ON THE METHOD OF SYMES FOR INTEGRATING SYSTEMS OF THE TODA TYPE

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ABSTRACT. Let $G = KL$ and $g = k + l$ be Lie group and Lie algebra decompositions. This identifies k^o with l^*. Any G-invariant function, f, on g^* induces by restriction a function $f|_{k^o} = l^*$. We prove a formula which says that the integral curve through $\alpha \in k^o$ is obtained as $b(t)\alpha$, where $a(t) = \exp t\xi$ with $\xi = L_f(\alpha)$,

(*) $\qquad a(t) = b(t)c(t)$

where (*) is the KL decomposition and where $L_f\colon g^* \to g$ is the Legendre transform. This generalizes a formula of Symes for the generalized Toda lattice.

Let g be a Lie algebra and k and l subalgebras of g with

$$\mathrm{g} = \mathrm{k} + \mathrm{l}. \tag{1}$$

This gives a corresponding decomposition of the dual spaces.

$$\mathrm{g}^* = \mathrm{l}^o + \mathrm{k}^o \tag{2}$$

and, in particular, an identification of k^o with l^*. Let O be a coadjoint orbit of L (a Lie group whose Lie algebra is l) regarded as a submanifold of k^o. A function f on g^* restricts to O and hence defines a Hamiltonian system relative to the natural symplectic structure on O. The purpose of this note is to explain the method of Symes for integrating this sytem when f is an invariant function on g^* and we have a global group theoretical decomposition

$$G = LK \tag{3}$$

corresponding to (1).

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In the case

$$g = gl(n) \qquad \mathbf{k} = O(n) \qquad \mathbf{l} = \{\text{lower triangular matrices}\} \tag{4}$$

we can identify $\mathbf{k}^0$ with symm (n) = {symmetric matrices}.

It was first observed by Kostant [5] that one can identify the set of Jacobi matrices as a coadjoint orbit **O** of L and that the (finite nonperiodic) Toda lattice equations as formulated by Flaschka [2] and studied by Moser [7] are just the restrictions to **O** of the function $f(A) = \operatorname{tr} A^2$ on $gl(n)$. This led him to a general principal for proving the complete integrability of such equations and to a systematic generalization of the Toda equations involving arbitrary semisimple Lie groups and to the detailed solutions of these equations, cf. [5]. Some of these results were also obtained by Symes [9] and others [1, 8].

More recently, Symes [10] has given a rather explicit method for solving the Hamiltonian system corresponding to an arbitrary invariant f and arbitrary orbit **O**. His proof in [10] makes use of an explicit global coordinate chart and, hence, might seem to be restricted to the special choice (4). We shall show that the method works whenever there is a global decomposition (3). We shall use the notion of collective motion as introduced in [3]. We briefly recall some of the basic facts in the theory of collective motion referring the reader to [3] for details and definition:

Suppose we are given a Hamiltonian action of a Lie group G on a symplectic manifold M with moment map $\Phi: M \to g^*$. A function F on M is called *collective* if it is of the form $F = f \circ \Phi$, where f: $\mathbf{g}^* \to \mathbb{R}$ is a smooth function.

The integration of the Hamiltonian system given by such a collective F proceeds in three steps:

(1) For point $m \in M$ calculate the G orbit points 0 through the point $\Phi(m)$.

(2) Solve the Hamiltonian system on **O** given by the function $f|_{\mathbf{O}}$. Let $\beta(t)$ be the solution curve with $\beta(0) = \Phi(m)$.

(3) The function f determines a map L_f (the Legendre transformation) of $\mathbf{g}^* \to \mathbf{g}$. The image $L_f(\beta(t) = \gamma(t)$ defines a curve in g. The curve γ can be regarded as a time-dependent vector field on G and, hence, determines a curve $a(t)$ in G with $a(0) = e$. Then $a(t)m$ is the trajectory through m of the Hamiltonian system given by F.

In case f is a G-invariant function, steps (2) and (3) simplify: In step (2) $f|_{\mathbf{O}}$ is a constant so $\beta(t)$ is the constant so $\beta(t) \equiv \Phi(m)$. In step (3) $\gamma(t)$ is a constant so $a(t)$ is a one-parameter group. Thus, for invariant f the solution curve through m is

$$(\exp t\xi)m \quad \text{where } \xi = \xi(m) = L_f(\Phi(m)). \tag{5}$$

The cotangent bundle T^*G may be identified with $G \times \mathbf{g}^*$ using the left invariant identification. Left multiplication by G on itself induces a Hamiltonian action on T^*G given by $a_1(a, \alpha) = (a_1 a, \alpha)$ and the moment map for this action, Φ_r: $T^*G \to \mathbf{g}^*$ is given by $\Phi_r(a, \alpha) = a \cdot \alpha$, where $\cdot$ denotes the coadjoint action. Right inverse multiplication of G on itself defines a Hamiltonian action on T^*G given by $a_2(a, \alpha) = (aa_2^{-1}, a_2 \cdot \alpha)$ and its moment map is given by $\Phi_l(a, \alpha) = -\alpha$. In particular, a function F on T^*G is left invariant if and only if $F(a, \alpha)$ does not depend on a, so that we can write it as $F(a, \alpha) = f(-\alpha)$ and so is collective for the right action. The function F is both right and left invariant if and only if f is invariant under the coadjoint representation. In that case, the trajectory through the point (a, α) is

$$(a \exp -t\xi, \alpha) \quad \text{where } \xi = L_f(-\alpha). \tag{6}$$

Now let K be a subgroup of G. The right (inverse) action of K on G induces a Hamiltonian action of K on T^*G whose moment map is given by $\Phi_r^K(a, \alpha) = -\pi_{k^*}\alpha$, where $\pi_{k^*}: g^* \to k^*$ is the projection dual to the injection $\mathbf{k} \to g$. In particular,

$$(\Phi_r^K)^{-1}(0) = \{(a, \alpha) \mid \alpha \in k^0\}. \tag{7}$$

This is a coisotropic submanifold, cf. [4] or [6] and its corresponding symplectic quotient is $(\Phi_r^K)^{-1}(0)/K$ and can be identified with $T^*(G/K)$. Any bi-invariant function on T^*G gives rise to a function on $T^*(G/K)$ via restriction and passage to the quotient.

Now suppose that (3) holds. This means that every $a \in G$ can be written as $a = cb$, $c \in L$, $b \in K$, and we can identify G/K with L. Thus, in the identification of $(\Phi_r^K)^{-1}(0)/K$ with T^*L we represent the equivalence class of the element (a, α) by $(c, b\alpha)$, where $a = cb$. Here α, and hence $b\alpha$, lies in k^o and we identify k^o with l^* using (2). A right invariant function F on T^*G induces a function F_L on T^*L by 'restriction' $F_L(c, \beta) = F(c, \beta)$, $c \in L$, $\beta \in k^o$. If F is left G invariant then F_L will be left L invariant; we can then write $F_L(c, \beta) = f(-\beta)$ but notice that the function f_L on l^* defined by $f_L = f|_{k^o}; k^o \sim l^*$ is not necessarily $\mathrm{Ad}^* L$-invariant. But we can now turn step (2) in the method of collective motion around for T^*L. We would like to solve the Hamiltonian system associated with the function f_L restricted to an orbit, O in l^*. Such an orbit will be the image under the moment map $\Phi: T^*L \to l^*$ of a solution curve $q(t)$ of F_L on T^*L. Under the identification of T^*L with $(\Phi^K)^{-1}(0)/K$ the curve $q(t)$ is the image of a solution curve, $p(t)$ of F on T^*G, given by (6). So we obtain the following procedure of Symes for solving the Hamiltonian system given by $f|_{k^o}$: for $\alpha \in k^o$ let $\xi = L_f(\alpha)$. Taking $(e, -\alpha)$ as the initial point in (6), write $\xi = L_f(\alpha)$ and $\exp -t\xi = c(t)b(t)$. Then $b(t)\alpha$ is the desired solution curve.

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