm{d}x', \quad \mu = 1, \ldots, \mu_1 .
\end{aligned}
\tag{18}
$$

In conclusion, we note that in the case when $v_1 = \cdots = v_{\mu_1} = 0$, systems (12) and (18) give us the well-known Kadomtsev–Petviashvili equation [10]. The Lax pair for this equation was first obtained in paper [11]. Further, when $\mu_1 = 1$ the solutions of system (7) independent of y satisfy the system obtained earlier in paper [12]. Similar solutions of system (14) for $\mu_1 = 1$ obey the system derived independently in papers [8, 13]. Finally, the solutions of system (11) independent of y obey for $\mu_1 = 1$ the system given in [8].

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