REGULAR * REPRESENTATIONS OF LIE ALGEBRAS

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ABSTRACT. We prove the existence of a * product on the cotangent bundle of a parallelizable manifold M. When M is a Lie group the properties of this * product allow us to define a linear representation of the Lie algebra of this group on $L^2(G)$, which is, in fact, the one corresponding to the usual regular representation of G.

0. INTRODUCTION

A * product on a symplectic manifold (M, ω) is a particular deformation of the associative algebra N of smooth real-valued functions on M. Such deformations, which in the case ($M = \mathbb{R}^{2n}$, $\omega = \omega_0$ = canonical symplectic form) reduce to the Moyal product [3], have been used to give a completely autonomous presentation of quantum mechanics in the framework of classical phase space [1].

The existence of * products have been proved for various mutually overlapping classes of symplectic manifolds. Let us mention (i) symplectic manifolds with a vanishing third De Rham cohomology group [4]; (ii) certain quotients of open sets of $\mathbb{R}^{2p}$ by a group of linear symplectic transformations [2]. This method has, in particular, given the existence on the torus T^{2n}, and on the cotangent bundle to the compact classical groups.

We prove the existence of a * product on the cotangent bundle of a parallelizable manifold M. When M is a connected Lie group G this * product is G-invariant and is a * representation of the Lie algebra $\mathscr{G}$ of g. The associated linear representation of $\mathscr{G}$ on the space of formal series on T^*G stabilizes a subspace isomorphic to $L^2(G)$ and is equivalent on this subspace to the differential of the usual regular representation of G.

1. * PRODUCTS, PARALLELIZABLE MANIFOLDS, HOCHSCHILD COHOMOLOGY

In this section we recall the definition of a * product and of the relevant Hochschild cohomology. We then study some elementary properties of the cotangent bundle to a parallelizable manifold. Finally we prove a technical proposition on Hochschild coboundaries.

1.1. Let (M, ω) be a symplectic manifold and let $N = \mathscr{C}^\infty(M, \mathbb{R})$. The symplectic structure ω

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induces an isomorphism between the N-module of smooth vector fields on M and the N-module of smooth 1-forms. In particular, if $f \in N$ one denotes by X_f, the Hamiltonian vector field associated to f by:

$$i(X_f)\omega = -\mathrm{d}f. \tag{1.1}$$

The *Poisson bracket* $\{f, g\}$ of elements f and g of N is a Lie algebra multiplication law on N; it has the expression:

$$\{f, g\} = X_f g = -X_g f = \omega(X_f, X_g). \tag{1.2}$$

Let $E(N, \lambda)$ be the space of formal power series in a complex parameter λ with coefficients in N.

DEFINITION 1 [1]. A $*$ product on (M, ω) is a bilinear map $N \times N \to E(N, \lambda)$: $(u, v) \to u * v = \sum_{r=0}^{\infty} \lambda^r C_r(u, v)$ where the so-called cochains C_r are bilinear, bidifferential operators with values in N and satisfy the following axioms:

(i) $C_0(u, v) = uv, \quad C_1(u, v) = \{u, v\}, \quad \forall u, v \in N,$

(ii) $C_r(u, v) = (-1)^r C_r(v, u), \quad \forall u, v \in N, \forall r \in \mathbb{N},$

(iii) $C_r(k, u) = 0, \quad \forall u \in N, \forall k \in \mathbb{R}, \forall r \geqslant 1,$

(iv) when extended to $E(N, \lambda)$, the product is associative, i.e., $(u * v) * w = u * (v * w), \forall u, v, w \in N.$

The general theory of deformations in the sense of Gerstenhaber relates the deformations of an associative algebra to the corresponding Hochschild cohomology.

DEFINITION 2. A *p-cochain* is a p-linear map $N^p \to N$. The *coboundary of a p-cochain C* is a $p+1$ cochain $\tilde{\delta} C$ defined by:

$$\tilde{\delta} C(u_0, \ldots, u_p) = \sum_{i=0}^{p-1} (-1)^i [u_i C(u_0, \ldots, \hat{u}_i, \ldots, u_p) - C(u_0, \ldots, u_{i-1}, u_i u_{i+1}, u_{i+2}, \ldots, u_p) + u_{i+1} C(u_0, \ldots, \hat{u}_{i+1}, \ldots, u_p)], \quad \forall u_j \in N. \tag{1.3}$$

A p-cochain is called a *p-cocycle* if $\tilde{\delta} C = 0$ and a *p-coboundary* if $C = \tilde{\delta} B$. As the operator $\tilde{\delta}$ is such that $\tilde{\delta}^2 = 0$, one defines the pth *Hochschild cohomology space* as the quotient of the space of p-cocycles by the space of p-coboundaries. It is denoted $\tilde{H}^p_{\mathrm{diff}}(N)$ because all cochains considered are multidifferential operators.

PROPOSITION 1. (Vey) [5]. *$\tilde{H}^p_{\mathrm{diff}}(N)$ is isomorphic to the space of p-contravariant, skew-symmetric,* *smooth tensor fields on M.*

Explicitly if C is a p-cocycle, there exists a skew-symmetric contravariant smooth p-tensor A such that:

$$C(u_1, \ldots, u_p) = A(\mathrm{d}u_1, \ldots, \mathrm{d}u_p) + \widetilde{\delta} E(u_1, \ldots, u_p).$$

In particular, a p-cocycle is exact if and only if its completely antisymmetric part vanishes. Furthermore, the skew-symmetric part of a p-cocycle is always a p-differential operator of order 1 in each argument.

When one studies by order the associativity relation (axiom (iv)) of a $*$ product one observes that at the order t $(t \geqslant 2)$ it has the following form:

$$\widetilde{\delta} C_t(u, v, w) = \sum_{\substack{r+s=t \\ r, s \geqslant 1}} [C_r(C_s(u, v)w) - C_r(u, C_s(v, w))] \underset{\mathrm{def}}{=} E_t(u, v, w).$$

Furthermore, by virtue of the associativity relations at order $t' < t$, one shows that the 3-cochain E_t is a 3-cocycle. This means that a $*$ product constructed up to order $(t-1)$ can be extended to order t provided E_t is a 3-coboundary. The philosophy of the proof of our existence theorem is to choose particular cochains C_r such that at each order E_t will be a 3-coboundary.

1.2. Let M be a parallelizable manifold of dimension m. Let X_i $(i \leqslant m)$ be smooth vector fields on M, which, at each point x, form a basis of the tangent space M_x. Let θ^i be the smooth 1-forms such that $\theta^i(X_j) = \delta^i_j$; denote by $c^k_{ij}(x)$ the smooth functions on M such that

$$[X_i, X_j](x) = \sum_k c^k_{ij}(x) X_k(x). \tag{1.4}$$

Let us introduce the functions p_i on T^*M by:

$$p_i(\xi) = \xi(X_i). \tag{1.5}$$

If $\Pi: T^*M \to M$ is the canonical projection, one checks that the $2m$ smooth 1-forms $\{\mathrm{d}p_i, \Pi^*\theta^i; i \leqslant m\}$ form at each point ξ a basis of the dual of the tangent space $(T^*M)_\xi$. The classical Liouville 1-form λ and the corresponding symplectic structure $\omega = \mathrm{d}\lambda$ can be expressed in terms of these 1-forms. Indeed:

$$\lambda_\xi = \sum_i p_i(\xi)(\Pi^*\theta^i)_\xi, \tag{1.6}$$

$$\omega = \sum_i \mathrm{d}p_i \wedge \Pi^*\theta^i - \tfrac{1}{2} \sum_{i,j,k} p_i(\Pi^* c^i_{jk}) \Pi^*\theta^j \wedge \Pi^*\theta^k. \tag{1.7}$$

It is useful to introduce the $2m$ vector fields $(Z^i, Y_i; i \leqslant m)$ on T^*M such that:

$$\mathrm{d}p_i(Z^j) = \Pi^*\theta^j(Y_i) = \delta^j_i, \qquad \mathrm{d}p_i(Y_j) = \Pi^*\theta^i(Z^j) = 0. \tag{1.8}$$

They form in each point ξ a basis of $(T^*M)_\xi$ and furthermore:

$$\Pi_* Z^i = 0, \qquad \Pi_* Y_i = X_i. \tag{1.9}$$

The commutators of these vectors fields read:

$$[Z^i, Z^j] = [Z^i, Y_j] = 0, \qquad [Y_i, Y_j] = \sum_k \Pi^* C^k_{ij} Y_k. \tag{1.10}$$

The Poisson bracket of two functions f and g on T^*M can be expressed in terms of the action of these vector fields; explicitly:

$$\{f, g\} = \sum_i (Z^i(f) Y_i(g) - Z^i(g) Y_i(f)) + \sum_{r,i,j} p_r(\Pi^* C^r_{ij}) Z^i(f) Z^j(g). \tag{1.11}$$

1.3. We recall here some results concerning Hochschild cohomology. More precisely, we show that an exact p-cocycle is the coboundary of a $(p-1)$ cochain which is 'given in terms of the p-cocycle'. For the sake of simplicity, we consider here only differentiable 3-coboundaries, null on the constants, defined on the cotangent bundle of a parallelizable manifold M.

To express a 3-cochain E on T^*M we use (cf. Section 1.2) the global vector fields Z^i, Y_i $(i \leqslant m)$; to simplify the notation we denote them by T^a $(T^a = Z^a, T^{m+a} = Y_a, a \leqslant m)$ and we omit the summation signs. Then if $u, v, w \in \mathscr{C}^\infty(T^*M, \mathbb{R})$:

$$E(u, v, w) = \sum_{0 < a, b, c < K} \frac{1}{a!b!c!} E_{i_1 \dots i_a, j_1 \dots j_b, k_1 \dots k_c}(T^{i_1} \dots T^{i_a} u) \times \times (T^{j_1} \dots T^{j_b} v) \times (T^{k_1} \dots T^{k_c} w), \tag{1.11a}$$

where $E_{i_1 \dots i_a, j_1 \dots j_b, k_1 \dots k_c}$ are smooth functions on T^*M which are symmetric in $(i_1 \dots i_a)$, in $(j_1 \dots j_b)$ and in $(k_1 \dots k_c)$. The *order* of a term in E is, by definition, the triple of strictly positive integers (a, b, c). We consider on these triples the *lexicographic ordering* and call the *symbol* of E, denoted $\sigma(E)$, the terms of maximal order in E relative to this ordering. Thus:

$$\sigma(E)(u, v, w) = \frac{1}{r!s!t!} E_{i_1 \dots i_r, j_1 \dots j_s, k_1 \dots k_t}(T^{i_1} \dots T^{i_r} u)(T^{j_1} \dots T^{j_s} v)(T^{k_1} \dots T^{k_t} w).$$

LEMMA 1. *Let E by a 3-cocycle. Then the order of its symbol is $(r, s, 1)$.*

Proof. Assume it is of order (r, s, t) with $t > 1$. Then the terms of $\tilde{\delta} E$ of order $(r, s, t-1, 1)$ come only from the symbol of E and one has thus the condition:

$$E_{i_1 \dots i_r, j_1 \dots j_s, k_1 \dots k_{t-1} l}(\xi)(T^{i_1} \dots T^{i_r} u)(T^{j_1} \dots T^{j_s} v)(T^{k_1} \dots T^{k_{t-1}} w) T^l z = 0,$$

$\forall \xi \in T^*M$ and $\forall u, v, w, z \in \mathscr{C}^\infty(T^*M, \mathbb{R})$. Hence $E_{i_1 \dots i_r, j_1 \dots j_s, k_1 \dots k_t} = 0$ and the conclusion.

LEMMA 2. *If the symbol of a 3-cocycle E is of order $(r, s, 1)$ with $s > 1$ then it coincides with the symbol of $\tilde{\delta} C$ where*

$$C(u, v) = \frac{1}{r!(s+1)!} E_{i_1 \dots i_r, j_1 \dots j_s, j_{s+1}} (T^{i_1} \dots T^{i_r} u)(T^{j_1} \dots T^{j_{s+1}} v). \tag{1.12}$$

Proof. The terms of $\tilde{\delta} E$ of order $(r, s-1, 1, 1)$ come from terms in E of order $(r, s, 1)$ and of order $(r, s-1, 2)$. Hence, the cocycle condition implies that:

$$(E_{i_1 \dots i_r, j_1 \dots j_{s-1} k, l} + E_{i_1 \dots i_r, j_1 \dots j_{s-1}, kl}) \times$$
$$\times (T^{i_1} \dots T^{i_r} u)(T^{j_1} \dots T^{j_{s-1}} v)(T^k w)(T^l z) = 0$$

$\forall u, v, w \in \mathscr{C}^\infty(T^*M, \mathbb{R})$. Hence $E_{i_1 \dots j_r, j_1 \dots j_{s-1} k, l}$ is symmetric in k and l; thus symmetric in all its $(s+1)$ last indices. The cochain C given by (1.12) is well defined and the result follows.

LEMMA 3. *If the symbol of a 3-cocycle E is of order $(r, 1, 1)$, then it coincides with the symbol of $\tilde{\delta} C$ with*

$$C(u, v) = \frac{1}{2r!} \quad \tfrac{1}{2} E_{i_1 \dots i_r, k_1, k_2} + \tfrac{1}{2} E_{i_1 \dots i_r, k_2, k_1} +$$
$$+ \frac{1}{r+2} \sum_{p=1}^{r} (E_{i_1 \dots \hat{i}_p \dots i_r k_2, k_1, i_p} + E_{i_1 \dots \hat{i}_p \dots i_r k_1, k_2, i_p})](T^{i_1} \dots T^{i_r})(T^{k_1} T^{k_2} v) +$$
$$+ \frac{2}{(r+2)!} \sum_{p=1}^{r+1} E_{i_1 \dots \hat{i}_p \dots i_{r+1}, i_p, k} (T^{i_1} \dots T^{i_{r+1}} u)(T^k v). \tag{1.13}$$

We include the case $r = 1$ by assuming, in this situation, that the completely antisymmetric part of E vanishes (cf. Proposition 1).

Proof. If $r > 1$ the terms in E of order $(r-1, 1, 1, 1)$ come from terms in E of order $(r, 1, 1)$, $(r-1, 2, 1)$ and $(r-1, 1, 2)$. The cocycle condition implies that:

$$(E_{i_1 \dots i_{r-1} j, k, l} + E_{i_1 \dots i_{r-1}, jk, l} + E_{i_1 \dots i_{r-1}, j, kl})(T^{i_1} \dots T^{i_{r-1}} u)(T^j v)(T^k w)(T^l z) = 0,$$

$\forall u, v, w \in \mathscr{C}^\infty(T^*M, \mathbb{R})$. In particular, $E_{i_1 \dots i_{r-1} j, k, l}$ antisymmetrized over its last three indices vanishes. This is also the case for $r = 1$.

Let us define $a_{i_1 \dots i_r, k, l}$ by:

$$E_{i_1 \dots i_r, k, l} \underset{\text{def}}{=} \tfrac{1}{2}(E_{i_1 \dots i_r, k, l} + E_{i_1 \dots i_r, l, k}) + a_{i_1 \dots i_r, k, l}.$$

Clearly one has:

$$a_{i_2 \dots i_r j, k, l} + a_{i_2 \dots i_r k, l, j} + a_{i_2 \dots i_r l, j, k} = 0.$$

Thus, as $a_{i_1 \dots i_r k, l}$ is antisymmetric in its last 2 indices, one has:

$$a_{i_2 \dots i_r j, k, l} = 2a_{i_2 \dots i_r k, j, l} + (a_{i_2 \dots i_r l, k, j} + a_{i_2 \dots i_r k, l, j}).$$

Hence:

$$(r+2)a_{i_1 \dots i_r, k, l} = 2 \sum_{p=1}^{r} a_{i_1 \dots \hat{i}_p \dots i_r k, i_p, l} + 2a_{i_1 \dots i_r k, l} +$$

$$+ \sum_{p=1}^{r} (a_{i_1 \dots \hat{i}_p \dots i_r l, k, i_p} + a_{i_1 \dots \hat{i}_p \dots i_r k, l, i_p}).$$

Define then:

$$E'_{i_1 \dots i_r k, l} = \frac{2}{r+2} \sum_{p=1}^{r+1} a_{i_1 \dots \hat{i}_p \dots i_{r+1}, i_p, l},$$

$$E''_{i_1 \dots i_r, kl} = \tfrac{1}{2} E_{i_1 \dots i_r k, l} + \tfrac{1}{2} E_{i_1 \dots i_r, l, k} + \frac{1}{r+2} \sum_{p=1}^{r} \times$$

$$\times (a_{i_1 \dots \hat{i}_p \dots i_r k, l} + a_{i_1 \dots \hat{i}_p \dots i_r l, k}).$$

One sees that E' is symmetric in its $(r+1)$ first indices and that E'' is symmetric in its r first indices and in its 2 last indices. Furthermore:

$$E_{i_1 \dots i_r k, l} = E'_{i_1 \dots i_r k, l} + E''_{i_1 \dots i_r, kl}.$$

The conclusion follows easily.

If one applies one of the above lemmas to a 3-cocycle E one constructs a 2-cochain C', whose coefficients are linear combinations of the coefficients of E corresponding to the same set of indices, and such that $\sigma(E) = \sigma(\tilde{\delta} C')$. Using a recursive procedure one gets:

PROPOSITION 2. *Let E be a differentiable 3-cocycle, null on the constants, on the cotangent bundle of a parallelizable manifold M. Then if E is a 3-coboundary, one can choose a 2-cochain C such that* $E = \tilde{\delta} C$ *and*:

$$C(u, v) = \sum_{0 < p, q \leq K} C_{i_1 \dots i_p, j_1 \dots j_q} (T^{i_1} \dots T^{i_p} u)(T^{j_1} \dots T^{j_q} v)$$

where the coefficients $C_{i_1 \dots i_p, j_1 \dots j_q}$ *are linear combinations of the coefficients*

$E_{k_1 \dots k_a, l_1 \dots l_b, m_1 \dots m_c}$ *of* E *with* $\{i_1 \dots i_p, j_1 \dots j_q\} = \{k_1 \dots k_a, l_1 \dots l_b, m_1 \dots m_c\}$.

2. EXISTENCE OF A $*$ PRODUCT ON THE COTANGENT BUNDLE OF A PARALLELIZABLE MANIFOLD

Let M be a parallelizable manifold, T^*M its cotangent bundle. A $*$ product on T^*M is given by a formal power series

$$u * v = uv + \sum_{r=1}^{\infty} \lambda^r C_r(u, v)$$

where the C_r are bidifferential operators which, with the notations of Section 1, are written as:

$$C_r(u, v) = \sum_{1 \leqslant a, b \leqslant K} C^{(r)}_{i_1 \dots i_a, j_1 \dots j_b} (T^{i_1} \dots T^{i_a} u)(T^{j_1} \dots T^{j_b} v).$$

DEFINITION 1. The *Z-order of a term* of C_r is the number of indices $i_1 \dots i_a, j_1 \dots j_b$ which are $\leqslant m$, i.e., the number of Z vector fields arising in the bidifferential operator.

We want to prove the existence of a $*$ product on T^*M whose cochains satisfy axioms (i) – (iv) and in addition:

H1 (resp. H1$'$) *For* $r \geqslant 1$, C_r *is given as a sum of terms of Z-order varying from* r *to* $2r$; *a term of Z-order* $(r + i)$ *has a coefficient which is the product of a homogeneous polynomial in* p_k *of degree* i *by a function* $\Pi^* f$, $f \in \mathscr{C}^\infty(M, \mathbb{R})$ (resp. *by a constant*).

H2 *For* $r > 1$ *and for all* $1 \leqslant i, j \leqslant m$, $C_r(p_i, p_j) = 0$.

REMARK 1. These assumptions are satisfied by C_1. Indeed:

$$C_1(u, v) = \{u, v\} = Z^i(u) Y_i(v) - Z^i(v) Y_i(u) + p_r \Pi^* C^r_{ij} Z^i(u) Z^j(v).$$

The $2m$ first terms have Z-order 1 and their coefficients are constants. The $m(m-1)/2$ last terms have Z-order 2 and their coefficients are the product of a polynomial in p of degree 1 by the functions $\Pi^* C^r_{ij}$. In the case where the C^r_{ij} are constants, assumption H1$'$ is satisfied.

REMARK 2. H1 implies H2 for $r > 2$. Indeed a term of Z-order > 2 is automatically zero for $u = p_i$, $v = p_j$.

LEMMA 1. *If the cochains* C_r *satisfy* H1 (resp. H1$'$) *for* $r \leqslant n$ *then* E_{n+1} *is a sum of terms of Z-order varying from* $(n + 1)$ *to* $2(n + 1)$ *and the coefficient of a term of order* $n + 1 + i$

$(0 \leqslant i \leqslant n+1)$ *has the form* $P(p_k)\Pi^*(f)$ *where* P *is a homogeneous polynomial of degree* i *and* $f \in \mathscr{C}^\infty(M)$ (resp. *is a constant*).

Proof. Recall that:

$$E_{n+1}(u, v, w) = \sum_{\substack{r+s=t \\ r, s \geqslant 1}} (C_r(C_s(u, v), w) - C_r(u, C_s(v, w))).$$

As the vector fields Z^i commute with each other and with the vector fields Y_j, and as $[Y_i, Y_j] = (\Pi^* C_{ij}^k) Y_k$, the Z-order of a term in a given cochain does not depend on the particular way of writing the bidifferential operators in terms of the vector fields Y_i, Z^j. In particular, it does not change if one symmetrizes the coefficients. Consider a term of C_r of Z-order $r+i$ $(0 \leqslant i \leqslant r)$ and a term of C_s of order $s+j$ $(0 \leqslant j \leqslant s)$. They give rise to terms in E_{n+1} of Z-order $r+s+i+j-k$ $(0 \leqslant k \leqslant j)$ with coefficients which are homogeneous polynomials in p of degree $(i+j-k)$ multiplied by a $\Pi^* f$ (resp. a constant). Hence, the conclusion.

COROLLARY. *If the cochains* C_r *satisfy* H1 (resp. H1$'$) *for* $r \leqslant n$ *and if* E_{n+1} *is exact then* $E_{n+1} = \tilde{\delta} C_{n+1}$ *where* C_{n+1} *satisfy* H1 (resp. H1$'$).

Proof. The result follows immediately from Lemma 1 and Proposition 2.

LEMMA 2. *There exists on* T^*M *a* $*$ *product up to order* 3 *whose cochains satisfy* H1 *and* H2 (resp. H1$'$ *and* H2 *if* C_{ij}^k *are constants*).

Proof. We know that $C_1 = \{\,,\}$ satisfies H1 (resp. H1$'$) and E_2 is exact because its antisymmetric part is zero by Jacobi's identity. Thus $E_2 = \tilde{\delta} C_2$ where C_2 satisfies H1 (resp. H1$'$) by virtue of the Corollary. We can assume that, $\forall i, j = 1, \ldots, m$; $C_2(p_i, p_j) = 0$. Indeed, it would be satisfied if we use the construction described in Section 1.3. Another argument is that we can subtract from C_2 a term of the form $a_{ij} Z^i Z^j$ which is the only one contributing to $C_2(p_i, p_j)$. Finally, we can assume C_2 to be symmetric. Indeed $E_2(u, v, w) = -E_2(w, v, u)$ and if $C(u, v) = (-1)^r C(v, u)$ then $\tilde{\delta} C(u, v, w) = -(-1)^r \tilde{\delta} C(w, v, u)$.

The cochain E_3 is then automatically exact because its antisymmetric part is zero. Using once more the Corollary we have $E_3 = \tilde{\delta} C_3$ and C_3 satisfies H1 (resp. H1$'$). Finally, as above, $C_3(u, v) = -C_3(v, u)$.

LEMMA 3. *If the cochains* C_r *satisfy* H1 (resp. H1$'$), H2 *and the parity assumption* (ii) *for* $r \leqslant n$, *then* E_{n+1} *is exact and* $E_{n+1} = \tilde{\delta} C_{n+1}$ *where* C_{n+1} *satisfies* H1 (resp. H1$'$) *and* (ii).

Proof. The assumptions imply immediately that:

$$C_r(\Pi^* f, \Pi^* g) = 0, \quad \forall f, g \in \mathscr{C}^\infty(M), \forall r \geqslant 1,$$

$$C_r(p_i, p_j) = 0, \quad \forall i, j = 1, \ldots, m, \forall r > 1,$$

$$C_r(p_i, \Pi^* f) = 0, \quad \forall i = 1, \ldots, m, \forall f \in \mathscr{C}^\infty(M), \forall r > 1,$$

$$C_r(\{p_i, p_j\}, p_k) = 0, \quad \forall i, j, k = 1, \ldots, m, \forall r > 2,$$

$$C_r(\{p_i, p_j\}, \Pi^* f) = 0, \quad \forall i, j = 1, \ldots, m, \forall f \in \mathscr{C}^\infty(M), \forall r > 2.$$

This implies that the antisymmetric part of E_{n+1}, $(n \geqslant 3)$, which one knows, *a priori*, to be a 1-differential operator in all its arguments (Proposition 1), vanishes identically. Hence E_{n+1} is exact and using the Corollary one sees that $E_{n+1} = \widetilde{\delta} C_{n+1}$ where C_{n+1} satisfies H1 (resp. H1$'$). By the argument used in Lemma 2, one can assume C_{n+1} to satisfy the parity assumption, replacing C_{n+1} by its symmetrization or its antisymmetrization which still satisfies H1 (resp. H1$'$).

Using Lemmas 2 and 3 we get by induction:

THEOREM 1. *Let M be a parallelizable manifold, T^*M its cotangent bundle. Then there exists a $*$ product on T^*M:*

$$u * v = uv + \lambda \{u, v\} + \sum_{r=2}^{\infty} \lambda^r C_r(u, v).$$

where the C_r are bidifferential operators satisfying assumptions H1 *and* H2. *If the functions C_{ij}^k are constants, then the C_r satisfy* H1$'$ *and* H2.

When M is a connected Lie group G, the above results apply obviously. If one uses for vector fields X_i on G the left invariant vector fields corresponding to a basis $\overline{X}_i$ of the Lie algebra $\mathscr{G}$ of G one gets:

COROLLARY. *Let G be a connected Lie group, T^*G its cotangent bundle and Π: $T^*G \to G$ the canonical projection. There exists a $*$ product on T^*G, invariant by the lift of the left translations of G, such that:*

(∗) *If P and Q are homogeneous polynomials in p of degree r and s and if f, $g \in \mathscr{C}^\infty(G)$*

$$(\Pi^*f \cdot P) * (\Pi^*g \cdot Q) = \sum_{i=0}^{r+s} \lambda^i (\Pi^* h_i) R_i$$

where $h_i \in \mathscr{C}^\infty(G)$ and R_i is a homogeneous polynomial of p in degree $(r + s - i)$.

The 'left' action of G on T^*G has a momentum map ξ and one sees easily that if $J(\xi)(\overline{X}_i) = -\Sigma_k (\mathrm{Ad}\, \Pi(\xi))^{-1} k_i p_k(\xi) =_{\mathrm{def}} \Lambda_i(\xi)$, then:

$$\Lambda_i * \Lambda_j - \Lambda_j * \Lambda_i = 2\lambda \{\Lambda_i, \Lambda_j\}.$$

The $*$ product is thus, with the terminology of [1], a $*$ representation of $\mathscr{G}$, which we call the *regular $*$ representation.* The linear representation ρ of $\mathscr{G}$ on $E(N, \lambda)$ defined by:

$$\rho(\overline{X}_i)s = \frac{1}{2\lambda}(\Lambda_i * s - s * \Lambda_i), \quad \forall s \in E(N, \lambda)$$

contains, among the invariant subspaces, the space $\Pi^*L^2(G)$. One checks that on this subspace,

ρ is equivalent to the differential of the regular representation of G.

It thus seems reasonable to assume that the regular $*$ representation plays an important role among the $*$ representations of G. This point will be studied elsewhere, in particular in the case of a compact group.

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