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Boundary Layer Phenomena for Differential-Delay Equations with State-Dependent Time Lags, I.

JOHN MALLET-PARET & ROGER D. NUSSBAUM

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Abstract

In this paper we begin a study of the differential-delay equation

$$\varepsilon x'(t) = -x(t) + f(x(t-r)), \quad r = r(x(t)).$$

We prove the existence of periodic solutions for $0 < \varepsilon < \varepsilon_0$, where ε_0 is an optimal positive number. We investigate regularity and monotonicity properties of solutions $x(t)$ which are defined for all t and of associated functions like $\eta(t) = t - r(x(t))$. We begin the development of a Poincaré-Bendixson theory and phase-plane analysis for such equations. In a companion paper these results will be used to investigate the limiting profile and corresponding boundary layer phenomena for periodic solutions as ε approaches zero.

0. Introduction

The scalar differential-delay equation

$$\varepsilon x'(t) = -x(t) + f(x(t-1)) \qquad (0.1)_\varepsilon$$

has arisen in several different areas of science over the past fifteen years. We refer to the introduction of [34] for further details, but we note in passing that the equation arises in the study of an "optically bistable device" (see [7, 8, 17–20]) and in a variety of models for physiological processes or diseases (see [28–30, 51] for discussion of models which treat production of blood cells, respiration and cardiac arrythmias). The equation has appeared also in interesting recent work of LONGTIN [26] and LONGTIN & MILTON [23–25] on the so-called human pupil-light reflex.

Over the past several years it has become apparent that there is a need for a theory of equations similar to $(0.1)_\varepsilon$ but containing delays that are functions of the state of the system. Among the first rigorous treatments of state-dependent delays are those in the pioneering work of DRIVER [9–12] concerning models arising in classical electrodynamics (see also DRIVER & NORRIS [13]). Models leading to such equations have been introduced by MACKEY [27] and BÉLAIR & MACKEY [1] to describe commodity price fluctuations and by MACKEY & MILTON [30] to describe the dynamics of blood cell production. Furthermore, discussions of the authors with A. LONGTIN, J. MILTON and M. MACKEY suggest that equations with state-dependent delay are of interest in theories of the pupil-light reflex.

In this paper we study a variant of $(0.1)_\varepsilon$ in which the constant time lag is replaced by a time lag which depends on $x(t)$:

$$\varepsilon x'(t) = -x(t) + f(x(t-r)), \quad r = r(x(t)). \tag{$0.2)_\varepsilon$}$$

The given function r is such that $r(x(t)) \geqq 0$ for all relevant t. As in [34], we are interested in solutions of $(0.2)_\varepsilon$ (usually periodic solutions) which oscillate about a steady-state constant solution c, and we normalize by looking for solutions which oscillate about zero. (Since c is typically not explicitly known, nontrivial problems may arise in normalizing $c = 0$. We refer the reader to [36] for a discussion of this point.) Assuming $c = 0$, we shall always suppose that $r(0) = 1$ and that f satisfies a negative feedback condition, $xf(x) < 0$ for $x \in [-B, A]$ and $x \neq 0$. We shall then seek slowly oscillating periodic solutions or SOP solutions of $(0.2)_\varepsilon$. Recall (cf. [34]) that a periodic solution $x(t)$ of $(0.2)_\varepsilon$ is called an SOP solution if there exist numbers $q_1 > 1$ and $q_2 > q_1 + 1$ such that $x(0) = 0, x(q_1) = 0, x(q_2) = 0, x(t) < 0$ for $0 < t < q_1, x(t) > 0$ for $q_1 < t < q_2$ and $x(t + q_2) = x(t)$ for all t. Of course, if $x(t)$ is an SOP solution, for each fixed τ, then $x(t + \tau)$ is also called an SOP solution. SOP solutions are called P_2-solutions in references [35] and [47].

Our goal in this paper and in the sequel [38] is to prove existence of SOP solutions of $(0.2)_\varepsilon$, to establish theorems about the shape of general periodic solutions, and to study their limiting profile as $\varepsilon \to 0^+$. We shall show that a variety of new mathematical phenomena arise which are not present in the constant time-lag case [3, 33–35, 47]. The simple example

$$\varepsilon x'(t) = -x(t) - kx(t-r), \quad r = 1 + cx(t), \tag{$0.3)_\varepsilon$}$$

with $k > 1$ and $c > 0$, illustrates some of the new phenomena. In [38] we shall prove that the limiting profile Γ as $\varepsilon \to 0^+$ of SOP solutions $x_\varepsilon(t)$ of $(0.3)_\varepsilon$ is given by a sawtooth function of period $k + 1 : \Gamma$ is a straight line $y = c^{-1}x$ for $-1 \leqq x \leqq k$, has vertical segments at $x = -1$ and $x = k$, and is extended periodically. This, of course, contrasts markedly with the constant time-lag case treated in [34]. There the limit is a step function with jumps at the integers.

For reasons of length we defer discussion of the limiting behavior of solutions of $(0.2)_\varepsilon$ as $\varepsilon \to 0^+$ to the subsequent paper [38]. Here we prove existence of SOP solutions of $(0.2)_\varepsilon$ (for ε small enough) and discuss qualitative properties of SOP solutions and other solutions of $(0.2)_\varepsilon$. Our existence theorems for SOP solutions are essentially nonconstructive and provide little information about the shape of SOP solutions. Thus "regularity" theorems concerning SOP solutions are philosophically akin to regularity theorems for weak solutions of partial differential equations. The results we obtain, particulary in Section 2 below, play a crucial role in [38].

This paper is long and, in parts, technical, so we want to provide the reader with a detailed guide. For the most part we assume one of two possible sets of hypotheses, labelled H1 and H2 or H1′ and H2′ on f and r:

H1. A and B are given positive real numbers and $f:[-B,A]\to[-B,A]$ is a Lipschitz map with $xf(x)<0$ for all $x\in[-B,A]$, $x\neq 0$.

H2. For A and B as in H1, $r:[-B,A]\to\mathbb{R}$ is a Lipschitz map with $r(0)=1$ and $r(u)\geqq 0$ for all $u\in[-B,A]$.

H2′. B is a positive real number and $r:[-B,\infty)\to\mathbb{R}$ is a locally Lipschitz map with $r(0)=1$, $r(u)\geqq 0$ for all $u\geqq -B$, and $r(-B)=0$.

H1′. $f:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitzian map, and if B is as in H2′ and $A=\sup\{|f(u)|: -B\leqq u\leqq 0\}$, then $uf(u)<0$ for all $u\in[-B,A]$, $u\neq 0$.

The letters A and B are always constants as in H1 and H2 or H1′ and H2′. Notice that the example in $(0.3)_\varepsilon$ satisfies H1′ and H2′, but not H1 and H2.

With this background, we can state part of the basic theorem in Section 1 (see Theorem 1.1 in Section 1).

Theorem. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′. *Assume that f is in C^1 near 0 and $f'(0)=-k<-1$. Let ν_0, $\frac{\pi}{2}<\nu_0<\pi$, be the unique solution of $\cos(\nu_0)=-1/k$ and define $\lambda_0=\nu_0/\sqrt{k^2-1}$. Then for each $\lambda>\lambda_0$ the equation*

$$x'(t)=-\lambda x(t)+\lambda f(x(t-r)),\qquad r=r(x(t)) \tag*{$(0.4)_\lambda$}$$

has an SOP solution $x_\lambda(t)$ such that $-B<x_\lambda(t)<A$ for all t.

In fact, we prove considerably more in Theorem 1.1. We actually prove that there is a global continuum of periodic solutions of $(0.4)_\lambda$ and that these bifurcate from the zero solution at $\lambda=\lambda_0$. Furthermore, we show that if x is an SOP solution of $(0.4)_\lambda$ such that $-B<x(t)<A$ for all t, if the minimal period of $x(t)$ is $p(x,\lambda)$, and if $p(0,\lambda_0)=2\pi/\nu_0$, then the map p is (in a precise sense given in Theorem 1.1) continuous.

If f is an odd function and r is an even function, one expects special SOP solutions, called S-solutions or P_1-solutions, to $(0.4)_\lambda$. A periodic solution $x(t)$ of $(0.4)_\lambda$, normalized so that $x(0)=0$, is called an S-solution or P_1-solution, if there exists $q_1>1$ such that $x(t+q_1)=-x(t)$ for all t and $x(t)>0$ for $0<t<q_1$. The existence of such solutions is discussed in Theorem 1.2, Remark 1.7, and Remark 1.8.

The argument proving Theorem 1.1 is somewhat technical so it may be worthwhile to discuss the basic strategy of the proof. We fix a large $\Lambda>0$ and consider $(0.4)_\lambda$ for $0<\lambda<\Lambda$. For $R\geqq\Lambda(A+B)$ and $M\geqq\sup\{r(u): -B\leqq u\leqq A\}$, we define $X=C([-M,0])$ and define a compact, convex set $C_R\subset X$ of Lipschitzian functions of Lipschitz constant less than or equal to R. See (1.6). We then define a continuous map $F: C_R\times(0,\Lambda)\to C_R$ and

$F_\lambda(\phi) \equiv F(\phi, \lambda)$. We prove that nonzero fixed points of F_λ are in one-one correspondence with SOP solutions $x(t)$ of $(0.4)_\lambda$ such that $-B < x(t) < A$ for all t. The fact that we can define such a continuous map F involves elementary arguments and estimates. However, caution is necessary: it is precisely at this point that our approach may fail for variant equations.

Much of the literature for proving existence of SOP solutions [16, 43, 45] emphasizes the role of proving that 0 is an "attractive fixed point" or "ejective fixed point" of F_λ. However, the central point of all proofs is actually a calculation of the fixed-point index of F_λ on certain relatively open sets in C_R. Here, we use the trick of homotoping the equation $(0.4)_\lambda$ to one with a constant time lag:

$$x'(t) = -\lambda x(t) + \lambda f(x(t - r_s)), \quad r_s = (1 - s) + sr(x(t)).$$

This homotopy leads to a corresponding homotopy of the map $F_{\lambda,s}: C_R \to C_R$ corresponding to r_s, $0 \leqq s \leqq 1$. We prove that this homotopy enables one to reduce the problem of computing the fixed-point index of F_λ to the well-understood problem of computing the fixed-point index when $r(u)$ is a constant. We suspect that this trick will prove useful for other problems concerning functional differential equations.

We note in passing that certain of the lemmas used in the proof of Theorem 1.1 are of independent interest and play a role in our future work. Specifically, Lemma 1.5 establishes (under our usual assumptions on f and r) that there is a constant $\delta > 0$ such that $(0.4)_\lambda$ has no SOP solution for $0 < \lambda < \delta$. There is also a constant C, independent of λ, such that the minimal period p of any SOP solution $x(t)$ of $(0.4)_\lambda$ satisfies $p \leqq C$.

Section 2 of this paper treats the problem of understanding more precisely the appearance of solutions (particularly SOP solutions) of equation $(0.4)_\lambda$. Assume always that f and r satisfy H1 and H2 or H1′ and H2′. The starting point of our results is Proposition 2.1. Suppose that $\bar{q}$ and $q \geqq \bar{q} + 1$ are given real numbers and that $\theta: [\bar{q}, q] \to \mathbb{R}$ is a Lipschitz map such that $\theta(\bar{q}) = \theta(q) = 0$ and $-B \leqq \theta(t) \leqq A$ for $\bar{q} \leqq t \leqq q$. Then there is a unique solution $x(t) = x(t, \theta, \lambda)$ of the equation

$$x'(t) = -\lambda x(t) + \lambda f(x(t - r)), \quad r = r(x(t)), \quad t \geqq q, \qquad (0.5)_\lambda$$

$$x \mid [\bar{q}, q] = \theta,$$

and $x(t)$ is defined for all $t \geqq q$ and satisfies $-B \leqq x(t) \leqq A$ for all $t \geqq q$ and $\eta(t) = t - r(x(t)) \geqq \bar{q}$ for all $t \geqq q$. If $\theta(t) > 0$ for $\bar{q} < t < q$ and if we define $q_2 = q$, then we let q_j, $j \geqq 2$, denote the consecutive zeros of $x(t)$ (allowing $q_{j+1} = \infty$ if $|x(t)| > 0$ for $t > q_j$). Then $q_{j+1} - q_j > 1$ for all $j \geqq 1$, and if $\eta(t) = t - r(x(t))$, then $\eta(t) > q_{j-1}$ for $q_j \leqq t \leqq q_{j+1}$ and $j \geqq 2$.

The function $\eta(t) = t - r(x(t))$ plays a crucial role in our theory of $(0.4)_\lambda$. If $x(t)$ is as above and $K = \{t \geqq q : x'(t) = 0\}$, we prove in Theorem 2.1 that the restriction η/K of η to K is a strictly increasing function on K. In fact, if $\alpha \in K$ and $t > \alpha$, we prove that $\eta(t) > \eta(\alpha)$. Note, however, that we do *not* prove in Theorem 2.1 that $\eta'(t) > 0$ for all $t \geqq q$.

Monotonicity properties of periodic solutions $x(t)$ of $(0.4)_\lambda$ play an important role in our theory. If c and d are positive numbers, we say that $x(t)$ satisfies Property M between $-d$ and c if, roughly speaking, $x(t)$ has nice monotonicity properties on intervals J such that $-d < x(t) < c$ for $t \in J$. A precise definition is given in Section 2 below and in Section 3 of [34]. In Theorem 2.2 we prove that, under certain conditions on f and under very mild assumptions on r, if $x(t)$ is an SOP solution of $(0.2)_\varepsilon$ for any $\varepsilon > 0$, then $x(t)$ satisfies Property M between $-d$ and c. The point is that the positive numbers c and d depend only on f, and are defined exactly as they are for the corresponding theorem in the constant time-lag case. See Theorem 3.1 in [34]. The lag $r(x(t))$ plays no role here.

One important special case of Theorem 2.2 should be noted (see Remark 2.1 below). Suppose that f and r satisfy H1 and H2 or H1′ and H2′ and that f is in C^1 on $(-B, A)$ with $f'(0) = -k < -1$. Assume that $|f(f(u))| > |u|$ for $-B < u < A$ and $u \neq 0$ and that $f'(u) < 0$ for $-B < u < A$. Then, if $x_\varepsilon(t)$ is an SOP solution of $(0.2)_\varepsilon$, there are exactly two values of t such that $x_\varepsilon'(t) = 0$ on each half-open interval of length equal to the period of p. We do not know, for general r, whether the same theorem is true without the assumption that $|f(f(u))| > |u|$ for $u \in (-B, A)$, $u \neq 0$.

The idea of integer-valued Lyapunov functions has played a useful role in studying differential-delay equations with a single constant time lag (see [31, 32]): Given a function $y(t)$ associated in a natural way with a problem, one tries to prove that $N(t)$, the number of zeros of $y(s)$ on $(t-1, t]$, is a decreasing function on $Z = \{t : y(t) = 0\}$. The remainder of Section 2 involves the extension of this idea to the nonconstant time-lag case. Here we try to count the number of zeros of some function $y(s)$ on the interval $(\eta(t), t]$, for $t \in Z = \{t \mid y(t) = 0\}$. However, many serious technical problems arise which are not present in the constant time-lag case. For example, in contrast to the constant time-lag case (see [42]), if f and r are real-analytic, it is not known whether an SOP solution of $(0.2)_\varepsilon$ is necessarily real-analytic.

The above-mentioned integer-valued Lyapunov functions play a crucial role in Theorem 2.3 and Corollaries 2.2 and 2.3. Suppose that f and r satisfy H1 and H2 or H1′ and H2′ and that $f(u)$ satisfies a mild nondegeneracy condition on $[-B, A]$ (see Definition 2.1). Assume that $x(t)$ satisfies $(0.4)_\lambda$ for all $t \in \mathbb{R}$, x is not identically zero, and $-B < x(t) < A$ for all t. Then it is proved in Corollary 2.2 that $K = \{t \in \mathbb{R} : x'(t) = 0\}$ has no accumulation points and that there does not exist t such that $x'(\eta^j(t)) = 0$ for all $j \geqq 0$, where $\eta(t) = t - r(x(t))$ and $\eta^j(t) = \eta(\eta^{j-1}(t))$. Furthermore, if $f'(u) < 0$ for $-B < u < A$, there exists T such that if $x'(t) = 0$ and $t \geqq T$, then $x''(t) \neq 0$. In Corollary 2.3, we prove, without the assumption that $f'(u) < 0$ for $-B < u < A$, that there exists T_1 such that if $x(t) = 0$ and $t \geqq T_1$, then $x'(t) \neq 0$. Of course, if $x(t)$ is periodic, we conclude that if $x(t) = 0$, then $x'(t) \neq 0$ and if $x'(t) = 0$, then $x''(t) \neq 0$.

As we have already noted, the function $\eta(t) = t - r(x(t))$ plays an important role in our theory. Suppose that f and r satisfy H1 and H2 or H1′ and H2′ and that r is in C^2 on $[-B, A]$. Assume that $x(t)$ satisfies $(0.4)_\lambda$ for all real t and $-B \leqq x(t) \leqq A$ for all t. Assume that there is a constant $D \geqq 0$ such that

$$r''(u) \leqq D(r'(u))^2 \quad \text{for } -B \leqq u \leqq A$$

and $\lambda > D$. It is a special case of Theorem 2.5 that if $\eta'(\rho) > 0$, then $\eta'(t) > 0$ for all $t \geqq \rho$. The question of finding optimal conditions which ensure that $\eta'(t) > 0$ for all t remains open. However, numerical studies suggest that, in general, one may have $\eta'(t) < 0$ for some t, even if $x(t)$ is an SOP solution of $(0.4)_\lambda$ and λ is large.

If one knows that $\eta'(t) > 0$ for all t, where $x(t)$ is as above (see Theorem 2.5), then one can refine Theorem 2.3 and Corollary 2.2 considerably. Thus suppose that f and r satisfy H1 and H2 or H1′ and H2′ and that f and r are in C^1 on $(-B, A)$ with $f'(u) < 0$ for $-B < u < A$. Assume that $x(t)$ is a periodic solution of $(0.2)_\varepsilon$, $-B < x(t) < A$ for all t and $x(t)$ is not identically zero. Assume that $\eta'(t) > 0$ for all t, where $\eta(t) = t - r(x(t))$. Let q_j, $j \in \mathbb{Z}$, denote the consecutive zeros of $x(t)$. If p is the minimal period of $x(t)$, we must have that $q_{j+N} - q_j = p$ for some even integer N. However, it is a special case of Theorem 2.6 that in fact $N = 2$. Furthermore, for each $j \in \mathbb{Z}$ there is a unique number t such that $q_j < t < q_{j+1}$ and $x'(t) = 0$. In the constant time-lag case $r(u) \equiv 1$ this result has been obtained earlier by SELL & MALLET-PARET in an unpublished work.

The reader should note that the latter half of Section 2 and, in particular, Theorem 2.6, represents a first step in developing a kind of Poincaré-Bendixson theory and phase-plane analysis for equation $(0.4)_\lambda$. Related earlier work can be found in [21, 22, 32, 39, 48]. In future work, we hope to show that these ideas can be pushed much further.

The fact that under the hypotheses of Theorem 2.6 the equation $x'(t) = 0$ has precisely one solution t such that $q_j < t < q_{j+1}$ may seem unremarkable and "obvious". However, even if $r(u)$ is a constant, this is not the case: If f is not monotone, it is proved in [34] and [47] that the equation $x'(t) = 0$ may have many solutions on $[q_j, q_{j+1}]$. On the other hand, if $x(t)$ is an SOP solution of the equation

$$x'(t) = \lambda f(x(t - r)), \quad r = r(x(t)), \quad \lambda > 0,$$

and $xf(x) < 0$ for all $x \neq 0$, it is relatively easy to prove that $x'(t) = 0$ has a unique solution on any interval (q_j, q_{j+1}), where q_j and q_{j+1} are successive zeros of $x(t)$.

1. Existence of slowly oscillating periodic solutions

In this section we prove the existence of slowly oscillating periodic solutions of equations of the form

$$\varepsilon x'(t) = -x(t) + f(x(t - r)), \quad r = r(x(t)) \qquad (1.1)_\varepsilon$$

or, equivalently,

$$x'(t) = -\lambda x(t) + \lambda f(x(t - r)), \quad r = r(x(t)). \qquad (1.2)_\lambda$$

Recall (see [34]) that a periodic solution of $(1.1)_\varepsilon$, normalized so that $x(0) = 0$, is called an SOP solution if there exist numbers $q_1 > 1$ and $q_2 > q_1 + 1$, such that $x(t) > 0$ for $0 < t < q_1$, $x(t) < 0$ for $q_1 < t < q_2$, and

$x(t+q_2)=x(t)$ for all t. Of course, if $x(t)$ is a SOP solution and τ is a fixed real, then $y(t)=x(t+\tau)$ is also called an SOP solution. Much of what we say extends to equations of the form

$$x'(t)=g\big(x(t),x(t-r)\big), \quad r=r\big(x(t)\big),$$

but for simplicity we restrict attention to $(1.2)_\lambda$.

We begin by discussing the initial value problem for $(1.2)_\lambda$. If, in the following, we were to assume that $r(u)>0$ for all relevant values of u, then the argument could be reduced to standard results about ordinary differential equations. However, with little additional work, we can treat the case $r(u)\geqq 0$, and we take this approach.

Proposition 1.1. *Let A and $-B$ be real numbers with $-B<A$ and let $f:[-B,A]\to[-B,A]$ be a Lipschitz map. Let $r:[-B,A]\to\mathbb{R}$ be a Lipschitz map such that $r(u)\geqq 0$ for $-B\leqq u\leqq A$, and select $M\geqq\sup\{r(u):-B\leqq u\leqq A\}$. Let $\phi:[-M,0]\to[-B,A]$ be a Lipschitz map. Then, if $\lambda>0$, there exists a unique Lipschitz function $x:[-M,\infty)\to\mathbb{R}$, such that $x\,|\,[-M,0]=\phi$, $-B\leqq x(t)\leqq A$ for all t, x is continuously differentiable on $[0,\infty)$ and $x(t)$ satisfies $(1.2)_\lambda$ for all $t\geqq 0$. If $x(t)<A$ for some $t_0\geqq 0$, then $x(t)<A$ for all $t\geqq t_0$; and if $x(t_0)>-B$ for some $t_0\geqq 0$, then $x(t)>-B$ for all $t\geqq t_0$.*

Proof. Fix a number $T>0$, let R be a Lipschitz constant for ϕ, and assume $R\geqq\lambda(A+B)$. Let $X=C[-M,T]$ be the Banach space of continuous maps $y:[-M,T]\to\mathbb{R}$ in the sup norm. Define $J=[-M,T]$ and $C=\{y\in X:y\,|\,[-M,0]=\phi,\ -B\leqq y(t)\leqq A$ for all $t\in J$ and $|y(t)-y(s)|\leqq R|t-s|$ for all $t,s\in J\}$. It is easy to see that C is a closed, bounded convex subset of X, and the Ascoli-Arzelà theorem implies that C is compact.

Define a map $F:C\to X$ by

$$(Fx)(t)=\phi(t) \qquad \text{for } -M\leqq t\leqq 0, \tag{1.3}$$

$$(Fx)(t)=e^{-\lambda t}\phi(0)+e^{-\lambda t}\int_0^t \lambda e^{\lambda s}f\big(x(s-r(x(s)))\big)\,ds, \qquad 0\leqq t\leqq T.$$

We leave it to the reader to prove that if $x\in C$ is a fixed point of F (so $F(x)=x$), then $x\,|\,[-M,0]=\phi$ and $x(t)$ satisfies $(1.2)_\lambda$ on $[0,T]$. If we can prove that F is continuous and $F(C)\subset C$, the Schauder fixed-point theorem implies that F has a fixed point.

To prove continuity of $F:C\to X$, suppose that $x_j\in C$ for $j\geqq 1$ and $\|x_j-x\|_\infty\to 0$. Let K_1 be a Lipschitz constant for f and K_2 a Lipschitz constant for r, and for notational convenience write $\rho_j(s)=r\big(x_j(s)\big)$, $\rho(s)=r\big(x(s)\big)$. Then we obtain for $0\leqq t\leqq T$,

$$\begin{aligned}|(Fx_j)(t)-(Fx)(t)| &\leqq e^{-\lambda t}\int_0^t \lambda e^{\lambda s}\big[|f(x_j(s-\rho_j(s)))-f(x(s-\rho_j(s)))|\\ &\qquad +|f(x(s-\rho_j(s)))-f(x(s-\rho(s)))|\big]\,ds\\ &\leqq e^{-\lambda t}\int_0^t \lambda e^{\lambda s}[K_1\|x_j-x\|_\infty+K_1RK_2\|x_j-x\|_\infty]\,ds\\ &=[1-e^{-\lambda t}]\,[K_1+K_1K_2R]\,\|x_j-x\|_\infty.\end{aligned}$$

We conclude that

$$\|F(x_j) - F(x)\|_\infty \leqq [1 - e^{-\lambda T}][K_1 + K_1 K_2 R]\|x_j - x\|_\infty,$$

so F is actually Lipschitzian on C with Lipschitz constant $[1 - e^{-\lambda T}]$ $[K_1 + K_1 K_2 R]$.

It remains to prove that $F(C) \subset C$. If $x \in C$ and $\rho(s) = r(x(s))$, we have for $0 \leqq t \leqq T$,

$$(Fx)(t) \leqq e^{-\lambda t}A + e^{-\lambda t}\int_0^t \lambda e^{\lambda s} A\, ds = A,$$

$$(Fx)(t) \geqq -e^{-\lambda t}B + e^{-\lambda t}\int_0^t \lambda e^{\lambda s}(-B)\, ds = -B.$$

To prove that $F(x)$ is Lipschitzian with Lipschitz constant R, it suffices to prove that $|(Fx)'(t)| \leqq R$ for $0 \leqq t \leqq T$. However, we have

$$(Fx)'(t) = -\lambda e^{-\lambda t}\phi(0) - \lambda e^{\lambda t}\int_0^t \lambda e^{\lambda s} f(x(s - \rho(s)))\, ds + \lambda f(x(t - \rho(t)))$$

$$\leqq \lambda e^{-\lambda t}B + \lambda(1 - e^{-\lambda t})\, B + \lambda A = \lambda(A + B) \leqq R.$$

A similar argument shows that

$$(Fx)'(t) \geqq -\lambda(A + B) \geqq -R,$$

which gives the desired estimate.

It follows that for each integer $T = n \geqq 1$, there exists a Lipschitz function $x_n(t)$ with Lipschitz constant R, for which $-B \leqq x_n(t) \leqq A$ for $-M \leqq t \leqq n$, $x_n \mid [-M, 0] = \phi$, and x_n satisfies $(1.2)_\lambda$ on $[0, n]$. By using the Cantor diagonalization procedure and the Ascoli-Arzelà theorem, one can see that there exists a subsequence $x_{n_i}(t)$ which converges uniformly to a continuous function $x(t)$ on any compact interval $[-M, T]$. It is not hard to see that $-B \leqq x(t) \leqq A$ for all $t \geqq -M$, $x \mid [-M, 0] = \phi$ and x satisfies $(1.2)_\lambda$ for $t \geqq 0$.

It remains to prove the uniqueness of the solution $x(t)$ above. If $y(t)$ is a second such solution, define

$$\tau = \sup\{t \geqq 0 : x(s) = y(s) \quad \text{for } 0 \leqq s \leqq t\}.$$

If $\tau < \infty$, by replacing $x(t)$ by $x_1(t) = x(t - \tau)$ we can assume that $\tau = 0$. If $T > 0$ is chosen sufficiently small, our previous remarks show that F is a contraction mapping on C and hence $x(t) = y(t)$ on $[0, T]$. It follows that $x(s) = y(s)$ for $0 \leqq s \leqq \tau + T$, which contradicts the choice of τ.

If $x(t_0) < A$ for some $t_0 \geqq 0$, we have for $t \geqq t_0$,

$$x(t) = x(t_0)\, e^{-\lambda(t-t_0)} + e^{-\lambda(t-t_0)}\int_{t_0}^t \lambda e^{\lambda(s-t_0)} f(x(s - r(x(s))))\, ds$$

$$\leqq x(t_0)\, e^{-\lambda(t-t_0)} + e^{-\lambda(t-t_0)}\int_{t_0}^t \lambda e^{\lambda(s-t_0)} A\, ds$$

$$= [x(t_0) - A]\, e^{-\lambda(t-t_0)} + A < A.$$

A similar argument applies if $x(t_0) > -B$. $\square$

Remark 1.1. The argument in Proposition 1.1 applies to classes of equations more general than $(1.2)_\lambda$. However, note that the uniqueness part of the argument fails if ϕ is only continuous, although a slight variant of the proof here still yields existence.

Remark 1.2. Suppose that $\phi_j : [-M, 0] \to [-B, A]$ is a sequence of Lipschitz functions all of which have Lipschitz constant R and suppose that $\|\phi_j - \phi\| \to 0$ for some $\phi \in C([-M, 0])$. Assume that λ_j is a sequence of positive reals and that $\lambda_j \to \lambda > 0$. Assume that $x_j : [-M, \infty) \to [-B, A]$ is a Lipschitz map, that $x_j \mid [-M, 0] = \phi_j$, and that

$$x_j'(t) = -\lambda_j x_j(t) + \lambda_j f(x(t - r(x_j(t)))) \quad \text{for } t \geqq 0.$$

(Here f and r are as in Proposition 1.1.) Let $x : [-M, \infty] \to [-B, A]$ be a Lipschitz map with $x \mid [-M, 0] = \phi$ and

$$x'(t) = -\lambda x(t) + \lambda f(x(t - r(x(t)))) \quad \text{for } t \geqq 0.$$

Then we assert that for any compact interval $[-M, T]$, $x_j(t) \to x(t)$ uniformly in $t \in [-M, T]$.

To prove this, we suppose not; so by taking a subsequence we can assume that for some $\delta > 0$ and all $j \geqq 1$,

$$\|x_j - x\|_\infty = \sup_{-M \leqq t \leqq T} |x_j(t) - x(t)| \geqq \delta.$$

If $R_1 \geqq R$ and $R_1 \geqq \lambda_j(A + B)$ for all $j \geqq 1$, then each x_j is Lipschitzian with Lipschitz constant R_1. Thus the Ascoli-Arzelà theorem implies that by taking a further subsequence, which we again label $\langle x_j \rangle$, we can assume that $\|x_j - y\|_\infty \to 0$ for some continuous map y. It is easy to see that y satisfies $(1.2)_\lambda$ and $y \mid [-M, 0] = \phi$; this contradicts the uniqueness of solutions of the initial value problem for $(1.2)_\lambda$.

In [38] we consider examples like

$$\varepsilon x'(t) = -x(t) - kx(t - r), \quad r = 1 + cx(t),$$

where $\varepsilon > 0$, $k > 1$, and $c > 0$. The function $f(x) = -kx$ does not leave any nontrivial bounded interval $[-B, A]$ invariant, so Proposition 1.1 does not directly apply. Nevertheless, a simple trick allows us to use Proposition 1.1.

Corollary 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitzian function such that $xf(x) < 0$ for all $x \neq 0$. Let $r : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitzian map and assume there are numbers $B \in (0, \infty)$ and $A \in (0, \infty]$ such that $r(-B) = 0$ and $r(u) \geqq 0$ for $-B \leqq u < A_1$ and such that $r(A_1) = 0$ if $A_1 < \infty$. Define $A_2 = \max\{f(u) \mid -B \leqq u \leqq 0\}$ and $A = \min(A_1, A_2)$. If $M \geqq \max\{r(u) : -B \leqq u \leqq A\}$, $\lambda > 0$, and $\phi : [-M, 0] \to [-B, A]$ is a Lipschitz map, there is a unique Lipschitz map $x : [-M, \infty) \to [-B, A]$, continuously differentiable on $[0, \infty)$, with $x \mid [-M, 0] = \phi$ and*

$$x'(t) = -\lambda x(t) + \lambda f(x(t - r)), \quad r = r(x(t)), \quad \text{for } t \geqq 0.$$

Proof. Define $\tilde{f}(u) = f(u)$ for $u \geqq -B$ and $\tilde{f}(u) = f(-B)$ for $u \leqq -B$. Similarly, define $\tilde{r}(u) = r(u)$ for $-B \leqq u \leqq A_1$, $\tilde{r}(u) = 0$ for $u \notin [-B, A_1]$. Define $-\tilde{B}$ by

$$-\tilde{B} = \min\left(\min\{f(u) \mid 0 \leqq u \leqq A_2\}, -B\right).$$

Our hypotheses imply that $\tilde{f}([-\tilde{B}, A_2]) \subset [-\tilde{B}, A_2]$, so Proposition 1.1 implies that there is a Lipschitz map $x: [-M, \infty) \to [-\tilde{B}, A_2]$, in C^1 on $[0, \infty)$, such that $x \mid [-M, 0] = \phi$ and

$$x'(t) = -\lambda x(t) + \lambda \tilde{f}(x(t - \tilde{r})), \quad \tilde{r} = \tilde{r}(x(t)), \quad \text{for } t \geqq 0.$$

It suffices to prove that $-B < x(t) \leqq A$ for all $t > 0$. We know that $-B \leqq x(0) \leqq A$. If $x(0) = -B$, we have

$$x'(0) = \lambda B + \lambda f(-B) > 0.$$

Thus there exists a $\delta > 0$ with $x(t) > -B$ for $0 < t \leqq \delta$. If $\tau > 0$ is the first time $t > 0$ with $x(t) = -B$, we obtain a contradiction: necessarily $x'(\tau) \leqq 0$, but $(1.2)_\lambda$ implies that

$$x'(\tau) = \lambda B + \lambda f(-B) > 0.$$

Thus $x(t) > -B$ for all $t > 0$.

If $A = A_2$, Proposition 1.1 implies that $x(t) \leqq A$ for all $t \geqq 0$. Thus, we assume that $A = A_1$. However, in this case, the exact argument used above shows that $x(t) < A_1$ for all $t > 0$. □

Remark 1.3. Because the proof of Corollary 1.1 involves a simple reduction to Proposition 1.1, the obvious analogue of Remark 1.2 holds for Corollary 1.1.

Remark 1.4. Let notation be as in Corollary 1.1 and suppose that $R > 0$, where R is a Lipschitz constant for ϕ. There exists a number $B_* = B_*(R)$, $0 < B_* < B$, such that if $\phi: [-M, 0] \to [-B_*, A]$ and $x(t)$ is the solution of $(1.2)_\lambda$ with $x \mid [-M, 0] = \phi$, then $-B_* < x(t) \leqq A$ for all $t > 0$. To prove this it suffices to prove that if $x(\tau) = -B_*$ for some $\tau \geqq 0$, then $x'(\tau) > 0$. The number R arises in estimating

$$|f(x(\tau - r)) - f(x(\tau))|, \quad r = r(x(\tau)).$$

Details are left to the reader.

Under the hypotheses of Proposition 1.1, we shall write $x(t) = x(t; \phi, \lambda)$ for the unique solution of $(1.2)_\lambda$ with $x \mid [q - M, q] = \phi$. Here $q \in \mathbb{R}$ and ϕ are given and $-B \leqq x(t) \leqq A$ for all $t \geqq q - M$.

Lemma 1.1. *Assume that $\lambda > 0$ and that hypotheses* H1 *and* H2 *are satisfied or that* H1′ *and* H2′ *are satisfied. Select $M \geqq \sup\{r(u) : -B \leqq u \leqq A\}$ and let $\phi: [-M, 0] \to [-B, A]$ be a Lipschitz map with $\phi(0) = 0$, $\phi(t_0) > 0$ for some $t_0 \in [-1, 0]$ and $\phi(t) \geqq 0$ for all $t \in [-1, 0]$. If $x(t) = x(t; \phi, \lambda)$, define $q_0 = q_0(\phi, \lambda)$ by*

$$q_0 = \sup\{t : x(s) = 0 \text{ for } 0 \leqq s \leqq t\}.$$

Define $q_1 = q_1(\phi, \lambda)$ *by*

$$q_1(\phi, \lambda) = \inf\{t > q_0 : x(t; \phi, \lambda) = 0\}$$

and $q_1 = \infty$ *if* $x(t) < 0$ *for all* $t > q_0$.

If $q_k = q_k(\phi, \lambda)$ *is finite, define*

$$q_{k+1}(\phi, \lambda) = \inf\{t > q_k : x(t; \phi, \lambda) = 0\}$$

and $q_{k+1} = \infty$ *if* $x(t) \neq 0$ *for all* $t > q_k$. *Then* $q_0 < 1$, $q_1 - q_0 > 1$ *and* $q_{k+1} - q_k > 1$ *for all* k *such that* $q_k < \infty$. *If* $q_k = \infty$ *for some* k, *then* $\lim_{t\to\infty} x(t; \phi, \lambda) = 0$.

Proof. We first assert that $q_0 < 1$. For if $q_0 \geqq 1$, then $x(t) = 0$ for $0 \leqq t \leqq 1$, from which we derive that $x'(t) = 0$ for $0 \leqq t \leqq 1$, $\eta(t) = t - 1$ for $0 \leqq t \leqq 1$, and

$$x'(t) = -\lambda f(x(t-1)) = -\lambda f(\phi(t-1)), \quad 0 \leqq t \leqq 1.$$

However, this last equation implies that $\phi(s) = 0$ for $-1 \leqq s \leqq 0$, contrary to our assumptions.

We next assert that there exists $\delta > 0$ such that $x(t) < 0$ on $(q_0, q_0 + \delta]$. By definition of q_0, there exists $t_j \to q_0^+$ such that $x(t_j) < 0$. On the other hand, if we define $\eta(s) = s - r(x(s))$ and $t > q_0$, then

$$e^{\lambda(t-q_0)} x(t) = \int_{q_0}^{t} \lambda e^{\lambda(s-q_0)} f(x(\eta(s)))\, ds. \tag{1.4}$$

Since $\eta(q_0) = q_0 - 1 < 0$, if $t \leqq q_0 + \delta$ and if δ is sufficiently small, it follows that $x(\eta(s)) \geqq 0$ for $q_0 \leqq s \leqq t$, and the right-hand side of (1.4) is a decreasing function of t for $q_0 \leqq t \leqq q_0 + \delta$. Since the left-hand side of (1.4) is negative at points $t_j \to q_0^+$, we must have $x(t) < 0$ for $q_0 < t \leqq q_0 + \delta$.

It follows that $q_1 > q_0$. We know in general that $x'(q_1) \geqq 0$. If $q_1 - q_0 \leqq 1$, we have

$$x'(q_1) = \lambda f(x(q_1 - 1)) \leqq 0,$$

and we must have $x'(q_1) = 0$. We conclude (even through r is only Lipschitz continuous) that $\eta'(q_1)$ exists and $\eta'(q_1) = 1$, so there exists $\delta > 0$ with

$$-1 \leqq \eta(t) < \eta(q_1) = q_1 - 1 \leqq q_0 \quad \text{for } q_1 - \delta \leqq t < q_1.$$

For this range of t, we obtain from $(1.2)_\lambda$ that

$$-e^{\lambda(t-q_1)} x(t) = \int_{t}^{q_1} \lambda e^{\lambda(s-q_1)} f(x(\eta(s)))\, ds. \tag{1.5}$$

The right-hand side of (1.5) is less than or equal to zero, while the left-hand side is positive, a contradiction.

If $q_{k+1} = \infty$ and $q_k < \infty$ for some even k, $x(t) < 0$ for $t > q_k$ and

$$x'(t) \geqq -\lambda x(t) \quad \text{for } t \geqq q_k + M,$$

which implies that $\lim_{t\to\infty} x(t) = 0$. The proof is similar for odd k.

The proof that $q_{k+1} - q_k > 1$ is essentially the same as the proof that $q_1 - q_0 > 1$ and is left to the reader. □

Lemma 1.2. *Let f and r be continuous maps of $\mathbb{R}$ to $\mathbb{R}$ with r locally Lipschitzian, $r(0) = 1$ and $f(0) = 0$. For a given $M \geqq 1$ and for a given $q \in \mathbb{R}$, let $\phi : [q-M, q] \to \mathbb{R}$ be a Lipschitzian function. Assume that there exists $\bar{q} \in [q-M, q]$ with $q - \bar{q} \geqq 1$ and $\phi(q) = \phi(\bar{q}) = 0$. If $\lambda > 0$, assume that there exists a Lipschitz function $x(t)$, defined for all $t \geqq q - M$, C^1 for $t \geqq q$, and satisfying*

$$x'(t) = -\lambda x(t) + \lambda f(x(t-r)), \quad r = r(x(t)),\ t \geqq q,$$

$$x \mid [q - M, q] = \phi.$$

Then $\eta(t) = t - r(x(t))$ satisfies $\eta(t) > \bar{q}$ for all $t > q$. In particular, if assumptions are as in Lemma 1.1, then $\eta(t) > q_{j-1}$ for $t \geqq q_j$.

Proof. If $q = \bar{q} + 1$, then $x'(q) = 0$ and it is easy to show that $\eta'_+(q)$ is defined and $\eta'_+(q) = 1$ (because r is Lipschitzian). It follows in this case that there exists $\delta > 0$ such that $\eta(t) > \bar{q}$ for $q < t < q + \delta$. Of course, the same is true if $q - \bar{q} > 1$. We shall prove Lemma 1.2 by contradiction; so assume the lemma is false and define τ by

$$\tau = \inf\{t > q : \eta(t) = \bar{q}\}.$$

It follows that

$$x'(\tau) = -\lambda x(\tau).$$

Since $q - \bar{q} \geqq 1$, we must have $x(\tau) \neq 0$; otherwise $\eta(\tau) = \tau - 1 > \bar{q}$. It follows from the equation for $x'(\tau)$ and the assumption $x(q) = 0$ that there exists a number s, $q < s < \tau$, with $x(s) = x(\tau)$. The choice of s implies that

$$\eta(s) < \eta(\tau) = \bar{q}.$$

We know from our previous remarks that there exists t, $q < t < s$, with $\eta(t) > \bar{q}$. We conclude that there exists a number τ_1, $q < \tau_1 < s$, with $\eta(\tau_1) = \bar{q}$. Since $q < \tau_1 < \tau$, we have a contradiction. □

Lemma 1.2 shows that the initial value problem for $(1.2)_\lambda$ is well-defined if f and r are as in Proposition 1.1, $x \mid [\bar{q}, q] = \phi$, ϕ is Lipschitzian, $\phi(q) = \phi(\bar{q}) = 0$ and $q - \bar{q} \geqq 1$.

If $M \geqq \sup\{r(u) : -B \leqq u \leqq A\}$ (where r satisfies H2 or H2′), we shall work in the Banach space $X = C([-M, 0])$ of continuous maps $\phi : [-M, 0] \to \mathbb{R}$ with

$$\|\phi\| = \sup\{|\phi(t)| : -M \leqq t \leqq 0\}.$$

We shall always use the letter M as above, i.e., $M \geqq \sup\{r(u) : -B \leqq u \leqq A\}$. If R is a given positive number and A is as in H1 or H1′, we define $C_R \subset X$ by

$$C_R = \{\phi \in X : 0 \leqq \phi(t) \leqq A \text{ for } -M \leqq t \leqq 0,\ \phi(0) = 0, \text{ and } \phi \text{ is Lipschitz continuous with Lipschitz constant } R\}. \quad (1.6)$$

Of course, C_R also depends on A and M, but we do not indicate this dependence. We define U_R by

$$U_R = \{\phi \in C_R : \phi(t) > 0 \text{ for some } t \in [-1, 0]\}. \tag{1.7}$$

It is clear that C_R is a closed, bounded convex subset of X, and the Ascoli-Arzelà theorem implies that C_R is compact. The set U_R is a relatively open subset of C_R.

Assume now that f and r satisfy H1 and H2 or H1′ and H2′, that Λ is a given positive number and that $R \geqq \Lambda(A+B)$, where A and B are as in H1 and H2 or H1′ and H2′. Then for $0 < \lambda \leqq \Lambda$ we can define a map $F_\lambda : C_R \to C_R$ thus: Define $F_\lambda(\phi) = 0$ if $\phi \mid [-1, 0]$ is identically zero or if $q_2(\phi, \lambda) = \infty$. If $\phi \in U_R$, $q_2(\phi, \lambda) < \infty$, and $q_2(\phi, \lambda) - q_1(\phi, \lambda) < M$, write $x(t) = x(t; \phi, \lambda)$ and define $F_\lambda(\phi) = \psi \in U_R$, where $\psi(s) = x(q_2 + s)$ for $q_1 - q_2 \leqq s \leqq 0$, and $\psi(s) = 0$ for $-M \leqq s \leqq q_1 - q_2$. If $q_2(\phi, \lambda) - q_1(\phi, \lambda) \geqq M$ and $q_2(\phi, \lambda) < \infty$, define $F_\lambda(\phi) = \psi \in U_R$, where $\psi(s) = x(q_2 + s)$ for $-M \leqq s \leqq 0$. By using Lemma 1.2, we see that if $\phi \in U_R$ and $F_\lambda(\phi) = \phi$, the corresponding solution $x(t) = x(t; \phi, \lambda)$, considered as defined on $[0, \infty)$, extends to an SOP solution of $(1.2)_\lambda$ defined for all $t \in \mathbb{R}$ and of period $q_2(\phi, \lambda)$. Thus our problem is to find further conditions which ensure that F_λ has a fixed point in U_R.

As a preliminary technical problem, we must prove that $(\phi, \lambda) \in C_R \times (0, \Lambda] \to F_\lambda(\phi) \in C_R$ is continuous. We shall also write $F(\phi, \lambda) = F_\lambda(\phi)$. The argument for continuity of F is straightforward but tedious. We begin with a lemma.

Lemma 1.3. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′. *Let Λ be a positive number, let $J = (0, \Lambda]$, and let R be such that $\Lambda(A+B) \leqq R$. If $(\phi, \lambda) \in U_R \times J$ (see (1.6)) is such that $q_2(\phi, \lambda) < \infty$ (see Lemma 1.1) and if $(\phi_k, \lambda_k) \in C_R \times J$, $k \geqq 1$, is a sequence which converges to (ϕ, λ), then $q_1(\phi_k, \lambda_k) \to q_1(\phi, \lambda)$ and $q_2(\phi_k, \lambda_k) \to q_2(\phi, \lambda)$. If α and η are given positive numbers, there exists $T = T(\eta, \alpha) > 0$ such that if $(\phi, \lambda) \in C_R \times J$ and $\lambda \geqq \alpha$ and $q_2(\phi, \lambda) \geqq 2T$, then $\|F(\phi, \lambda)\| \leqq \eta$.*

Proof. For notational convenience, write $x_k(t) = x(t; \phi_k, \lambda_k)$, $x(t) = x(t; \phi, \lambda)$, $q_{1k} = q_1(\phi_k, \lambda_k)$, $q_1 = q_1(\phi, \lambda)$, $q_{2k} = q_2(\phi_k, \lambda_k)$, and $q_2 = q_2(\phi, \lambda)$. Lemma 1.1 implies that $q_0 = q_0(\phi, \lambda) < 1$, $q_{0k} = q_0(\phi_k, \lambda_k) < 1$, $q_{1k} - q_{0k} > 1$, and $q_1 - q_0 > 1$. It follows that $q_{1k} > 1$ and $q_1 > 1$ for all $k \geqq 1$ and $x(1) < 0$. Continuous dependence on initial data (see Remark 1.2) implies that given any positive number μ and any number T with $q_2 < T < q_3(\phi, \lambda)$, there exists $k(\mu, T)$ with

$$\sup\{|x_k(t) - x(t)| : 1 \leqq t \leqq T\} < \mu \quad \text{for } k \geqq k(\mu, T).$$

For a given positive number $\theta < \min(\frac{1}{2}, T - q_2)$, we can assume (by reducing μ) that $x(t) \leqq -\mu$ for $1 \leqq t \leqq q_1 - \theta$, $x(t) \geqq \mu$ for $q_1 + \theta \leqq t \leqq q_2 - \theta$, and that $x(t) \leqq -\mu$ for $q_2 + \theta \leqq t \leqq T$. For $k \geqq k(\mu, T)$, it follows that $x_k(t) \neq 0$ for $1 \leqq t \leqq T$ and $|q_1 - t| \geqq \theta$ or $|q_2 - t| \geqq \theta$ and that $x_k(t)$ has a zero on

$[q_1 - \theta, q_1 + \theta]$ and on $[q_2 - \theta, q_2 + \theta]$. It follows by using Lemma 1.1 that $|q_{1k} - q_1| < \theta$ and $|q_{2k} - q_2| < \theta$ for k sufficiently large.

It remains to prove the last part of the lemma. If $q_2 = q_2(\phi, \lambda) \geqq 2T$, then either $q_1 = q_1(\phi, \lambda) \geqq T$ or $q_2 - q_1 \geqq T$. If $q_2 - q_1 \geqq T > 2M$, then $x(t)$ is decreasing on $[q_2 - M, q_2]$ and for $t \geqq q_1 + M$ we have

$$x'(t) \leqq -\lambda x(t) \leqq -\alpha x(t),$$

$$x(t) \leqq x(q_1 + M) \exp(\alpha(q_1 + M - t)) \leqq A \exp(\alpha(q_1 + M - t)).$$

It follows that

$$\sup\{x(t) : q_2 - M \leqq t \leqq q_2\} \leqq x(q_2 - M) \leqq A \exp(\alpha(q_1 - q_2 + 2M)) \leqq A \exp(\alpha(2M - T)).$$

Obviously, for $T \geqq T(\eta, \alpha)$ this implies that

$$\|F_\lambda(\phi)\| \leqq \eta.$$

If $q_1 \geqq T > 2M$ and η_1 is a given positive number, an argument like that above shows that

$$\sup\{|x(t)| : q_1 - M \leqq t \leqq q_1\} \leqq \eta_1$$

if $T \geqq T(\eta_1, \alpha)$. It follows by continuous dependence that if η_1 is sufficiently small and $\alpha \leqq \lambda \leqq \Lambda$, then

$$\sup\{|x(t)| : q_1 \leqq t \leqq q_1 + M\} \leqq \eta. \tag{1.8}$$

If $q_2 > q_1 + M$, then $x(t)$ is decreasing on $[q_1 + M, q_2]$; it follows from (1.8) that

$$\sup\{x(t) : q_1 \leqq t \leqq q_2\} \leqq \eta,$$

which completes the proof. □

Lemma 1.4. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′. *Let Λ be a positive number, let $J = (0, \Lambda]$ and select $R \geqq \Lambda(A + B)$. Then the map $F : C_R \times J \to C_R$ is continuous.*

Proof. Suppose that $(\phi, \lambda) \in C_R \times J$, that $(\phi_k, \lambda_k) \in C_R \times J$ is a sequence with $(\phi_k, \lambda_k) \to (\phi, \lambda)$, and that the notation is as in the proof of Lemma 1.3. If $\phi \in U_R$ and $q_2(\phi, \lambda) < \infty$, Lemma 1.3 implies that $q_{1k} \to q_1$ and $q_{2k} \to q_2$. Note also that x_k and x are Lipschitz maps with Lipschitz constant R on the interval $[-M, \infty)$. Using these facts one can prove that

$$\lim_{k\to\infty} \|F(\phi_k, \lambda_k) - F(\phi, \lambda)\| = 0.$$

The straightforward details are left to the reader.

If $\phi \in U_R$ and $q_2(\phi, \lambda) = \infty$ (so $F(\phi, \lambda) = 0$), then by continuous dependence on initial data one can prove as in Lemma 1.3 that for each real $T > 0$, $q_2(\phi_k, \lambda_k) \geqq T$ for all sufficiently large k. But then, for any given $\eta > 0$, Lemma 1.3 implies that for all sufficiently large k,

$$\|F(\phi_k, \lambda_k) - F(\phi, \lambda)\| \leqq \eta.$$

Finally, it remains to consider the case that $\phi \in C_R \setminus U_R$ and $x(t) = 0$ for all $t \geqq -1$. Take $\eta > 0$ and select T such that $\|F(\phi_k, \lambda_k)\| < \eta$ for all k such that $q_2(\phi_k, \lambda_k) \geqq T$. By continuous dependence on initial data, there exists k_0 (dependent on η and T) such that if $k \geqq k_0$, $\sup_{0 \leqq t \leqq T} |x_k(t)| < \eta$. It follows that if $k \geqq k_0$ and $q_{2k} \geqq T$ or $q_{2k} \leqq T$, $\|F(\phi_k, \lambda_k)\| < \eta$.

Thus we have proved continuity of F at (ϕ, λ) in all cases. □

Remark 1.5. In our original approach to this work, we used a modified version of C_R, namely,

$$\tilde{C}_R = \{\phi \in X : -B \leqq \phi(t) \leqq A \text{ for } -M \leqq t \leqq 0,\ \phi(0) = 0,\ \phi(t) \geqq 0 \text{ for } -1 \leqq t \leqq 0 \text{ and } \phi \text{ is Lipschitzian with Lipschitz constant } R\}.$$

We also defined $\tilde{U}_R$ by

$$\tilde{U}_R = \{\phi \in \tilde{C}_R : \phi(t) > 0 \text{ for some } t \in [-1, 0]\}.$$

If we follow standard notation and write x_t for the function $x_t(s) = x(t+s)$, $-M \leqq s \leqq 0$, it is natural to define $\tilde{F}(\phi, \lambda) = x_{q_2}$ if $q_2 = q_2(\phi, \lambda) < \infty$, and to define $\tilde{F}(\phi, \lambda) = 0$ if $q_2(\phi, \lambda) = \infty$ or $\phi \in \tilde{C}_R \setminus \tilde{U}_R$. One can prove that $\tilde{F} \mid \tilde{U}_R \times J$ is continuous, and nonzero fixed points of $\tilde{F}_\lambda$ give SOP solutions of $(1.2)_\lambda$. However, if M is large, $\tilde{F}$ need not be continuous on $\tilde{C}_R$. Originally, we circumvented this difficulty by working with $\tilde{F}$ only on $\tilde{U}_R$, but Lemma 1.2 allows us to use F and avoid some technical difficulties.

We also need a lemma establishing an upper bound on the period of SOP solutions of $(1.2)_\lambda$ for all $\lambda > 0$ and proving that $(1.2)_\lambda$ has no SOP solution for $\lambda > 0$ and λ small. This lemma extends Lemmas 1.4 and 1.5 in [34] to the case that $r(u)$ is not constant.

Lemma 1.5. *Suppose that A and B are positive reals (allowing $A = \infty$ or $B = \infty$) and that $f : (-B, A) \to \mathbb{R}$ is a Lipschitz map with Lipschitz constant K. Assume that $uf(u) < 0$ for all $u \in (-B, A)$, $u \neq 0$ and that $r : (-B, A) \to \mathbb{R}$ is a locally Lipschitzian map with $r(0) = 1$, $r(u) \geqq 0$ for all $u \in (-B, A)$ and*

$$M = \sup\{r(u) : -B < u < A\} < \infty.$$

Assume $x(t)$ is a SOP solution of $(1.2)_\lambda$ for some $\lambda > 0$ and that $-B < x(t) < A$ for all t. If $\delta > 0$ is chosen so that

$$[1 - e^{-\delta M}]K < 1,$$

it follows that $\lambda \geqq \delta$. Furthermore, if p is the minimal period of $x(t)$, there is a number $p_0 = p_0(K, M) \geqq p$. In fact, p must satisfy

$$K^2 \geqq \exp[\delta(\tfrac{1}{2}p - 2M)].$$

Proof. We can assume $x(\bar{q}_1) = 0 = x(0) = x(q_1) = x(q_2)$, $x(t) > 0$ on $(\bar{q}_1, 0)$, $x(t) < 0$ on $(0, q_1)$, and $x(t) > 0$ on (q_1, q_2); so q_2 is the period of $x(t)$. For convenience we write $\eta(s) = s - r(x(s))$. Equation $(1.2)_\lambda$ gives

$$x(t) = e^{-\lambda t} \int_0^t \lambda e^{\lambda s} f(x(\eta(s)))\, ds. \tag{1.9}$$

If we recall (see Lemma 1.2) that $\eta(s) \geqq \bar{q}_1$ for $0 \leqq s \leqq q_1$, that f is Lipschitzian with Lipschitz constant K, and that $uf(u) < 0$ for $u \in (-B, A)$, $u \neq 0$, then we obtain from (1.9) that

$$|x(t)| \leqq K(1 - e^{-\lambda t}) \Big(\sup_{\bar{q}_1 \leqq t \leqq 0} x(t) \Big) \quad \text{for } 0 \leqq t \leqq q_1. \tag{1.10}$$

For convenience we write $\phi = x \mid [\bar{q}_1, 0]$ and let $\|\phi\|$ denote the sup norm of ϕ. Thus (1.10) gives

$$|x(t)| \leqq K(1 - e^{-\lambda t}) \|\phi\|, \quad 0 \leqq t \leqq q_1. \tag{1.11}$$

If $q_1 \leqq M$, then (1.11) implies that

$$\sup_{0 \leqq t \leqq q_1} |x(t)| \leqq K(1 - e^{-\lambda M}) \|\phi\|. \tag{1.12}$$

If $q_1 > M$, we know that $x(t)$ is increasing on $[M, q_1]$, so inequality (1.12) remains valid for $q_1 > M$.

If we let $x \mid [0, q_1]$ take the role of ϕ and apply the same argument, we obtain

$$\sup_{q_1 \leqq t \leqq q_2} x(t) \leqq K(1 - e^{-\lambda M}) \Big(\sup_{0 \leqq t \leqq q_1} |x(t)| \Big) \leqq [K(1 - e^{-\lambda M})]^2 \|\phi\|. \tag{1.13}$$

If $\delta > 0$ is chosen so that

$$K(1 - e^{-\delta M}) < 1, \tag{1.14}$$

then (1.13) implies that if $0 < \lambda \leqq \delta$, $x(t)$ cannot be an SOP solution of $(1.2)_\lambda$.

Next suppose that $x(t)$ is an SOP solution of $(1.2)_\lambda$ (so $\lambda \geqq \delta$) and that $q_2 = 2T$ and $T \geqq 2M$. Note that (1.13) implies that $K > 1$. We must have $q_1 \geqq T$ or $q_2 - q_1 \geqq T$. Assume first that $q_1 \geqq T$. Equation (1.11) gives

$$\sup_{0 \leqq t \leqq M} |x(t)| \leqq K\|\phi\|,$$

and because $x'(t) \geqq -\lambda x(t)$ for $M \leqq t \leqq q_1$, we obtain

$$\begin{aligned} \sup_{q_1 - M \leqq t \leqq q_1} |x(t)| &\leqq \exp(-\lambda(q_1 - 2M)) \, |x(M)| \\ &\leqq \exp(-\lambda(T - 2M)) \, K\|\phi\|. \end{aligned}$$

Thinking of $x \mid [q_1 - M, q_1]$ as ϕ, we obtain as before that

$$\sup_{q_1 \leqq t \leqq q_2} x(t) \leqq K^2 \exp(-\lambda(T - 2M)) \|\phi\| \leqq K^2 \exp(-\delta(T - 2M)) \|\phi\|. \tag{1.15}$$

Inequality (1.15) implies that

$$K^2 \exp(-\delta(T - 2M)) \geqq 1, \tag{1.16}$$

which yields the estimate on the period q_2 in our lemma.

If $q_2 - q_1 \geqq T$, an analogous argument, which we leave to the reader, again yields inequality (1.16). □

For the remainder of our work in this section we shall need some more or less standard tools. We refer the reader to [2, 41, 43–45] for the definition of the fixed-point index and for some results concerning "attractive fixed points" and "ejective fixed points". (See also [34, p. 53] for definitions.)

Standard results concerning linear functional differential equations are summarized in [34, pp. 119–125]. The characteristic equation of

$$x'(t) = -\lambda x(t) - k\lambda x(t-1), \quad \lambda > 0, \quad k > 1, \tag{1.17}$$

is the equation

$$z = -\lambda - k\lambda e^{-z}, \quad \lambda > 0, \quad k > 1. \tag{1.18}$$

Equation (1.18) has a pure imaginary root (for some $\lambda > 0$, $k > 1$) if and only if

$$\lambda = \lambda_m = \frac{\nu_0 + 2\pi m}{\sqrt{k^2 - 1}} = \frac{\nu_m}{\sqrt{k^2 - 1}} \tag{1.19}$$

where ν_0 is the unique solution of

$$\cos(\nu_0) = -\frac{1}{k}, \quad \tfrac{\pi}{2} < \nu_0 < \pi \tag{1.20}$$

and m is a nonnegative integer. If $\lambda = \lambda_m$, (1.18) has only two pure imaginary solutions, namely, $\pm i\nu_m$, and these solutions are of multiplicity 1. (See Proposition A.2 of [34, p. 121].) If $x(t)$ is a periodic solution of (1.17) which is nonnegative on some interval of length greater than or equal to one and not identically zero, Proposition A.2 of [34] and the remarks on p. 120 of [34] imply that $\lambda = \lambda_0$ and

$$x(t) = \alpha \cos(\nu_0 t) + \beta \sin(\nu_0 t)$$

for some real numbers α and β.

Remark 1.6. For the remainder of this section, we shall usually assume that if f and r satisfy H1′ and H2′, then $f(u) = f(A)$ for $u \geqq A = \sup\{|f(u)|: -B \leqq u \leqq 0\}$ and $f(u) = f(-B)$ for $u \leqq -B$ and $r(u) = 0$ for $u \leqq -B$.

If f and r satisfy H1 and H2 or H1′ and H2′ and $0 \leqq \lambda \leqq \Lambda$ and $\Lambda(A+B) \leqq R$, we have defined a map $F: C_R \times (0, \Lambda] \to C_R$. Of course, F depends on f and r and the constant M in Lemma 1.1. For the statement of the following lemma, it is convenient to indicate the dependence on f and r by writing

$$F(\phi, \lambda) = F(\phi, \lambda; r, f). \tag{1.21}$$

For a given function r satisfying H2 or H2′ and for s a real number with $0 \leqq s \leqq 1$ write

$$r_s(u) = (1-s)\,u + sr(u). \tag{1.22}$$

Lemma 1.6. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′ (*note Remark 1.6*) *and that f is in C^1 near* 0 *with*

$$f'(0) = -k < -1.$$

If A and B are as in H1 *and* H2 *or* H1′ *and* H2′ *and $\Lambda > 0$, select R so that $R \geqq \Lambda(A+B)$ and let C_R be given by* (1.6) *and $J = (0, \Lambda]$. Suppose J_0 is a compact set of reals with $\lambda_0 \notin J_0$ (λ_0 as in* (1.19)*). There exists $\gamma > 0$ such that if $F(\phi, \lambda; r_s, f) = \phi$ for some $\phi \in C_R - \{0\}$, $\lambda \in J_0 \cap J$ and $0 \leqq s \leqq 1$ (see* (1.22)*), then*

$$\|\phi\| = \sup_{-M \leqq t \leqq 0} |\phi(t)| \geqq \gamma.$$

Furthermore, if $(\phi_j, \alpha_j, s_j) \in (C_R - \{0\}) \times (0, \Lambda] \times [0, 1]$ is any sequence with $\lim_{j\to\infty} \|\phi_j\| = 0$, and $F(\phi_j, \alpha_j; r_{s_j}, f) = \phi_j$, then $\lim_{j\to\infty} \alpha_j = \lambda_0$. If p_j is the minimal period of the SOP solution $x_j(t)$ corresponding to ϕ_j, α_j, and r_{s_j}, then $\lim_{j\to\infty} p_j = 2\pi/\nu_0$, with ν_0 as in (1.20).

Proof. We assume that the first part of the lemma is false, so there exists a sequence $(\phi_j, \beta_j, s_j) \in (C_R - \{0\}) \times J_0 \times [0, 1]$ such that

$$F(\phi_j, \beta_j; r_{s_j}, f) = \phi_j$$

and $\lim_{j\to\infty} \|\phi_j\| = 0$. By taking a subsequence, we can assume that $\lim_{j\to\infty} \beta_j = \beta \neq \lambda_0$ and $\lim_{j\to\infty} s_j = s$. Lemma 1.5 implies that $\beta > 0$. If $\bar{q}_{1j} = \inf\{t \leqq 0 : \phi_j(\rho) \geqq 0$ for $t \leqq \rho \leqq 0\}$, we know that there is an SOP solution $x_j(t)$ of the equation

$$x_j'(t) = -\beta_j x_j(t) - \beta_j f(x_j(t - r_{s_j}(x_j(t)))),$$
$$x_j \mid [\bar{q}_{1j}, 0] = \phi_j \mid [q_{1j}, 0].$$

If q_{1j} denotes the first $t > 0$ such that $x_j(t) = 0$ and q_{2j} denotes the second such zero, q_{2j} is the period of x_j, and Lemma 1.5 implies that q_{2j} is bounded, say $q_{2j} \leqq Q$ for all j. It follows by continuous dependence on initial data that

$$\|x_j\| \equiv \sup_{0 \leqq t \leqq q_{2j}} |x_j(t)| = \sup_{t \in \mathbb{R}} |x_j(t)| \to 0 \quad \text{as } j \to \infty.$$

By taking a further subsequence we can assume that

$$q_{1j} \to q_1 \quad \text{and} \quad q_{2j} \to q_2.$$

If we define $r_j = r_{s_j}(x_j(t))$ and

$$y_j(t) = x_j(t)\, \|x_j\|^{-1},$$

we easily see (because $f(0) = 0$ and $f'(0) = -k$) that

$$y_j'(t) = -\beta_j y_j(t) - k\beta_j y_j(t - r_j) - \mu_j(t)\, y_j(t - r_j), \tag{1.23}$$

where $\mu_j(t)$ in (1.23) is periodic of period q_{2j} and $\mu_j(t) \to 0$ uniformly in t as $j \to \infty$. Equation (1.23) implies that $|y_j'(t)|$ is uniformly bounded and $|y_j(t)| \leqq 1$ for all t. It follows by using the Ascoli-Arzelà theorem and by taking a further subsequence that we can assume that $\lim_{j\to\infty} y_j(t) = y(t)$ and that the convergence is uniform on a compact interval of reals. Thus $y(t)$ has period q_2, $\|y\| = 1$, $y(t) \leqq 0$ for $0 \leqq t \leqq q_1$ and $y(t) \geqq 0$ on $[q_1, q_2]$. If L_1 is an upper bound for $|y_j'(t)|$ and L_2 a Lipschitz constant for r, it is easy to see that

$$|y_j(t - r_j) - y_j(t - 1)| \leqq L_1 L_2 \|x_j\|. \tag{1.24}$$

Integrating (1.23) from 0 to t, using (1.24), and taking the limit as $j \to \infty$ we obtain

$$y(t) = -\beta \int_0^t y(\rho)\, d\rho - k\beta \int_0^t y(\rho - 1)\, d\rho. \tag{1.25}$$

Equation (1.25) implies that

$$y'(t) = -\beta y(t) - k\beta y(t - 1);$$

since y is nonnegative on an interval $[q_1, q_2]$ with $q_2 - q_1 \geqq 1$, y is periodic, and $\|y\| = 1$, we must have $\beta = \lambda_0$, a contradiction.

If we apply this same argument to the sequence (ϕ_j, α_j, s_j) of the second part of Lemma 1.6, the results of the first part of the lemma imply that $\alpha_j \to \lambda_0$. If $x_j(t)$ is the SOP solution corresponding to (ϕ_j, α_j, s_j), $y_j(t) = x_j(t)\|x_j\|^{-1}$, and p_j is the minimal period of x_j, suppose that p_j does not converge to $2\pi/\nu_0$. Then by taking a subsequence, we can assume that there exists $\delta > 0$ such that $|p_j - 2\pi/\nu_0| \geqq \delta$ for all j. As in the proof above, we can take a further subsequence for which $p_j \to p \neq 2\pi/\nu_0$, and $y_j(t)$ converges uniformly on compact subsets of $\mathbb{R}$ to a C^1 periodic function $y(t)$ which is nonnegative on an interval $[q_1, q_2]$, $q_2 - q_1 \geqq 1$, has minimal period p, and satisfies

$$y'(t) = -\lambda_0 y(t) - k\lambda_0 y(t - 1).$$

However, we have already noted that every nonzero periodic solution of this equation has minimal period $2\pi/\nu_0$, which is a contradiction. □

The whole point of Lemma 1.6 is that it enables us to reduce the problem of computing the fixed-point index of F_λ on a small neighborhood of the origin in C_R to the relatively well-understood case that $r(u) \equiv 1$ is a constant.

Lemma 1.7. *Let the notation and assumptions be as in Lemma 1.6. If λ_0 is as in (1.19), $0 < \lambda \leqq \Lambda$, and $\lambda \neq \lambda_0$, there exists $\rho = \rho(\lambda) > 0$ with $F_\lambda(\phi) \neq \phi$ for $0 < \|\phi\| \leqq \rho$ and $\phi \in C_R$. If ρ is any number with $F_\lambda(\phi) \neq \phi$ for $0 < \|\phi\| \leqq \rho$, $\phi \in C_R$, and $i_{C_R}(F_\lambda, B_\rho)$ denotes the fixed point index of $F_\lambda : B_\rho = \{\phi \in C_R : \|\phi\| < \rho\} \to C_R$, then*

$$n_\lambda = i_{C_R}(F_\lambda, B_\rho) = \begin{cases} 1, & \text{if } 0 < \lambda < \lambda_0, \\ 0, & \text{if } \lambda_0 < \lambda < \Lambda. \end{cases}$$

Proof. The additivity property of the fixed point index implies that n_λ is independent of the particular number ρ as above. For a fixed λ with $0 < \lambda \leqq \Lambda$ and $\lambda \neq \lambda_0$, define a homotopy $\Phi_s : C_R \to C_R$, $0 \leqq s \leqq 1$, by

$$\Phi_s(\phi) = F(\phi, \lambda; r_s, f).$$

Lemma 1.6 implies that there exists $\rho > 0$ such that

$$\Phi_s(\phi) \neq \phi \quad \text{if } 0 < \|\phi\| \leqq \rho,\ \phi \in C_R,\ 0 \leqq s \leqq 1.$$

The homotopy property of the fixed point index therefore implies that

$$i_{C_R}(\Phi_0, B_\rho) = i_{C_R}(\Phi_1, B_\rho) = i_{C_R}(F_\lambda, B_\rho). \tag{1.26}$$

Equation (1.26) reduces Lemma 1.7 to the case that $r(u) = 1$ for all u. However, we have still not reduced to the standard case as in [34] because, when $r(u) \equiv 1$, equation $(1.2)_\lambda$ is not studied in C_R in [34]. To reduce to a standard case, first define D_R by

$$D_R = \{\phi \in C[-1, 0] : \phi(0) = 0,\ 0 \leqq \phi(t) \leqq A \quad \text{for } -1 \leqq t \leqq 0,$$
$$\phi \text{ is Lipschitzian with Lipschitz constant } R\}.$$

Define $\pi : C_R \to D_R$ by $\pi(\phi) = \phi \mid [-1, 0]$ and define $i : D_R \to C_R$ by $i(\phi) = \psi$, where $\psi(t) = \phi(t)$ for $-1 \leqq t \leqq 0$ and $\psi(t) = \phi(-1)$ for $-M \leqq t \leqq -1$. It is easy to see that

$$(\Phi_0 \cdot i)\,(\pi(\phi)) = \Phi_0(\phi) \qquad \text{for all } \phi \in C_R. \tag{1.27}$$

If $V_\rho = \{\phi \in D_R : \|\phi\| < \rho\}$, one obtains from the commutativity property of the fixed-point index and from (1.27) that for all sufficiently small $\rho > 0$,

$$i_{C_R}(\Phi_0, B_\rho) = i_{C_R}(\Phi_0 \cdot i \cdot \pi, B_\rho) = i_{D_R}(\pi \cdot \Phi_0 \cdot i, V_\rho). \tag{1.28}$$

In using the commutativity property to obtain (1.28), it is useful to note that there exists a constant L such that if $(\Phi_0 \cdot i)\,(\pi\phi) = \phi$, then $\|\phi\| \leqq L\|\pi\phi\|$.

If $\theta \in D_R$ and $x(t; \theta) = x(t)$ is the solution of

$$x'(t) = -\lambda x(t) + \lambda f(x(t-1)), \quad t \geqq 0,$$
$$x \mid [-1, 0] = \theta, \tag{1.29}$$

define $q_0 = \sup\{t \geqq 0 : x(s) = 0 \text{ for } 0 \leqq s \leqq t\}$, $q_1 = \inf\{t > q_0 : x(t) = 0\}$, and $q_2 = \inf\{t > q_1 : x(t) = 0\}$. Lemma 1.1 implies that $q_1 - q_0 > 1$ and $q_2 - q_1 > 1$. If we define $\Psi(\theta) = \theta_1$, where

$$\theta_1(t) = x(q_2 + t), \quad -1 \leqq t \leqq 0,$$

it is easy to see that

$$\Psi = \pi \cdot \Phi_0 \cdot i,$$

so we obtain from (1.28) that

$$i_{C_R}(\Phi_0, B_\rho) = i_{D_R}(\Psi, V_\rho).$$

The set D_R is not used in Section 1 of [34], so we define E_R by

$$E_R = \{\phi \in C[0, 1] : \phi(0) = 0,\ e^{\lambda t}\phi(t) \text{ is monotone increasing},$$
$$0 \leqq \phi(t) \leqq A \text{ for } 0 \leqq t \leqq 1,$$
$$\phi \text{ is Lipschitzian with Lipschitz constant } R\}.$$

If $\theta \in D_R$, $x(t)$ is a solution of (1.29) and q_1 and q_2 are as defined in the preceding paragraph, define $G : D_R \to E_R$ by

$$G(\theta) = \phi,\ \phi(t) = x(q_1 + t) \quad \text{for } 0 \leqq t \leqq 1.$$

If $\theta \in E_R$, $\theta \neq 0$ and $y(t) = y(t;\, \theta)$ is the solution of (1.29), let $\tilde{q}_1$ denote the smallest $t > 1$ with $y(t) = 0$, let $\tilde{q}_2$ denote the smallest $t > \tilde{q}_1$ with $y(t) = 0$, and define $H: E_R \to D_R$ by $H(0) = 0$ and

$$H(\theta) = \phi,\ \phi(t) = y(\tilde{q}_1 + t), \quad -1 \leqq t \leqq 0.$$

One can easily verify that H and G are continuous and $HG = \Psi$. Thus if we define $T = GH: E_R \to E_R$ and $W_\rho = \{\phi \in E_R : \|\phi\| < \rho\}$, an application of the commutativity property for the fixed-point index shows that, for all sufficiently small $\rho > 0$,

$$i_{D_R}(\Psi, V_\rho) = i_{E_R}(T, W_\rho).$$

The map T is described directly by

$$T(\phi) = \psi,\ \psi(t) = y(\tilde{q}_2 + t).$$

The set E_R is essentially the same as the set K_λ defined by (1.16) [34, p. 50]. In defining K_λ, the conditions that $\phi(t) \leqq A$ and that ϕ has Lipschitz constant R are not used, but the presence or absence of these conditions does not affect the calculations of the fixed-point index in [34, p. 49–59]. Those calculations show that 0 is an attractive fixed point of T if $0 < \lambda < \lambda_0$ and an ejective fixed point of T if $\lambda_0 < \lambda \leqq \Lambda$. Therefore, we have

$$i_{E_R}(T, W_\rho) = \begin{cases} 1 & \text{if } 0 < \lambda < \lambda_0, \\ 0 & \text{if } \lambda_0 < \lambda \leqq \lambda. \end{cases} \qquad \square$$

With the aid of Lemma 1.7 and a global bifurcation theorem, we can prove our main result concerning the existence of SOP solutions of $(1.2)_\lambda$.

Theorem 1.1. *Let f and r be functions which satisfy conditions* H1 *and* H2 *or conditions* H1′ *and* H2′ *and let A and B be reals as in* H1 *and* H2 *or* H1′ *and* H2′. *Assume that f is in C^1 near 0 and that $f'(0) = -k < -1$; for this k, let λ_0 be given by* (1.19). *If $K > \lambda_0$ and $R \geqq K(A + B)$, let C_R be defined by* (1.6) *and let $F_\lambda : C_R \to C_R$, $0 < \lambda < K$, as in the paragraph following* (1.6). *Then nonzero fixed points of F_λ in C_R are in one-one correspondence with slowly oscillating periodic solutions of* $(1.2)_\lambda$. *There exists $\delta > 0$ such that* $(1.2)_\lambda$ *has no SOP solutions for $0 < \lambda \leqq \delta$. If J denotes the interval $(0, K)$ and S is defined by*

$$S = \{(\phi, \lambda) : \phi \in C_R - \{0\},\ \lambda \in J,\ F_\lambda(\phi) = \phi\} \cup \{(0, \lambda_0)\} \subset C_R \times J,$$

then S is closed in the topological space $C_R \times J$. If S_0 denotes the connected component of S which contains $(0, \lambda_0)$, then, for each $\lambda \in J$ with $\lambda > \lambda_0$, there exists $\phi_\lambda \in C_R - \{0\}$ with $(\phi_\lambda, \lambda) \in S_0$. In particular, for each $\lambda > \lambda_0$, $(1.2)_\lambda$ *has an SOP solution $x_\lambda(t)$ with $-B < x_\lambda(t) < A$ for all t. If a map $p: S \to \mathbb{R}$ is defined (in the notation of Lemma 1.3) by $p(\phi, \lambda) = q_2(\phi, \lambda)$ for $\phi \neq 0$ (so $p(\phi, \lambda)$ is the minimal period of the SOP solution corresponding to $(\phi, \lambda) \in S$, $\phi \neq 0$) and $p(0, \lambda_0) = 2\pi/\nu_0$ (with ν_0 as in* (1.20)*), then p is continuous.*

Proof. The first part of the theorem follows from Lemma 1.4 and 1.5 and our previous remarks. We know that C_R is a compact, convex subset of $X = C([-M, 0])$, and Lemma 1.4 implies that $F: C_R \times J \to C_R$ is continuous, where $F(\phi, \lambda) = F_\lambda(\phi)$. Lemmas 1.6 and 1.7 are precisely the results needed to verify the hypotheses of the global bifurcation result, Theorem 4.1 in [46, p. 91]. In the notation of Theorem 4.1 in [46], $\Lambda = \{\lambda_0\}$, $x_0 = 0$, and $J = (0, K)$, and Theorem 4.1 implies that S_0 is *not* compact in $C_R \times J$. It follows that there does not exist $K_1 < K$ such that $S_0 \subset C_R \times (0, K_1]$; for if this were the case, we would have that $S_0 \subset C_R \times [\delta, K_1]$ and S_0 is closed, which would imply that S_0 is compact. The connectivity of S_0 thus implies that

$$S_0 \cap (C_R - \{0\}) \times \{\lambda\} \text{ is nonempty}$$

for $\lambda_0 < \lambda < K$.

Note that Theorem 1.2 in [44] can also be applied to obtain these results. Theorem 1.2 in [44] requires a map $\Phi: C \times (a, \infty) \to C$, with C a closed convex subset of a Banach space. However, we can reduce to this case by defining $\Phi(\phi, \mu)$ for $\phi \in C_R$, $\mu > 0$, by

$$\Phi(\phi, \mu) = F(\phi, \lambda(\mu)), \quad \lambda(\mu) = K\mu(1 + \mu)^{-1}.$$

If $\lambda > \lambda_0$, we can assume that $K > \lambda$, and the existence of an SOP solution $x_\lambda(t)$ of $(1.2)_\lambda$, with x_λ corresponding to $(\phi_\lambda, \lambda) \in S_0$, follows from the properties of S_0.

Lemma 1.3 shows that $(\phi, \lambda) \to q_2(\phi, \lambda)$ is a continuous map on $S - \{(0, \lambda_0)\}$. Lemma 1.6 shows that if p is extended to S by defining $p(0, \lambda_0) = 2\pi/\nu_0$, then p remains continuous. □

Remark 1.7. Theorem 1.1 strongly suggests that some sort of local Hopf bifurcation theorem should hold for $(1.2)_\lambda$. Indeed, such results are well know in the constant time-lag case: see [4, 40, 50]. However, we know of no such theorems for $(1.2)_\lambda$.

In Section 2 of [34] it is proved that there exists a global continuum of periodic solutions of

$$x'(t) = -\lambda x(t) + \lambda f(x(t-1))$$

bifurcating from $(0, \lambda_m)$, λ_m as in (1.19), $m \in \mathbb{Z}$. However, the argument in [34] does not directly carry over to the case of $(1.2)_\lambda$, and even local bifurcation of periodic solutions from $(0, \lambda_m)$, $m \in \mathbb{Z}$, $m \neq 0$, has not yet been proved.

If f and r satisfy H1 and H2 or H1′ and H2′ and, in addition, f is an odd function and r is an even function, then it is natural to seek special SOP solutions of $(1.2)_\lambda$, so-called "S-solutions" (see [34, p. 59] and [35, 48]). We call a periodic function $x(t)$ with $x(0) = x(q) = 0$, $x(t) < 0$ for $0 < t < q$, $q > 1$, and $x(t + q) = -x(t)$ for all t an S-solution. For f odd and r even, we seek S-solutions of $(1.2)_\lambda$ for $\lambda > \lambda_0$. If $K > 0$, if C_R and U_R are as in (1.6) and (1.7) with $R \geq K(A + B)$, and if f and r are as above, we can define a continuous map T_λ, $0 < \lambda < K$, of C_R to C_R. If $\phi \in C_R$ and $\phi \notin U_R$, we define $T_\lambda(\phi) = 0$. If $\phi \in U_R$ and $q_1(\phi, \lambda) = \infty$ with $q_1(\phi, \lambda)$ as in Lemma 1.1, we define $T_\lambda(\phi) = 0$. If $q_1(\phi, \lambda) < \infty$ and $q_1(\phi, \lambda) - q_0(\phi, \lambda) \geq M$, we define

$T_\lambda(\phi) = \psi$, where

$$\psi(t) = -x(q_1 + t), \qquad -M \leqq t \leqq 0,$$

and $x(t) = x(t; \phi, \lambda)$. If $q_1(\phi, \lambda) - q_0(\phi, \lambda) < M$, define $T_\lambda(\phi) = \psi$ where

$$\psi(t) = \begin{cases} -x(q_1 + t) & \text{for } -(q_1 - q_0) \leqq t \leqq 0, \\ 0 & \text{for } -M \leqq t \leqq -(q_1 - q_0). \end{cases}$$

By using the oddness of f and the evenness of r, it is relatively easy to see that if $T_\lambda(\phi) = \phi$ for some $\phi \in C_R - \{0\}$, then $x \mid [0, \infty)$, where $x(t) = x(t; \phi, \lambda)$, is the restriction to $[0, \infty)$ of an S-solution of $(1.2)_\lambda$. Also, one can see that $T_\lambda^2 = F_\lambda$.

Minor variants of the argument which gives Theorem 1.1 yield the following theorem. We omit the proof, except to note that if f and r satisfy H1 and H2, we can clearly assume $A = B$, while if f and r satisfy H1′ and H2′, we can define

$$\tilde{r}(u) = \begin{cases} 0 & \text{for } |u| \geqq B, \\ r(u) & \text{for } |u| \leqq B, \end{cases}$$

$$\tilde{f}(u) = \begin{cases} f(B) & \text{for } \quad u \geqq B, \\ f(u) & \text{for } |u| \leqq B, \\ f(-B) & \text{for } \quad u \leqq -B, \end{cases}$$

and observe that SOP solutions $x(t)$ of

$$x'(t) = -\lambda x(t) + \lambda \tilde{f}(x(t - \tilde{r}(x(t)))) \tag{1.30$_\lambda$}$$

satisfy $|x(T)| < B$ for all t and satisfy $(1.2)_\lambda$.

Theorem 1.2. *Let notation and assumptions be as in Theorem 1.1 and assume that f is odd and r is even. Assume that $A = B$ if f and r satisfy* H1 *and* H2 *and let B be as in* H2′ *if f and r satisfy* H1′ *and* H2′. *For $0 < \lambda < K$ let $T_\lambda : C_R \to C_R$ be as defined above. Then nonzero fixed points of T_λ in C_R are in one-one correspondence with* S-*solution of* $(1.2)_\lambda$ *for $0 < \lambda < K$; and there exists $\delta > 0$ such that* $(1.2)_\lambda$ *has no* S-*solution for $0 < \lambda < \delta$. If $P \subset C_R \times J$ is defined by*

$$P = \{(\phi, \lambda) : \phi \in C_R - \{0\},\ \lambda \in J,\ T_\lambda(\phi) = \phi\} \cup \{(0, \lambda_0)\},$$

then P is closed in $C_R \times J$. If P_0 denotes the connected component of P which contains $(0, \lambda_0)$, then for each λ with $\lambda_0 < \lambda < K$ there exists $\phi_\lambda \in C_R - \{0\}$ with $(\phi, \lambda) \in P_0$. In particular, for each $\lambda > \lambda_0$, $(1.2)_\lambda$ *has an* S-*solution $x_\lambda(t)$ with $-B < x_\lambda < B$ for all t.*

Remark 1.8. If $r(u)$ is a locally Lipschitzian, even function with $r(0) = 1$, $r(-B) = 0$ for some $B > 0$, and $r(u) > 0$ for $-B < u < B$, we can (as already noted) define $\tilde{r}(u) = r(u)$ for $|u| \leqq B$ and $\tilde{r}(u) = 0$ for $|u| > B$. Then $\tilde{r}$ satisfies H2′ and the S-solution $x(t)$ of $(1.30)_\lambda$, which we obtain by Theorem 1.2, satisfies $|x(t)| < B$ and hence actually satisfies $(1.2)_\lambda$. In view of examples like $r(u) = 1 - cu^{2m}$, $c > 0$ $m \geqq 1$, this trivial observation is important.

2. Monotonicity and regularity properties of solutions

We study here monotonicity and regularity properties of solutions of

$$\varepsilon x'(t) = -x(t) + f(x(t-r)), \quad r = r(x(t)), \quad \varepsilon > 0. \qquad (2.1)_\varepsilon$$

As the results of Section 1 are nonconstructive, we seek information about the shape of SOP solutions. We give an analogue for $(2.1)_\varepsilon$ of "Property M" (compare Section 3 of [34]), which plays a crucial role in the constant-time-lag case and in our further work on $(2.1)_\varepsilon$ (see [38]). We begin the development of a theory of Poincaré-Bendixson type for $(2.1)_\varepsilon$ and give an important application of that theory in Theorem 2.6.

For the reader's convenience we begin by summarizing Lemmas 1.1 and 1.2 from Section 1.

Proposition 2.1. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′, *let $M \geqq \sup\{r(u) : -B \leqq u \leqq A\}$ and for a real number q, let $\phi : [q-M, q] \to [-B, A]$ be a Lipschitzian map. Let $x(t; \phi, \lambda) = x(t)$ be the unique solution of*

$$x'(t) = -\lambda x(t) + \lambda f(x(t-r)), \quad r = r(x(t)),\ \lambda > 0,\ t \geqq q,$$

$$x \,|\, [q-M, q] = \phi.$$

If $x(q) = 0$, then $r(x(t)) > 0$ for all $t \geqq q$. If, in addition, there exists $\bar{q}$, $q - M \leqq \bar{q} \leqq q - 1$, such that $x(\bar{q}) = 0$, then $\eta(t) > \bar{q}$ for all $t > q$, where $\eta(t) = t - r(x(t))$. If ϕ is as in Lemma 1.1 (so $q = 0$, $\phi(0) = 0$, $\phi(t) \geqq 0$ for $-1 \leqq t \leqq 0$ and $\phi(t_0) > 0$ for some $t_0 \in [-1, 0]$), and if $q_j, j \geqq 0$, denote the successive zeros of $x(t)$ as defined in Lemma 1.1, then $q_j - q_{j-1} > 1$ for $j \geqq 1$ and $\eta(t) > q_{j-1}$ for all $t \geqq q_j$.

Proof. Proposition 2.1 is simply Lemmas 1.1 and 1.2 except for the assertion that $r(x(t)) > 0$ for all $t \geqq q$. Assume, by way of contradiction, that $r(x(t)) = 0$ for some $t > q$ and let t_0 be the smallest $t > q$ such that $r(x(t)) = 0$, so $x(t_0) \neq 0$. Define ρ by

$$\rho = \sup\{t < t_0 : x(t) = 0\},$$

so $q \leqq \rho < t_0$. We can assume for definiteness that $x(t) > 0$ for $\rho < t \leqq t_0$. Then we have

$$\varepsilon x'(t_0) = -x(t_0) + f(x(t_0)) < 0.$$

Since $x(\rho) = 0$, it follows that there exists s_0, $\rho < s_0 < t_0$, such that $x(s_0) = x(t_0)$ and, therefore, $r(x(s_0)) = r(x(t_0))$. This contradicts the minimality of t_0. ☐

Suppose that $x(t)$ satisfies $(2.1)_\varepsilon$ for $t \geqq 0$ and $x \,|\, [-M, 0] = \phi$, where ϕ is as in Lemma 1.1. If K denotes the set of critical points of $x(t)$, we want to show that a refinement of the arguments in Lemmas 1.1 and 1.2 and Proposition 2.1 proves that η/K is strictly increasing. (Recall that we say that a real-valued function ψ, defined on a set A of reals, is "strictly increasing on A"

if $\psi(s) < \psi(t)$ for all $s, t \in A$ such that $s < t$; "ψ is increasing on A" if $\psi(s) \leqq \psi(t)$ for all $s, t \in A$ such that $s < t$.) The following lemma is crucial.

Lemma 2.1. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′ *and that A, B, and M denote the usual constants. For a real number q let $\phi: [q - M, q] \to [-B, A]$ be a Lipschitzian map and, for $\varepsilon > 0$, suppose that $x(t)$ satisfies* $(2.1)_\varepsilon$ *for $t \geqq q$ and $x \mid [q - M, q] = \phi$. Assume that α and β are real numbers with $q \leqq \alpha < \beta$, $x'(\alpha) = 0$ and either* (1) *$x'(t) \geqq 0$ for all $t \in [\alpha, \beta]$ or* (2)*$x'(t) \leqq 0$ for all $t \in [\alpha, \beta]$. Then*

$$t - r(x(t)) = \eta(t) > \eta(\alpha) \quad \text{for } \alpha < t \leqq \beta.$$

Proof. Let $\alpha_1 = \sup\{\tau : \alpha \leqq \tau \leqq \beta$ and $x(\tau) = x(\alpha)\}$. For $\alpha \leqq t \leqq \alpha_1$ we have $\eta(t) = t - r(x(\alpha))$, so $\eta(t) > \eta(\alpha)$ for $\alpha \leqq t \leqq \alpha_1$. If $\alpha_1 = \beta$, we are done, so assume that $\alpha_1 < \beta$ and note that $x(t) > x(\alpha_1)$ for $\alpha_1 < t \leqq \beta$ in case 1 and $x(t) < x(\alpha_1)$ for $\alpha_1 < t \leqq \beta$ in case 2. Also, we have that $x'(\alpha_1) = x'(\alpha) = 0$.

If K is a Lipschitz constant for r and $\zeta > 0$ is chosen so that $|x'(t)| < K^{-1}$ for $\alpha_1 \leqq t \leqq \alpha_1 + \zeta$, we see that $\eta(t)$ is strictly increasing on $[\alpha_1, \alpha_1 + \zeta]$. In fact, if $t_1, t_2 \in [\alpha_1, \alpha_1 + \zeta]$ and $t_1 < t_2$ we have that

$$\begin{aligned}\eta(t_2) - \eta(t_1) &\geqq t_2 - t_1 - K|x(t_2) - x(t_1)| \\ &\geqq t_2 - t_1 - K|x'(s)|\,|t_2 - t_1| > 0,\end{aligned}$$

where $s \in [t_1, t_2]$. It follows that if $\eta(t) \leqq \eta(\alpha_1)$ for some $t \in (\alpha_1, \beta]$, then there exists $\delta \in (\alpha_1, \beta]$ such that $\eta(\delta) = \eta(\alpha_1)$. In case 1 we obtain

$$0 \leqq \varepsilon x'(\delta) = -x(\delta) + f(x(\eta(\delta))) < -x(\alpha_1) + f(x(\eta(\alpha_1))) = \varepsilon x'(\alpha_1) = 0,$$

which is a contradiction. Similarly, in case 2 we find

$$0 \geqq \varepsilon x'(\delta) = -x(\delta) + f(x(\eta(\delta))) > -x(\alpha_1) + f(x(\eta(\alpha_1))) = 0,$$

which is again a contradiction. It follows that $\eta(t) > \eta(\alpha_1) \geqq \eta(\alpha)$ for $\alpha_1 < t \leqq \beta$. $\square$

As in Section 1, for given $\varepsilon = \lambda^{-1} > 0$, $q \in \mathbb{R}$, and Lipschitz function $\phi: [q - M, q] \to [-B, A]$, $x(t) = x(t; \phi, \varepsilon^{-1})$ denotes a Lipschitz function which is a solution of $(2.1)_\varepsilon$ for $t \geqq q$ and satisfies $x \mid [q - M, q] = \phi$.

Theorem 2.1. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′. *Let A and B be as in* H1 *and* H2 *or* H1′ *and* H2′ *and assume $M \geqq \sup\{r(u) : -B \leqq u \leqq A\}$. For a given real number q, let $\phi: [q - M, q] \to [-B, A]$ be a Lipschitzian map. For a given $\varepsilon > 0$, let $x(t) = x(t; \phi, \varepsilon^{-1})$. If $K = \{t \geqq q \mid x'(t) = 0\}$ and $\eta(t) = t - r(x(t))$, then $\eta \mid K$ is strictly increasing; if $\alpha \in K$ and $t > \alpha$, then $\eta(t) > \eta(\alpha)$.*

Proof. Define T_* by

$$T_* = \sup\{T \in K : \eta \text{ is strictly increasing on } [q, T] \cap K\}.$$

If K is empty or if $T_* = \infty$, there is nothing to prove, so we can assume that $q \leqq T_* < \infty$. One can easily see that $T_* \in K$ and $\eta \mid [q, T_*] \cap K$ is strictly increasing. If K contains no element $\tau > T_*$, we are done, so we can assume that there exists $\tau > T_*$ such that $x'(\tau) = 0$. The same argument given in the proof of Lemma 2.1 shows that there exists $\zeta > 0$ such that $\eta \mid [T_*, T_* + \zeta]$ is strictly increasing. Thus, if $(T_*, T_* + \zeta]$ contains an element of K, we contradict the maximality of T_*. Thus there is a maximal interval $J = (T_*, T_* + \zeta_1)$, $\zeta_1 > 0$, $\zeta_1 \leqq \tau - T_*$, such that either (a) $x'(t) > 0$ for all $t \in J$ or (b) $x'(t) < 0$ for all $t \in J$. The maximality of J implies that $x'(T_* + \zeta_1) = 0$, so $T_* + \zeta_1 \in K$. However, Lemma 2.1 implies that $\eta(T_* + \zeta_1) > \eta(T_*)$, and this contradicts the maximality of T_*. Thus we have proved that $\eta \mid K$ is strictly increasing.

It remains to prove that if $\alpha \in K$ and $t > \alpha$, then $\eta(t) > \eta(\alpha)$. If $t \in K$, we are done, so assume $x'(t) \neq 0$. Define α_1 to be the infimum of numbers $\tau < t$ such that $x'(s)x'(t) > 0$ for all $s \in [\tau, t]$. Then $\alpha \leqq \alpha_1 < t$ and $x'(\alpha_1) = 0$; so Lemma 2.1 implies that $\eta(\alpha_1) < \eta(t)$. Since we have already proved that $\eta(\alpha) \leqq \eta(\alpha_1)$, the theorem is proved. □

We now wish to investigate Property M for slowly oscillating periodic solutions (or SOP solutions) of $(2.1)_\varepsilon$. For the reader's convenience, we repeat Definition 3.1 from [34, p. 68]. Suppose that $x(t)$ is a periodic function such that $x(0) = x(q_1) = x(q_2) = 0$, $x(t) > 0$ for $0 < t < q_1$, and $x(t) < 0$ for $q_1 < t < q_2$ and $x(t) = x(t + q_2)$ for all t. Select $\rho_1 \in (0, q_1)$ such that

$$x(\rho_1) = \max\{x(t) : 0 \leqq t \leqq q_1\}$$

and select $\rho_2 \in (q_1, q_2)$ such that

$$x(\rho_2) = \min\{x(t) : q_1 \leqq t \leqq q_2\}.$$

If c and d are positive numbers and $x(\rho_1) \geqq c$, define σ_1 to be the least $t \in (0, q_1)$ such that $x(t) = c$ and τ_1 to be the greatest $t \in (0, q_1)$ such that $x(t) = c$. If $x(\rho_1) < c$, define $\sigma_1 = \tau_1 = \rho_1$. Similarly, if $x(\rho_2) \leqq -d$, define σ_2 to be the least $t \in (q_1, q_2)$ such that $x(t) = -d$ and τ_2 to be the greatest $t \in (q_1, q_2)$ such that $x(t) = -d$. If $x(\rho_2) > -d$, define $\sigma_2 = \rho_2 = \tau_2$. With this notation x is said to satisfy property M between $-d$ and c if x is increasing on $[0, \sigma_1]$, $x(t) \geqq c$ for all t such that $\sigma_1 < t < \tau_1$, x is decreasing on $[\tau_1, \sigma_2]$, $x(t) < -d$ for $\sigma_2 < t < \tau_2$ and x is increasing on $[\tau_2, q_2]$.

The definition of Property M can also be extended to any continuous function x with domain $\mathbb{R}$; see [34, Remark 3.4, p. 76].

We are now in a position to generalize Theorem 3.1 in [34, p. 70].

Theorem 2.2. *Let α and β be positive reals and $f : [-\beta, \alpha] \to \mathbb{R}$ be a Lipschitz map such that $xf(x) < 0$ for all $x \in [-\beta, \alpha] - \{0\}$. Let $r : [-\beta, \alpha] \to \mathbb{R}$ be a Lipschitz map such that $r(x) \geqq 0$ for all $x \in [-\beta, \alpha]$ and $r(0) = 1$. Suppose*

further that there are positive numbers $c \leqq \alpha$ *and* $d \leqq \beta$ *and a number* $\gamma > 1$ *such that*
(1) f *is decreasing on* $[-d, c]$;
(2) $f(x) \geqq c$ *if* $-\beta \leqq x \leqq -d$ *and* $f(x) \leqq -d$ *if* $c \leqq x \leqq \alpha$;
(3) $|f(f(x))| = |f^2(x)| \geqq \gamma|x|$ *for all* $x \in [-d, c]$ *such that* $f(x) \in [-\beta, \alpha]$.
Then, if $x(t)$ *is a slowly oscillating periodic solution of* $(2.1)_\varepsilon$ *for some* $\varepsilon > 0$ *and* $-\beta \leqq x(t) \leqq \alpha$ *for all* t, x *satisfies Property* M *between* $-d$ *and* c.

Proof. Let $x(t)$ be an SOP solution of $(2.1)_\varepsilon$ of period q_2 with $-\beta \leqq x(t) \leqq \alpha$ for all t. Suppose $x(0) = 0$, $x(t) > 0$ for $0 < t < q_1$, $x(t) < 0$ for $q_1 < t < q_2$, and $x(t) > 0$ for $q_2 < t < q_3 = q_2 + q_1$. Assume, by way of contradiction, that the theorem is false. Then either (a) there exist numbers t_j, $1 \leqq j \leqq 3$, such that $q_2 < t_1 < t_2 < t_3 < q_3$, $x(t_2) < c$ and $x(t_j) > x(t_2)$ for $j = 1, 3$ or (b) there exist numbers s_j, $1 \leqq j \leqq 3$, such that $q_1 < s_1 < s_2 < s_3 < q_2$, $x(s_2) > -d$ and $x(s_j) < x(s_2)$ for $j = 1, 3$. For definiteness, assume that case (a) holds and let

$$S = \{\tau \in (q_2, q_3) : x(\tau) < c; \text{ there exist } \tau_1, \tau_3 \in (q_2, q_3) \text{ with } \tau_1 < \tau < \tau_3 \text{ and } x(\tau_j) > x(\tau) \text{ for } j = 1, 3\}.$$

Define $\mu = \inf\{x(\tau) \mid \tau \in S\}$ (so that $0 < \mu < c$). Select γ_1, $1 < \gamma_1 < \gamma$, with $\gamma_1\mu < c$ and select $\tau = \tau_2 \in S$ with $x(\tau_2) \leqq \gamma_1\mu$. Let $\tau_1, \tau_3 \in (q_2, q_3)$ be such that $\tau_1 < \tau_2 < \tau_3$ and $x(\tau_j) > x(\tau_2)$ for $j = 1, 3$. We can replace τ_2 by a number $\tau_2^* \in (\tau_1, \tau_3)$ with

$$x(\tau_2^*) = \min\{x(t) : \tau_1 \leqq t \leqq \tau_3\}. \tag{2.2}$$

For a given $\delta < x(\tau_1) - x(\tau_2^*)$, define τ_1^* to be the least number $t \geqq q_2$ with $x(t) = x(\tau_2^*) + \delta$, so $\tau_1^* < \tau_1 < \tau_2^*$, $x'(\tau_1^*) \geqq 0$ and $x(\tau_1^*) > x(\tau_2^*)$. If we define $\eta(t) = t - r(x(t))$, then by taking $\delta > 0$ sufficiently small we can ensure that $\eta(\tau_2^*) - \eta(\tau_1^*) \geqq \tau_2^* - \tau_1 > 0$. Finally, define τ_3^* to be the least number $t \geqq \tau_2^*$ with $x(t) = x(\tau_3)$.

The above construction yields numbers τ_j^*, $1 \leqq j \leqq 3$, with $\tau_j^* \in (q_2, q_3)$, $\tau_1^* < \tau_2^* < \tau_3^*$, $x'(\tau_j^*) \geqq 0$ for $1 \leqq j \leqq 3$ and $x'(\tau_2^*) = 0$, and $x(\tau_j^*) > x(\tau_2^*)$ for $j = 1, 3$. We have already seen that $\eta(\tau_1^*) < \eta(\tau_2^*)$, and Proposition 2.1 and Theorem 2.1 imply that $\eta(\tau_3^*) > \eta(\tau_2^*)$ and $\eta(\tau_j^*) > q_1$ for $j = 1, 2, 3$. Using equation $(2.1)_\varepsilon$ we see that

$$f(x(\eta(\tau_j^*))) \geqq x(\tau_j^*) \quad \text{for } j = 1, 3, \tag{2.3}$$

$$f(x(\eta(\tau_2^*))) = x(\tau_2^*). \tag{2.4}$$

If we define $\sigma_j = \eta(\tau_j^*)$ and use the above inequalities together with properties (1) and (2) of f, we conclude that

$$x(\sigma_j) < x(\sigma_2), \quad j = 1, 3,$$

$$x(\sigma_2) > -d.$$

Furthermore, we already know that

$$q_1 < \sigma_1 < \sigma_2 < \sigma_3 < q_2.$$

Now we argue as in the first step of the proof and find numbers σ_1^*, σ_2^* and σ_3^* with

$$q_1 < \sigma_1^* < \sigma_2^* < \sigma_3^* < q_2,$$

$x(\sigma_j^*) < x(\sigma_2^*)$ for $j = 1, 3$, $x(\sigma_2^*) \geqq x(\sigma_2) > -d$, $x'(\sigma_j^*) \leqq 0$ for $j = 1, 3$ and $x'(\sigma_2^*) = 0$ and $0 < \eta(\sigma_1^*) < \eta(\sigma_2^*) < \eta(\sigma_3^*) < q_1$. Using this information and $(2.1)_\varepsilon$ we find that

$$f(x(\eta(\sigma_j^*))) \leqq x(\sigma_j^*) \quad \text{for } j = 1, 3, \tag{2.5}$$

$$f(x(\eta(\sigma_2^*))) = x(\sigma_2^*). \tag{2.6}$$

If we use (2.5) and (2.6), define $\rho_j = \eta(\sigma_j^*)$ and use the properties of f, we conclude that $x(\rho_2) < c$ and $x(\rho_j) > x(\rho_2)$ for $j = 1, 3$. Since x is periodic of period q_2, we have that $\rho_2 + q_2 \in S$. However, we also know that

$$\begin{aligned} &f(x(\rho_2)) = x(\sigma_2^*) \geqq x(\sigma_2), \text{ so} \\ &f^2(x(\rho_2)) \leqq f(x(\sigma_2)) = x(\tau_2^*) \leqq x(\tau_2). \end{aligned} \tag{2.7}$$

Property 3 of f and (2.7) imply that

$$x(\rho_2 + q_2) \leqq \gamma^{-1} x(\tau_2) \leqq \gamma^{-1}\gamma_1 \mu < \mu, \tag{2.8}$$

which contradicts the definition of μ. □

If one has slightly more information about f, the conclusion of Theorem 2.2 can be sharpened:

Corollary 2.1. *Let assumptions and notation be as in Theorem 2.2. In addition assume that $f'(u)$ exists for $-d < u < c$ and $f'(u) < 0$ for $-d < u < c$. If x is an SOP solution of $(2.1)_\varepsilon$ for some $\varepsilon > 0$ and $\sup_t x(t) \geqq c$, then $x'(t) \neq 0$ for all t such that $0 \leqq x(t) < c$. If $\sup_t x(t) < c$, there is exactly one real number t with $0 < t < q_1$ and $x'(t) = 0$. Similarly, if $\inf_t x(t) \leqq -d$, then $x'(t) \neq 0$ for all t such that $-d < x(t) \leqq 0$; and if $\inf_t x(t) > -d$, there is exactly one real number t with $q_1 < t < q_2$ and $x'(t) = 0$.*

Proof. Let σ_j, ρ_j, and τ_j, $j = 1, 2$, be as given in the definition of Property M. Note that ρ_1 and ρ_2 are, *a priori*, not necessarily uniquely determined. It suffices to prove that $x'(t) > 0$ for $0 \leqq t < \sigma_1$ and $x'(t) < 0$ for $\tau_1 < t \leqq q_1$; the proof is essentially the same for $q_1 \leqq t < \sigma_2$ and for $\tau_2 < t \leqq q_2$. Thus assume, by way of contradiction, that there exists t with $0 \leqq t < \sigma_1$ or $\tau_1 < t \leqq q_1$ and $x'(t) = 0$, and define S by

$$S = \{t \mid x'(t) = 0,\ 0 \leqq t < \sigma_1 \text{ or } \tau_1 < t \leqq q_1\}.$$

Define μ by

$$\mu = \inf\{x(t) \mid t \in S\},$$

and note that $\mu > 0$. If γ_1 is such that $1 < \gamma_1 < \gamma$ (for γ as in Theorem 2.2), select $\tau \in S$ so that

$$x(\tau) \leqq \gamma_1 \mu.$$

We divide the proof into two cases.

Case 1. $x(\rho_1) = \max\{x(t) : 0 \leqq t \leqq q_1\} < c$. We know that $x(t)$ satisfies Property M between $-d$ and c and $\tau \neq \rho_1$, so there exists $\delta > 0$ such that $x \mid (\tau - \delta, \tau + \delta)$ is increasing (if $\tau < \rho_1$) or decreasing (if $\tau > \rho_1$). Since $x'(\tau) = 0$, it follows that $x''(\tau) = 0$ (otherwise $x'(t)$ changes sign on $(\tau - \delta, \tau + \delta)$), and using $(2.1)_\varepsilon$ we conclude that

$$f'(x(\eta(\tau)))\, x'(\eta(\tau))\, \eta'(\tau) = 0, \quad \text{where } \eta(\tau) = \tau - r(x(\tau)). \tag{2.9}$$

Note that because $x'(\tau) = 0$, we know (even though r is only Lipschitzian) that $\eta'(\tau)$ exists, $\eta'(\tau) = 1$ and also

$$x(\tau) = f(x(\eta(\tau))). \tag{2.10}$$

Thus our assumptions on f imply that $-d < x(\eta(\tau)) < 0$ and $f'(x(\eta(\tau))) < 0$. Equation (2.9) now implies that

$$x'(\eta(\tau)) = 0.$$

The fact that $x'(\rho_1) = 0$ implies that

$$x(\rho_1) = f(x(\eta(\rho_1))), \tag{2.11}$$

so $-d < x(\eta(\rho_1)) < 0$ and the properties of f imply

$$x(\eta(\rho_1)) \leqq x(\eta(\tau)). \tag{2.12}$$

If $x(\tau) < x(\rho_1)$, strict inequality must hold in (2.12) and $\eta(\rho_1) \neq \eta(\tau)$. If $x(\tau) = x(\rho_1)$, we find that

$$\eta(\rho_1) - \eta(\tau) = \rho_1 - \tau \neq 0.$$

We conclude that $\eta(\rho_1) \neq \eta(\tau)$ in any case. Equation (2.12) and the fact that x satisfies Property M between $-d$ and c now imply that there exists $\delta^* > 0$ such that either $x \mid (\eta(\tau) - \delta^*, \eta(\tau) + \delta^*)$ is increasing or $x \mid (\eta(\tau) - \delta^*, \eta(\tau) + \delta^*)$ is decreasing.

Since we also know that $x'(\eta(\tau)) = 0$, we can now repeat the above argument to find that

$$x'(\eta^2(\tau)) = 0, \quad f(x(\eta^2(\tau))) = x(\eta(\tau)), \tag{2.13}$$

where $\eta^2(\tau) = \eta(\eta(\tau))$. Equations (2.13) and (2.10) give

$$f^2(x(\eta^2(\tau))) = x(\tau),$$

and we conclude from the properties of f that

$$x(\eta^2(\tau)) \leqq \gamma^{-1} x(\tau). \tag{2.14}$$

Theorem 2.1 and Proposition 2.1 imply that $\eta^2(\tau) + q_2 \in (0, q_1)$ so $\eta^2(\tau) + q_2 \in S$. On the other hand, (2.14) implies

$$x(\eta^2(\tau) + q_2) \leqq \gamma^{-1}x(\tau) \leqq (\gamma^{-1}\gamma_1) \inf\{x(t) \mid t \in S\},$$

which is a contradiction $(\gamma^{-1}\gamma_1 < 1)$.

Case 2. $\max\{x(t) : 0 \leqq t \leqq q_1\} \geqq c$. Essentially the same argument as in case 1 (except that $x(\tau) = x(\rho_1)$ is now impossible) shows that $\eta^2(\tau) + q_2 \in S$ and

$$f^2(x(\eta^2(\tau))) = x(\tau).$$

Just as in case 1, this leads to a contradiction.

Since we have obtained a contradiction in cases 1 and 2, the proof is complete. □

Remark 2.1. Suppose that f and r satisfy H1 and H2 or H1′ and H2′ and that, in addition, f is in C^1 near 0 and $f'(0) = -k < -1$. If A and B are as in H1 and H2 or H1′ and H2′, the results of Section 1 imply that for $0 < \varepsilon < \varepsilon_0$ $(2.1)_\varepsilon$ has an SOP solution $x(t) = x_\varepsilon(t)$ with $-B < x_\varepsilon(t) < A$ for all t, where $\varepsilon_0 = \lambda_0^{-1}$ and λ_0 is given by (1.19). If $f'(u) < 0$ for $-B < u < A$ and $|f^2(u)| = |f(f(u))| > |u|$ for $-B < u < A$, $u \neq 0$, Corollary 2.1 implies that $x_\varepsilon'(t) = 0$ for exactly two values of t on each half-open interval of length equal to the period of x_ε. This result plays an important role in [38].

On the other hand, suppose that f and r are as in Proposition 2.1, that $f'(u) < 0$ for $-B < u < A$ and that $x(t)$ is an SOP solution of $(2.1)_\varepsilon$. If we do *not* assume that $|f(f(u))| \equiv |f^2(u)| > |u|$ for all $u \in (-B, A)$ (so f^2 may have multiple fixed points), then our previous theorems do not imply that $x(t)$ satisfies Property M on $(-B, A)$. (If $r(u) = 1$ for all u, unpublished results of MALLET-PARET & SELL imply that $x(t)$ does satisfy Property M on $(-B, A)$). More generally, if $x(t)$ satisfies $(2.1)_\varepsilon$ for all real t, but $x(t)$ is not assumed periodic, one can ask for information about the zeros of $x'(t)$. The last few results of this paper shed some light on these questions.

First, we need some definitions. We shall say that a continuous function y defined on an interval I "does not change sign on I" if there do not exist points $t_1, t_2 \in I$ such that $y(t_1)\, y(t_2) < 0$. We shall define $N_s(y; I)$, the number of sign changes of y on I, by

$$\begin{aligned} N_s(y; I) = \sup\{k \in \mathbb{N} : &\text{ there exist real numbers } t_j \in I,\ 0 \leqq j \leqq k \\ &\text{such that } t_j < t_{j+1} \text{ for } 0 \leqq j < k, \\ &y(t_j)\, y(t_{j+1}) < 0 \text{ for } 0 \leqq j < k\}. \end{aligned} \tag{2.15}$$

We are interested in counting the zeros of $x'(t)$, where $x(t)$ is a solution of $(2.1)_\varepsilon$ for some $\varepsilon > 0$. Thus assume that f, r, and x are as in Proposition 2.1 and define

$$\eta(t) = t - r(x(t)), \quad t \geqq q - M. \tag{2.16}$$

Define $m_0(t;x) = m_0(t)$ for $t \geqq q$ by

$$m_0(t;x) = \begin{cases} 0 & \text{if } x'(t) \neq 0, \\ 1 & \text{if } x'(t) = 0. \end{cases} \tag{2.17}$$

Also, define $N_0(t;x)$ by

$$N_0(t;x) = \sum_{\eta(t) < s \leqq t} m_0(s;x). \tag{2.18}$$

A priori, $N_0(t;x)$ may be infinite.

If f and r satisfy H1 and H2 or H1′ and H2′ and $x(t)$ is a C^1 function which is defined for all real t, which satisfies $-B < x(t) < A$ for all t, and which is a solution of $(2.1)_\varepsilon$ for all t, define $m(t;x) \equiv m(t)$ by $m(t;x) = 0$ if $x'(t) \neq 0$ and by

$$m(t;x) = \sup\{k+1 : x'(\eta^j(t)) = 0 \text{ for } 0 \leqq j \leqq k\} \tag{2.19}$$

otherwise, where $\eta^j(t)$ denotes the composition of η with itself j times and $\eta^0(t) = t$. If f and r are in C^∞, then x is in C^∞; if $f'(x(t)) \neq 0$ for all t, it is easy to prove that $m(t;x)$ is just the multiplicity of t as a zero of x'. Define $N(t;x)$ by

$$N(t;x) = \sum_{\eta(t) < s \leqq t} m(s;x). \tag{2.20}$$

Obviously, $N(t;x)$ also depends on η, but we do not indicate this dependence. *A priori*, it is possible that $m(t;x) = \infty$ or $N(t;x) = \infty$.

Finally, define $N_s(t;x)$ to be the number of sign changes of $x'(s)$ on $(\eta(t), t]$, i.e., by (2.15),

$$N_s(t;x) = N_s(x'; (\eta(t), t]). \tag{2.21}$$

Our strategy is to prove that the functions $N_0(t;x)$, $N(t;x)$, and $N_s(t;x)$ are decreasing functions on $K = \{t : x'(t) = 0\}$ and to use this fact to obtain more information about solutions of $(2.1)_\varepsilon$. We begin with some lemmas.

Lemma 2.2. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′ *and that A, B, and M denote constants as in Theorem 2.1. For given real numbers q and $\varepsilon > 0$, let $\phi : [q-M, q] \to [-B, A]$ be a Lipschitzian and differentiable function and let $x(t) = x(t; \phi, \varepsilon^{-1})$. Assume that either $\phi'(q) = x'(q)$ (so x has a two-sided derivative at q) or that x does not have a local extremum at q. Assume also that $f'(u)$ exists and $f'(u) < 0$ for all $u \in (-\beta, \alpha)$, where $-\beta \leqq x(t) \leqq \alpha$ for all $t \geqq q - M$. (Note that this condition is automatically satisfied if $f'(u) < 0$ for $-B < u < A$.) If $t_0, t_1 \geqq q$ are numbers with $t_0 < t_1$, $x'(t_0) = 0 = x'(t_1)$ and $x'(t) \leqq 0$ for $t_0 \leqq t \leqq t_1$, then there exists s with $\eta(t_0) < s < \eta(t_1)$, $x'(s) = 0$ and $x(s) = \max\{x(\sigma) : \eta(t_0) \leqq \sigma \leqq \eta(t_1)\}$, where $\eta(t) = t - r(x(t))$. Similarly, if $x'(t_0) = x'(t_1) = 0$ and $x'(t) \geqq 0$ for $t_0 \leqq t \leqq t_1$, there exists s with $\eta(t_0) < s < \eta(t_1)$, $x'(s) = 0$, and $x(s) = \min\{x(\sigma) : \eta(t_0) \leqq \sigma \leqq \eta(t_1)\}$.*

Proof. We claim that there does not exist a $\delta > 0$ with $x(\eta(t)) < x(\eta(t_1))$ for $t_1 - \delta \leqq t < t_1$. If not, take such a $\delta > 0$ and note that

$$f(x(\eta(t))) > f(x(\eta(t_1))) \quad \text{for } t_1 - \delta \leqq t < t_1.$$

The assumption that $x'(t_1) = 0$ implies that

$$x(t_1) = f(x(\eta(t_1))),$$

so that

$$\varepsilon x'(t) = -x(t) + f(x(\eta(t))) > -(x(t) - x(t_1)) \quad \text{for } t_1 - \delta \leqq t < t_1.$$

If we write $u(t) = x(t) - x(t_1)$, $\lambda = \varepsilon^{-1}$, we conclude that

$$\frac{d}{dt}\left(e^{\lambda(t-t_1)}u(t)\right) > 0 \text{ for } t_1 - \delta \leqq t < t_1.$$

It follows that $u(t) < 0$ for $t_1 - \delta \leqq t < t_1$ and

$$\varepsilon x'(t) > -u(t) > 0, \quad t_1 - \delta \leqq t < t_1,$$

which contradicts the assumption that $x'(t) \leqq 0$ for $t_0 \leqq t \leqq t_1$.

The above remarks show that for each $\delta > 0$ there exists t_*, $t_1 - \delta \leqq t_* < t_1$, so that $x(\eta(t_*)) \geqq x(\eta(t_1))$. Since $\eta'(t_1) = 1$, by taking δ small enough we can also ensure that $\eta(t_*) < \eta(t_1)$; Theorem 2.1 guarantees that $\eta(t_*) > \eta(t_0)$. We also know that $x(t_0) \geqq x(t_1)$ and $f(x(\eta(t_j))) = x(t_j)$ for $j = 0, 1$, so the monotonicity of f implies that

$$x(\eta(t_0)) \leqq x(\eta(t_1)) \leqq x(\eta(t_*)).$$

It follows from the above remarks that there exists s with $\eta(t_0) < s < \eta(t_1)$ and

$$x(s) = \max\{x(\sigma) : \eta(t_0) \leqq \sigma \leqq \eta(t_1)\} \geqq x(\eta(t_*)).$$

Of course, if x is differentiable at s, then $x'(s) = 0$. The only value of s for which x may not be differentiable at s is $s = q$. But our assumptions imply that if x is not differentiable at q, then x does not have a local maximum at q, so $s \neq q$.

The proof in the case that $x'(t) \geqq 0$ for $t_0 \leqq t \leqq t_1$ and $x'(t_0) = x'(t_1) = 0$ is analogous and is left to the reader. □

If $x(t)$ is as in Lemma 2.2, it is *a priori* possible that $\{t \geqq q : x'(t) = 0\}$ may have accumulation points. Our next lemma discusses this possibility of accumulation points. First, it is convenient to give a definition.

Definition 2.1. If J is an interval of reals and $f : J \to \mathbb{R}$ is a map, f is called "strictly nondegenerate on J" if for each $u \in J$ there exists $\delta = \delta_u > 0$ such that (1) $f \mid [u, u + \delta] \cap J$ is either strictly increasing or strictly decreasing and (2) $f \mid [u - \delta, u] \cap J$ is either strictly increasing or strictly decreasing.

If J_1 and J_2 are intervals of reals and $g : J_1 \times J_2 \to \mathbb{R}$ is a map, we say that $u \to g(x, u)$ is "strictly nondegenerate on J_2 uniformly on J_1" if, for each

$x \in J_1$, the map $u \to g(x, u)$ is strictly nondegenerate and the number δ_u in Definition 2.1 for the map $v \to g(x, v)$ can be chosen independent of $x \in J_1$ for each $u \in J_2$.

If $f: J \to \mathbb{R}$ is in C^∞ and if for each $u \in J$ there exists an integer $k \geqq 1$ such that $f^{(k)}(u) \neq 0$, then one can see by using Taylor's theorem (with the integral form of the remainder term) that f is strictly nondegenerate on J.

With future applications in mind, we prove our next lemma in greater generality than is immediately necessary for our work here.

Lemma 2.3. *Let J be an interval of reals and $g: J \times J \to \mathbb{R}$ a Lipschitz map such that $u \to g(\xi, u)$ is strictly nondegenerate on J uniformly for $\xi \in J$. Assume that there are nonnegative constants c_1 and c_2 such that for all ξ_1, ξ_2 and $v \in J$,*

$$-c_1(\xi_1 - \xi_2) \leqq g(\xi_1, v) - g(\xi_2, v) \leqq -c_2(\xi_1 - \xi_2). \tag{2.22}$$

(Note that these condition are all met if $f: J \to \mathbb{R}$ is strictly nondegenerate and Lipschitzian and $g(\xi, u) = -\xi + f(u)$). Let $r: J \to [0, \infty)$ be a Lipschitz map. Let $\rho_1 < \rho_2$ be reals, $\theta: [\rho_1, \rho_2] \to \mathbb{R}$ a differentiable function and $x \mid [\rho_1, \infty) \to \mathbb{R}$ a continuous map such that $x(t) \in J$ for all $t \geqq \rho_1$, $x(t) = \theta(t)$ for $\rho_1 \leqq t \leqq \rho_2$, $\eta(t) \equiv t - r(x(t)) \geqq \rho_1$ for all $t \geqq \rho_2$, $x \mid [\rho_2, \infty)$ is continuously differentiable and

$$x'(t) = g(x(t), x(\eta(t))) \quad \text{for } t \geqq \rho_2.$$

Let $K = \{t \geqq \rho_1 : D_+ x(t) = 0\}$ (where $D_+ x(t)$ denotes the right-hand derivative of x at t) and let K_{acc} denote the set of accumulation points of K. If $\tau \in K_{\text{acc}}$ and $\tau > \rho_2$, then it follows that $\eta(\tau) \in K_{\text{acc}}$.

Proof. Let $t_k \neq \tau$ be a sequence of points in K with $\lim_{k\to\infty} t_k = \tau$. By taking a subsequence, we can assume that either $\tau < t_{k+1} < t_k$ for all $k \geqq 1$ or $t_k < t_{k+1} < \tau$ for all k. For definiteness we assume that $\tau < t_{k+1} < t_k$ for all k. Because $\eta'(\tau) = 1$, it is not hard to prove that for some $\delta_1 > 0$ $\eta \mid [\tau - \delta_1, \tau + \delta_1]$ is strictly increasing. (Some caution is necessary since r is only assumed Lipschitzian). Furthermore, by decreasing δ_1 we can assume that $|x(\eta(t)) - x(\eta(\tau))| \leqq \delta$ for $|t - \tau| \leqq \delta_1$, where δ is as in the definition of strict nondegeneracy for $u \to g(\xi, u)$ at $u = x(\eta(\tau))$.

If k_0 is chosen so that $t_k - \tau \leqq \delta_1$ for $k \geqq k_0$, we claim that for each $k \geqq k_0$, there exists s_k with $\tau < s_k < t_k$ and $x'(\eta(s_k)) = 0$. Since $\eta(\tau) < \eta(s_k) < \eta(t_k)$, this will imply that $\eta(\tau) \in K_{\text{acc}}$. To prove the existence of s_k, we suppose not. Rolle's theorem then implies that

(1) There does not exist $t \in (\tau, t_k]$ with $x(\eta(t)) = x(\eta(\tau))$.

(2) There does not exist $t \in [\tau, t_k)$ with $x(\eta(t)) = x(\eta(t_k))$.

It follows that either (a) $x(\eta(\tau)) < x(\eta(t)) < x(\eta(t_k))$ for all $t \in (\tau, t_k)$ or (b) $x(\eta(\tau)) > x(\eta(t)) > x(\eta(t_k))$ for all $t \in (\tau, t_k)$. Because $u \to g(\xi, u)$ is strictly nondegenerate on J uniformly in $\xi \in J$, we conclude that either (α) $g(x(t), x(\eta(t))) > g(x(t), x(\eta(t_k)))$ for all $t \in [\tau, t_k)$ or (β) $g(x(t), x(\eta(t))) < g(x(t), x(\eta(t_k)))$ for all $t \in [\tau, t_k)$.

Assume first that case (α) holds. Writing $u(t) = x(t) - x(t_k)$ and recalling that $x'(t_k) = 0$, we obtain for $\tau \leqq t < t_k$ that

$$u'(t) = g(x(t), x(\eta(t))) > g(u(t) + x(t_k), x(\eta(t_k))) - g(x(t_k), x(\eta(t_k)))$$
$$\geqq -c_1 u(t).$$

It follows that

$$\frac{d}{dt}(\exp(c_1(t - t_k))\, u(t)) > 0 \quad \text{for } \tau \leqq t < t_k.$$

Since $u(t_k) = 0$, this implies that $u(t) < 0$ for $\tau \leqq t < t_k$. Because $c_1 \geqq 0$, we conclude that $u'(t) > 0$ for $\tau \leqq t < t_k$, which contradicts the fact that $u'(\tau) = \lim_{j \to \infty} x'(t_j) = 0$.

Suppose next that case (β) holds. For $\tau \leqq t < t_k$ we obtain

$$u'(t) = g(x(t), x(\eta(t))) < g(x(t), x(\eta(t_k))) - g(x(t_k), x(\eta(t_k)))$$
$$\leqq -c_2 u(t).$$

We conclude that

$$\frac{d}{dt}(\exp(c_2(t - t_k))\, u(t)) < 0 \quad \text{for } \tau \leqq t < t_k.$$

Since $u(t_k) = 0$, we must have that $u(t) > 0$ for $\tau \leqq t < t_k$. It follows that

$$u'(\tau) < -c_2 u(\tau) \leqq 0,$$

which contradicts $u'(\tau) = 0$. □

We also need a result concerning the multiplicity of zeros of $x(t)$.

Lemma 2.4. *Let $J = [-\beta, \alpha]$ be a bounded interval, let $g: J \times J \to \mathbb{R}$ be a Lipschitzian map and suppose that $r: J \to \mathbb{R}$ is a Lipschitzian map with $r(u) \geqq 0$ for $-\beta < u < \alpha$ and $r(u) > 0$ if $g(u, u) = 0$ and $-\beta < u < \alpha$. Assume that $x: \mathbb{R} \to (-\beta, \alpha)$ is a C^1 map with*

$$x'(t) = g(x(t), x(\eta(t))) \quad \textit{for all } t,$$

where $\eta(t) = t - r(x(t))$. If $x(t)$ is not constant and $\eta^j(t)$ denotes the j^{th} iterate of η for $j \geqq 0$, there does not exist $t_0 \in \mathbb{R}$ with $x'(\eta^j(t_0)) = 0$ for all $j \geqq 0$. If, in addition, $u \to g(\xi, u)$ is nondegenerate uniformly for $\xi \in J$ and g satisfies (2.22) *with nonnegative c_1 and c_2 and $K = \{t \in \mathbb{R} : x'(t) = 0\}$, then K has no accumulation points.*

Proof. We suppose that there exists t_0 with $x'(\eta^j(t_0)) = 0$ for all $j \geqq 0$. Define $t_j = \eta^j(t_0)$ for $j \geqq 0$ and define $y_j(t)$ by

$$y_j(t) = x(t_j + t) - x(t_j).$$

Recalling that $x'(t_j) = 0$ and that g is a Lipschitz map we obtain

$$\begin{aligned} |y_j'(t)| &= |g(x(t_j + t), x(\eta(t_j + t))) - g(x(t_j), x(\eta(t_j)))| \\ &\leqq c_3|x(t_j + t) - x(t_j)| + c_3|x(\eta(t_j + t)) - x(\eta(t_j))|, \end{aligned} \tag{2.23}$$

where c_3 depends only on g. Note that we can write

$$\eta(t_j + t) = t_{j+1} + t - \theta_j(t),$$

where

$$\theta_j(t) = r(x(t_j + t)) - r(x(t_j)).$$

If we also recall that x is a Lipschitz map with Lipschitz constant c_4,

$$c_4 = \sup_{u,v \in J} |g(u, v)|,$$

we obtain

$$\begin{aligned} |x(\eta(t_j+t)) - x(\eta(t_j))| &= |x(t_{j+1}+t-\theta_j(t)) - x(t_{j+1}+t) + x(t_{j+1}+t) - x(t_{j+1})| \\ &\leqq c_4|\theta_j(t)| + |y_{j+1}(t)|. \end{aligned} \tag{2.24}$$

Because r is Lipschitzian we obtain from (2.24) that

$$|x(\eta(t_j + t)) - x(\eta(t_j))| \leqq c_5|y_j(t)| + |y_{j+1}(t)|, \tag{2.25}$$

where $c_5 c_4^{-1}$ is the Lipschitz constant for r. Using (2.25) in (2.23) gives

$$|y_j'(t)| \leqq c_6|y_j(t)| + c_6|y_{j+1}(t)|, \tag{2.26}$$

where $c_6 = c_3 + c_3 c_5$. We also know that there is a constant M so that $|y_j(t)| \leqq M$ and $|y_j'(t)| \leqq M$ for all t and all $j \geqq 0$. Using this information we easily show that

$$Y(t) = (y_0(t), y_1(t), \ldots, y_m(t), \ldots) \in l^\infty$$

and that $t \to Y(t)$ is continuous from $\mathbb{R}$ to l^∞. Equation (2.26) gives

$$|y_j(t)| \leqq c_6 \int_0^t |y_j(s)| + |y_{j+1}(s)|\, ds \quad \text{for } t \geqq 0; \tag{2.27}$$

by the continuity of $t \to Y(t)$, it follows from (2.27) that

$$\|Y(t)\|_\infty \leqq (2c_6) \int_0^t \|Y(s)\|_\infty\, ds = 2c_6 R(t). \tag{2.28}$$

If we define $c = 2c_6$, then (2.28) implies that

$$\frac{d}{dt}(e^{-ct}R(t)) \leqq 0,$$

so that

$$R(t) \leqq e^{ct}R(0) = 0 \quad \text{for all } t \geqq 0.$$

Since $R(t) \geqq 0$ for all $t \geqq 0$, we must have $R(t) = 0$ for all $t \geqq 0$ and $x(t_j + t) = x(t_j)$ for all $t \geqq 0$. Because t_j is a decreasing sequence, it follows that $x(t_j) = x(t_0)$ for all $j \geqq 0$. Thus to prove that $x(s) = x(t_0)$ for all real s, it suffices to prove that $\inf_{j \geqq 0} t_j = -\infty$. If not, we must have

$$\lim_{j\to\infty} t_j = \tau > -\infty \quad \text{and} \quad x(\tau) = x(t_0).$$

It follows that $x(t) = x(\tau)$ for all $t \geqq \tau$ and $x'(\tau) = g(x(\tau), x(\eta(\tau))) = 0$. If $r(x(\tau)) = 0$ and if we set $u = x(\tau)$, we obtain that $r(u) = 0$ and $g(u, u) = 0$, which contradicts our assumptions. Therefore we must have that $\eta(\tau) < \tau$. However, this gives

$$\tau = \lim_{j\to\infty} \eta(\tau_j) = \eta(\tau) < \tau,$$

which is again a contradiction.

It remains to prove that if g satisfies the given additional conditions and x is not constant, then K has no accumulation points. However, if τ is an accumulation point of K, Lemma 2.3 implies that $\eta^j(\tau)$ is an accumulation point of K for all $j \geqq 0$, so $\eta^j(\tau) \in K$ for all $j \geqq 0$. This contradicts the first part of Lemma 2.4. □

We are now in a position to obtain more detailed information about the zeros of $x'(t)$, where $x(t)$ satisfies $(2.1)_\varepsilon$. The following theorem generalizes with little difficulty to equations of the form

$$\varepsilon x'(t) = g(x(t), x(\eta(t))),$$

but for simplicity we shall restrict attention to $(2.1)_\varepsilon$.

Theorem 2.3. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′ *and let A, B, and M denote the usual constants. For given reals q and $\varepsilon > 0$, let $\phi: [q-M, q] \to [-B, A]$ be a Lipschitzian, differentiable function and let $x(t) = x(t; \phi, \varepsilon^{-1})$. Assume that $\phi'(q) = x'(q)$ or that x does not have a local extremum at q. Finally, suppose that f is differentiable and that $f'(u) < 0$ for $-\beta \leqq u \leqq \alpha$, where $-\beta \leqq x(t) \leqq \alpha$ for all $t \geqq q - M$. If $K = \{t > q : x'(t) = 0\}$ and $N_0(t; x) = N_0(t)$ is defined by* (2.18), *then $N_0 \mid K$ is a decreasing function. If there exists $\sigma \in k$ with $N_0(\sigma) < \infty$, then there exists $T \geqq \sigma$ such that $x'(\eta(t)) \neq 0$ and $\varepsilon x''(t) = f'(x(\eta(t)))\, x'(\eta(t)) \neq 0$ for all $t \in K$ with $t \geqq T$.*

Proof. If $\sigma, \tau \in K$ and $\sigma < \tau$, we must prove that $N_0(\sigma) \geqq N_0(\tau)$. If $N_0(\sigma) = \infty$, this is immediate, so we assume that $N_0(\sigma) < \infty$. We claim that σ is not an accumulation point of K. If it were (since $N_0(\sigma) < \infty$), there would exist a sequence $s_k \in K$, $s_k > \sigma$, with $\lim s_k = \sigma$. The argument in Lemma 2.3 would then imply that there exists a sequence $\rho_k \in K$, with $\eta(\sigma) < \rho_k$ and $\lim \rho_k = \eta(\sigma)$, which would contradict $N_0(\sigma) < \infty$.

Thus if we define $\sigma_1 = \inf\{t \in K : t > \sigma\}$, then $\sigma_1 > \sigma$. If $\eta(\sigma_1) \geqq \sigma$, then $N_0(\sigma) \geqq N_0(\sigma_1) = 1$. If $\eta(\sigma_1) \leqq \sigma$, Lemma 2.2 implies that

$$N_0(\sigma_1; x) - N_0(\sigma; x) = 1 - \sum_{\eta(\sigma) < s \leqq \eta(\sigma_1)} m_0(s; x) \leqq 0. \tag{2.29}$$

Furthermore, in either case, if $x'(\eta(\sigma_1)) = 0$ (so $\eta(\sigma_1) \leqq \sigma$), Lemma 2.2 implies that

$$N_0(\sigma) > N_0(\sigma_1). \tag{2.30}$$

We can now proceed inductively. For $k \geqq 1$, define

$$\sigma_{k+1} = \inf\{t \in K : t > \sigma_k\}. \tag{2.31}$$

It is a simple argument by induction that $\sigma_k < \sigma_{k+1}$ and

$$N_0(\sigma_k) \geqq N_0(\sigma_{k+1}), \tag{2.32}$$

and strict inequality holds in (2.32) if $\eta(\sigma_k) \in K$.

We claim that σ_k is not bounded above. If σ_k were bounded above, there would exist $\gamma \in K_{\text{acc}}$ with $\sigma_k < \gamma$ for all k and $\lim_{k\to\infty} \sigma_k = \gamma$. Lemma 2.3 implies that $\eta(\gamma) \in K_{\text{acc}}$, and Theorem 2.1 implies that $\eta(\sigma) < \eta(\gamma) < \gamma$. By our construction, there are only finitely many elements of K contained in any compact subset of $(\eta(\sigma), \gamma)$, which contradicts the fact that $\eta(\gamma) \in K_{\text{acc}}$.

It follows that

$$\{t \in K : t > \sigma\} = \{\sigma_k : k \geqq 1\},$$

so there exists $k \geqq 1$ with $\tau = \sigma_k$ and

$$N_0(\tau) = N_0(\sigma_k) \leqq N_0(\sigma).$$

Because strict inequality holds in (2.32) for all k with $\eta(\sigma_k) \in K$ and because $N_0(\sigma) < \infty$, it must be true that for all sufficiently large k, $\eta(\sigma_k) \notin K$. It follows that there exists T with $x'(\eta(t)) \neq 0$ for all $t \in K$ with $t \geqq T$. Since $\eta'(t) = 1$ for $t \in K$, we conclude from $(2.1)_\varepsilon$ that

$$\varepsilon x''(t) = f'(x(\eta(t)))\, x'(\eta(t)) \neq 0. \quad \square$$

Corollary 2.2. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′ *and that f is strictly nondegenerate on $[-B, A]$ (see Definition 2.1). Suppose that $x : \mathbb{R} \to [-B, A]$ is a C^1 function which satisfies $(2.1)_\varepsilon$ for all $t \in \mathbb{R}$ and is not identically zero. If $K = \{t \in \mathbb{R} : x'(t) = 0\}$, then K has no accumulation points and there does not exist $t \in K$ with $x'(\eta^j(t)) = 0$ for all $j \geqq 0$. (Here $\eta(t) = t - r(x(t))$ and η^j denotes the composition of η with itself j times.) If $-\beta \leqq x(t) \leqq \alpha$ for all t, f is differentiable on $[-\beta, \alpha]$ and $f'(u) < 0$ for $-\beta \leqq u \leqq \alpha$, then there exists T with $x''(t) \neq 0$ for all $t \in K$, $t \geqq T$. If, in addition, $x(t)$ is periodic and $x'(t) = 0$, it follows that $x''(t) \neq 0$.*

Proof. The first part of Corollary 2.2 is a special case of Lemma 2.4. Since K has no accumulation points, it follows, in the notation of Theorem 2.3, that $N_0(t) < \infty$ for all $t \in K$. The final statements of Corollary 2.2 now follow from Theorem 2.3. $\square$

The same techniques used to prove Corollary 2.2 also provide information about $\tilde{K} = \{t : x(t) = 0\}$ and show that there exists T with $x'(t) \neq 0$ for all $t \in \tilde{K}$, $t \geqq T$.

Corollary 2.3. *Assume that f and r satisfy* H1 *and* H2 *or* H1′ *and* H2′ *and that f is strictly nondegenerate on $[-B, A]$. Suppose that $x: \mathbb{R} \to [-B, A]$ is a C^1 function which satisfies $(2.1)_\varepsilon$ for all $t \in \mathbb{R}$ and is not identically zero. If $\tilde{K} = \{t \in \mathbb{R}: x(t) = 0\}$, $\tilde{K}$ has no accumulation points. If, for $t \in \tilde{K}$, $v_0(t; x) = v_0(t)$ denotes the number of zeros of $x(t)$ on the interval $(t-1, t]$ (not counting multiplicity of zeros), then $v_0 \mid \tilde{K}$ is a decreasing function. There exist T and an odd integer $m \geqq 1$ such that if $x(t) = 0$ and $t \geqq T$, then $x'(t) \neq 0$ and $v_0(t) = m$. In particular, if x is periodic, $v_0(t) = m$ and $x'(t) \neq 0$ for all $t \in \tilde{K}$.*

Proof. Corollary 2.2 implies that $K = \{t: x'(t) = 0\}$ has no accumulation points, so given any $t \in \mathbb{R}$ there exists $\delta > 0$ with $x'(s) \neq 0$ for $0 < |t-s| < \delta$. This implies that $\tilde{K}$ has no accumulation points. To prove that $v_0 \mid \tilde{K}$ is decreasing, it thus suffices to prove that if $x(q_1) = x(q_2) = 0$ and $x(t) \neq 0$ for $q_1 < t < q_2$, then $v_0(q_2) \leqq v_0(q_1)$. For definiteness we can assume that $x(t) > 0$ for $q_1 < t < q_2$. We know then that there exists $\delta > 0$ with $x'(t) > 0$ for $q_1 < t < q_1 + \delta$, so $(2.1)_\varepsilon$ gives

$$f(x(\eta(t))) > x(t) > 0, \quad x(\eta(t)) < 0, \quad q_1 < t < q_1 + \delta.$$

If $x'(q_1) > 0$, we also conclude from $(2.1)_\varepsilon$ that

$$f(x(q_1 - 1)) > 0, \quad x(q_1 - 1) < 0.$$

It follows, when $x'(q_1) > 0$, that there exists $\delta_1 > 0$ with

$$x(s) < 0 \quad \text{for } q_1 - 1 < s < q_1 - 1 + \delta_1.$$

However, if $x'(q_1) = 0$, then $\eta'(q_1) = 1$ and we deduce from the fact that $x(\eta(t)) < 0$ for $q_1 < t < q_1 + \delta$ that there exists $\delta_1 > 0$ for which we have

$$x(s) < 0 \quad \text{for } q_1 - 1 < s < q_1 - 1 + \delta_1.$$

If $x'(q_2) < 0$, the same reasoning, using $(2.1)_\varepsilon$, shows that $x(q_2 - 1) > 0$. Thus, in this case we deduce from the intermediate value theorem that $x(\tau) = 0$ for some τ with $q_1 - 1 < \tau < q_2 - 1$. This implies that $v_0(q_2) \leqq v_0(q_1)$ when $x'(q_2) \neq 0$.

If $x'(q_2) = 0$, we have $x(q_2 - 1) = 0$. Because η is strictly increasing on some open neighborhood of q_2 and because $q_2 - 1$ is an isolated zero of x, there exists $\delta > 0$ with $x(\eta(t)) \neq 0$ for $q_2 - \delta \leqq t < q_2$. We claim that $x(\eta(t)) > 0$ for $q_2 - \delta \leqq t < q_2$. Writing $\lambda = \varepsilon^{-1}$, we obtain from $(2.1)_\varepsilon$ that

$$-e^{\lambda(t-q_2)} x(t) = \int_t^{q_2} \lambda e^{\lambda(s-q_2)} f(x(\eta(s)))\, ds.$$

If $x(\eta(s)) \leqq 0$ for $q_2 - \delta \leqq t < q_2$, we deduce that the left-hand side of the equation is negative and the right-hand side is greater than or equal to zero for $q_2 - \delta \leqq t < q_2$, a contradiction. We conclude that if $x'(q_2) = 0$, there exists $\delta_1 > 0$ with $x(s) > 0$ for $q_2 - 1 - \delta_1 < s < q_2 - 1$ and $x(s) < 0$ for $q_1 - 1 < s < q_1 - 1 + \delta_1$. The intermediate value theorem implies that $x(\tau) = 0$ for some τ with $q_1 - 1 < \tau < q_2 - 1$. Since $x(q_2 - 1) = 0$ also, we have $v_0(q_2) < v_0(q_1)$ in this case.

Since $v_0 \mid \tilde{K}$ is a decreasing integer-valued positive function, there exists an integer $m \geqq 1$ and a number T_1 so that $\nu_0(t) = m$ for $t \in \tilde{K}$, $t \geqq T_1$. Because $\nu_0(q_1) > \nu_0(q_2)$ whenever q_1 and q_2 are consecutive zeros of x and $x'(q_2) = 0$, there exists T_2 with $x'(t) \neq 0$ for all $t \in \tilde{K}$, $t \geqq T_2$. If $t = \max(T_1, T_2)$ and $x(q) = 0$ for some $q \geqq T$ and $x(t)$ is positive on $(q - \delta, q]$ for some $\delta > 0$, it follows that

$$0 > x'(q) = f(x(q-1)),$$

so $x(q-1) > 0$. However, it is easy to see that $x(q-1) > 0$ implies that m is odd. A similar argument applies if $x(t) < 0$ on $(q - \delta, q]$ for some $\delta > 0$. □

Our next theorem follows by arguments analogous to those used in Theorem 2.4. The usefulness of Theorem 2.4 is that $N_s(t; x)$ may be finite when $N_0(t; x)$ is infinite. For reasons of length, we omit the proof.

Theorem 2.4. *Let notation and assumptions be as in Theorem 2.3. If $N_s(t; x)$ is defined by* (2.15) *and* (2.21), *then $N_s(t; x)$ is a decreasing function of t for $t \in K$. Furthermore, if r is in C^1 on $[-B, A]$ and if $N_s(t_0; x) < \infty$ for some $t_0 \in K$, there exists T with $x''(t) \neq 0$ for all $t \in K$, $t \geqq T$.*

Our results up to this point have used only crude assumptions about the function r. If we assume more about r, we obtain considerably more information about solutions $x(t)$ of $(2.1)_\varepsilon$ and about $\eta(t) = t - r(x(t))$:

Theorem 2.5. *Let q and $M > 0$ be given real numbers and α and β real numbers with $-\beta < \alpha$. Assume that $\phi : [q - M, q] \to [-\beta, \alpha]$ is a Lipschitz function, $f : [-\beta, \alpha] \to \mathbb{R}$ is a Lipschitz function, $r : [-\beta, \alpha] \to [0, \infty)$ is a C^2 function and $\lambda > 0$. Suppose that $x : [q - M, \infty) \to [-\beta, \alpha]$ is a Lipschitz function such that $\eta(t) = t - r(x(t)) \geqq q - M$ for all $t \geqq q$, $x \mid [q - M, q] = \phi$, $x \mid [q, \infty)$ is in C^1 and, for all $t \geqq q$,*

$$x'(t) = -\lambda x(t) + \lambda f(x(t - r)), \quad r = r(x(t)).$$

Assume that there is a constant D with

$$r''(u) \leqq D(r'(u))^2 \quad \text{for} \quad -\beta \leqq u \leqq \alpha. \tag{2.33}$$

If $\rho \geqq q$ and $\eta'(\rho) > 0$ and if $\lambda > D$, it follows that $\eta'(t) > 0$ for all $t \geqq \rho$.

Note that if $x'(\rho) = 0$ for some $\rho \geqq q$, $\eta'(\rho) = 1 > 0$.

Proof. Assume the theorem is false and define

$$t_1 = \inf\{t > \rho : \eta'(t) = 0\}.$$

At $t = t_1$, $f(x(\eta(t)))$ is differentiable with derivative zero (even though f is only Lipschitzian), so

$$\varepsilon x''(t_1) = -x'(t_1). \tag{2.34}$$

It follows that $\eta''(t_1)$ exists and

$$\eta''(t_1) = -r''(x(t_1))\,(x'(t_1))^2 - r'(x(t_1))\,x''(t_1). \tag{2.35}$$

Because $\eta'(t_1) = 0$, we have

$$1 = r'(x(t_1))\, x'(t_1), \tag{2.36}$$

and using (2.34) and (2.36) in (2.35) gives

$$\eta''(t_1) = -r''(x(t_1))\,(r'(x(t_1)))^{-2} + \lambda. \tag{2.37}$$

The definition of t_1 implies that $\eta''(t_1) \leqq 0$, so it follows from (2.37) and (2.36) that $r'(u) \neq 0$ and

$$r''(u) \geqq \lambda (r'(u))^2, \quad u = x(t_1). \tag{2.38}$$

If $\lambda > D$, (2.38) contradicts (2.33). □

It is unclear at present what conditions on f and r are optimal to ensure that $\eta'(t) > 0$ for all t. However, if $r(u) = 1 + cu^2$ and $c > 0$ (so (2.33) fails) and if f satisfies H1, numerical studies suggest that SOP solutions $x(t)$ of $(2.1)_\varepsilon$ fail to satisfy the condition that $\eta'(t) > 0$ for all t, no matter how small ε is.

If $r'(u) \neq 0$ for all $u \in [-B, A]$, r is in C^2 on $[-B, A]$, and f and r satisfy H1 and H2 or H1′ and H2′, then Theorem 2.5 implies that there exists $\varepsilon_1 > 0$ $\left(r''(u) \leqq \varepsilon_1^{-1}(r'(u))^2 \text{ for } u \in [-B, A]\right)$ such that if $0 < \varepsilon < \varepsilon_1$ and if $x(t)$ is an SOP solution of $(2.1)_\varepsilon$, then

$$r(x(t)) < t + r(x(0)) \quad \text{for } t > 0,$$

$$r(x(t)) > t + r(x(0)) \quad \text{for } t < 0.$$

If, for example, $r(u) = 1 + cu$ and $x(t)$ is any SOP solution of $(2.1)_\varepsilon$ with $x(0) = 0$, then

$$cx(t) < t \text{ for } t > 0, \qquad cx(t) > t \quad \text{for } t < 0.$$

As we remarked earlier, even if $f(u)$ is strictly decreasing on an interval $(-\beta, \alpha)$ which contains the range of an SOP solution $x(t)$ of $(2.1)_\varepsilon$, Theorem 2.2 does not necessarily imply that $x(t)$ satisfies Property M on $(-\beta, \alpha)$. However, if, as in Theorem 2.5, we know that $\eta'(t) > 0$ for all t, where $\eta(t) = t - r(x(t))$, then we can prove that $x(t)$ satisfies Property M on $(-\beta, \alpha)$. The following lemma will play a crucial role.

Lemma 2.5. *Let α and β be positive reals and suppose that f and r are C^1 maps from $(-\beta, \alpha)$ to $\mathbb{R}$ with $f(0) = 0$, $r(0) = 1$, and $f'(u) < 0$ and $r(u) \geqq 0$ for all $u \in (-\beta, \alpha)$. Assume that $\varepsilon > 0$, that $x(t)$ is a periodic solution of $(2.1)_\varepsilon$ with $-\beta < x(t) < \alpha$ for all t and that $x(t)$ has minimal period p. If $\eta(t) = t - r(x(t))$, assume that $\eta'(t) > 0$ for all t. (See Theorem 2.5 for conditions which ensure that $\eta'(t) > 0$ for all t.) Define $w_\lambda(t)$ by*

$$w_\lambda(t) = \lambda^{-1}[x(t) - x(t-\lambda)], \quad \lambda \neq 0,$$

$$w_0(t) = x'(t).$$

Then, if $0 \leqq \lambda < p$ and $w_\lambda(t) = 0$, it follows that $w_\lambda'(t) \neq 0$.

Proof. Define $w(t, \lambda) = w_\lambda(t)$ and note that w is a C^1 function of (t, λ). We can extend f and r so that they satisfy H1 and H2. Thus Theorem 2.3 and Corollary 2.2 imply that if $w_0(t) = x'(t) = 0$, then $x''(t) \neq 0$. By the continuity of $x''(t)$ and the periodicity of $x(t)$, there exists $\delta > 0$ such that if $x'(t) = 0$ for any $t \in \mathbb{R}$ and $|s - t| \leqq \delta$, then $x''(s) \neq 0$. If $0 < \lambda \leqq \delta$ and $w_\lambda(t) = 0$, we claim that $w'_\lambda(t) \neq 0$. To see this, observe that if $w_\lambda(t) = 0$, the mean value theorem implies that there exists t_1, with $t - \lambda < t_1 < t$, such that $x'(t_1) = 0$, so $x''(s) \neq 0$ for $t - \lambda \leqq s \leqq t$. On the other hand, if $w'_\lambda(t) = 0$, so

$$x'(t) = x'(t - \lambda),$$

then the mean value theorem implies that there exists s, $t - \lambda < s < t$, with $x''(s) = 0$, which is a contradiction.

Motivated by the above observation, we define λ_0 by

$$\lambda_0 = \sup\{\lambda : 0 \leqq \lambda < p;\ \text{if } w_\mu(t) = 0 \text{ for } 0 \leqq \mu \leqq \lambda \text{ and } t \in \mathbb{R}, \text{ then } w'_\mu(t) \neq 0\}.$$

We must prove that $\lambda_0 = p$. We shall assume that $\lambda_0 < p$ and obtain a contradiction. Select $\tau_1 \in \mathbb{R}$ with

$$x(\tau_1) = \max\{x(t) : t \in \mathbb{R}\}.$$

Let $\{\tau_1, \tau_2, \ldots, \tau_m\}$ be $\{t : t \in [\tau_1, \tau_1 + p), x'(t) = 0\}$ and assume $\tau_j < \tau_{j+1}$ for $1 \leqq j < m$. Since $x'(t)$ changes sign at each τ_j, m must be an even integer. If $0 < \lambda < \lambda_0$ and $w(t, \lambda) = 0$ for some $t \in [\tau_1, \tau_1 + p]$, note that $t \neq \tau_1$ and $t \neq \tau_1 + p$. For if $t = \tau_1$ and $w_\lambda(t) = 0$, the definition of τ_1 implies that $x'(\tau_1) = 0 = x'(\tau_1 - \lambda)$ and $w'_\lambda(\tau_1) = 0$, contrary to our assumptions. If $0 < \lambda_1 < \lambda_0$, the implicit function theorem implies that there exist positive numbers ρ and δ such that if $w(\bar{t}, \bar{\lambda}) = 0$ for some $\bar{t} \in \mathbb{R}$ and $\bar{\lambda} \in [0, \lambda_1]$, then for $|\mu - \bar{\lambda}| < \delta$, $\mu \in [0, \lambda_1]$, there exists a unique solution $t = t(\mu)$ of the equation $w(t, \mu) = 0$ with $|t - \bar{t}| < \rho$. The map $\mu \to t(\mu)$, defined for $|\mu - \bar{\lambda}| < \delta$, $\mu \in [0, \lambda_1]$, is continuously differentiable.

Using the above remarks, we see that there exist C^1 functions $t_j(\lambda)$, $1 \leqq j \leqq m$, defined for $0 \leqq \lambda < \lambda_0$, such that $t_j(0) = \tau_j$ and $\tau_1 < t_j(\lambda) < \tau_1 + p$ for $0 < \lambda < \lambda_0$. Notice also that we must have $t_j(\lambda) \neq t_k(\lambda)$ for $1 \leqq j < k \leqq m$ and $0 \leqq \lambda < \lambda_0$, or we would contradict the local uniqueness of solutions of $w(t, \lambda) = 0$ given by the implicit function theorem. It follows that we must have $t_j(\lambda) < t_{j+1}(\lambda)$ for $1 \leqq j < m$ and $0 \leqq \lambda < \lambda_0$. Finally, if $w(\sigma, \lambda_1) = 0$ for some $\sigma \in (\tau_1, \tau_1 + p)$ and $0 < \lambda_1 < \lambda_0$, the same argument using the implicit function theorem shows that there is a C^1 function $t(\lambda)$, defined for $0 \leqq \lambda \leqq \lambda_1$, with $t(\lambda_1) = \sigma$ and $w(t(\lambda), \lambda) = 0$ for $0 \leqq \lambda \leqq \lambda_1$. If $\sigma \neq t_j(\lambda_1)$ for some j, $1 \leqq j \leqq m$, we argue as above that $t(\lambda) \neq t_j(\lambda)$ for $1 \leqq j \leqq m$ and $0 \leqq \lambda \leqq \lambda_1$ and $t(\lambda) \in (\tau_1, \tau_1 + p)$ for $0 < \lambda \leqq \lambda_1$. It follows that $t(0) \in [\tau_1, \tau_1 + p]$. However, one can argue that $t(0) \neq \tau_1 + p$, because if $t(0) = \tau_1 + p$, then $t'(0) > 0$ and $t(\lambda) > \tau_1 + p$ for $\lambda > 0$. Thus $t(0) \in [\tau_1, \tau_1 + p)$ and $t(0) = \tau_j$ for some j, $1 \leqq j \leqq m$, contradicting our previous assertions.

It follows that the zeros of $w_\lambda(t)$ on the interval $[\tau_1, \tau_1 + p)$ are precisely the numbers $t_j(\lambda)$, $1 \leqq j \leqq m$. Because $w'_\lambda(t) \neq 0$ if $w_\lambda(t) = 0$, $w_\lambda(t)$ changes sign at each of these zeros. Thus, the sign of $w_\lambda(t)$ on $(t_k(\lambda), t_{k+1}(\lambda))$ is constant for $0 \leqq \lambda < \lambda_0$. If λ_k is a sequence of reals approaching λ_0 from the left, we can, by choosing a subsequence, assume that $\lim_{j\to\infty} t_j(\lambda_k) = \sigma_j$, $1 \leqq j \leqq m$, and $\tau_1 \leqq \sigma_j \leqq \sigma_{j+1} \leqq \tau_1 + p$ for $1 \leqq j \leqq m-1$. If $v(t) = w_{\lambda_0}(t)$, we know that $w_{\lambda_k}(t)$ converges uniformly to $v(t)$ on $\mathbb{R}$. We conclude by a simple limiting argument that if $\sigma_j < \sigma_{j+1}$, then (i) $v(t) \geqq 0$ for $\sigma_j \leqq t \leqq \sigma_{j+1}$ if $w_\lambda(t) > 0$ on $(t_j(\lambda), t_{j+1}(\lambda))$ for some λ (and hence all λ) with $0 < \lambda < \lambda_0$ and (ii) $v(t) \leqq 0$ for $\sigma_j \leqq t \leqq \sigma_{j+1}$ if $w_\lambda(t) < 0$ on $(t_j(\lambda), t_{j+1}(\lambda))$ for some $\lambda \in (0, \lambda_0)$. A similar remark applies to the intervals $[\tau_1, \sigma_1]$ and $[\sigma_m, \tau_1 + p]$.

It remains to prove that if $v(t) = 0$, then $v'(t) \neq 0$. Notice that our observations above prove that $N_s(v; J) < \infty$ for any finite interval J and for N_s defined by (2.15). The defining equation for $x(t)$ shows that $v(t) = \lambda_0^{-1}[x(t) - x(t-\lambda_0)]$ satisfies

$$\varepsilon v'(t) = -v(t) + \lambda_0^{-1}[f(x(\eta(t))) - f(x(\eta(t-\lambda_0)))]. \tag{2.39}$$

Define $g(u) = f(x(u))$ and define continuous functions $F(u_1, u_2)$, $G(u_1, u_2)$ and $R(u_1, u_2)$ on $\mathbb{R}\times\mathbb{R}$ by

$$H(u_1, u_2) = \begin{cases} \dfrac{h(u_1) - h(u_2)}{u_1 - u_2} & \text{for } u_1 \neq u_2, \\ h'(u) & \text{for } u_1 = u_2 = u, \end{cases}$$

where $h = f$ if $H = F$, $h = g$ if $H = G$, and $h = r$ if $H = R$. An easy calculation shows

$$\begin{aligned} \varepsilon v'(t) &= -v(t) - G(\eta(t) - \lambda_0, \eta(t-\lambda_0))\, R(x(t-\lambda_0), x(t))\, v(t) \\ &\quad + F(x(\eta(t)), x(\eta(t) - \lambda_0))\, v(\eta(t)) \\ &= -a(t)\, v(t) - b(t)\, v(\eta(t)). \end{aligned} \tag{2.40}$$

The functions $v(t)$, $a(t)$, and $b(t)$ in (2.40) are continuous and periodic of period p, and $b(t) > 0$ for all t (because $f'(u) < 0$ for $-\beta < u < \alpha$).

For each $t \in K = \{t : v(t) = 0\}$, we can let J_t be the largest interval containing t with $v(s) = 0$ for all $s \in J_t$. We assert that there exists a finite collection S of real numbers $t \in [\tau_1, \tau_1 + p]$ such that

$$K \cap [\tau_1, \tau_1 + p] = \bigcup_{t\in S} J_t \cap [\tau_1, \tau_1 + p]. \tag{2.41}$$

In order to prove this, select $t_0 \in [\tau_1, \tau_1 + p]$ with $v(t_0) \neq 0$. Such a point exists because $0 < \lambda_0 < p$. For definiteness, we can assume that $v(t) > 0$ and let $J = [c, d]$ denote the largest interval of reals containing t_0 such that $v(t) \geqq 0$ for all $t \in J$. In order to establish (2.41) it suffices to prove that $K \cap J$ can be decomposed into a finite union of intervals of the form $J_t \cap J$, $t \in K \cap J$.

We first claim that c and d are finite. To see this, observe that if $A'(t) = a(t)$ for all t, then (2.40) gives

$$\frac{d}{dt}\left(\exp(\varepsilon^{-1}A(t))\, v(t)\right) = -\varepsilon^{-1}\exp(\varepsilon^{-1}A(t))\, b(t)\, v(\eta(t)). \tag{2.42}$$

If $c = -\infty$, we derive from (2.42) that $\exp(\varepsilon^{-1}A(t))\, v(t)$ is decreasing on $(-\infty, d]$, so if $v(t_1) = 0$ for any $t_1 < d$, $v(t) = 0$ for all $t \in [t_1, d]$. However,

$v(t_1 - np) = 0$ for all positive integers n, so $v(t) = 0$ for all $t \in (-\infty, d]$ and hence, by periodicity of v, for all t. This proves $c > -\infty$ and a similar argument gives $d < \infty$.

Now let

$$U = \{t \in [c, d] : v(t) > 0\} \subset (c, d),$$

so U is an open set. It follows that U can be written as a countable union of disjoint open intervals $I_j, 1 \leqq j \leqq n$, where we allow $n = \infty$. To prove (2.41), it suffices to prove $n < \infty$. If $n = \infty$ and we write $I_j = (c_j, d_j)$, we have

$$\sum_{j=1}^{\infty} (d_j - c_j) \leqq (d - c), \quad c < c_j < d_j < d.$$

Thus we conclude that $\lim_{j \to \infty} (d_j - c_j) = 0$ and that by taking a subsequence we can assume $\lim_{j \to \infty} c_j = \gamma = \lim_{j \to \infty} d_j$. By taking a further subsequence we can also assume that c_j is either a strictly decreasing sequence or a strictly increasing sequence.

We now use the above assumptions to contradict the fact that $N_s(v; L)$ is finite for any finite interval L. Our construction ensures that

$$v(c_k) = v'(c_k) = 0, \quad v(d_k) = v'(d_k) = 0 \quad \text{for } k \geqq 1,$$

so (2.40) implies

$$v(\eta(c_k)) = 0 = v(\eta(d_k)) \quad \text{for } k \geqq 1.$$

Equation (2.42) implies that for $t > c_k$ and t close to c_k,

$$\exp(\varepsilon^{-1} A(t))\, v(t) = -\varepsilon^{-1} \int_{c_k}^{t} \exp(\varepsilon^{-1} A(s))\, b(s)\, v(\eta(s))\, ds. \tag{2.43}$$

Since $-b(s) < 0$ for all s and $v(t) > 0$ on (c_k, d_k), equation (2.43) implies that there exists $\gamma_k \in (c_k, d_k)$ such that $v(\eta(\gamma_k)) < 0$; γ_k can be chosen as close as desired to c_k. If $t < d_k$ and t is close to d_k, we obtain

$$-\exp(\varepsilon^{-1} A(t))\, v(t) = -\varepsilon^{-1} \int_{t}^{d_k} \exp(\varepsilon^{-1} A(s))\, b(s)\, v(\eta(s))\, ds. \tag{2.44}$$

Equation (2.44) implies that there exists $\delta_k \in (c_k, d_k)$ with $v(\eta(\delta_k)) > 0$; δ_k can be chosen as close as desired to d_k. For γ_k and δ_k close enough to c_k and d_k, respectively, our assumption that η is strictly increasing implies that either $\eta(\gamma_k) < \eta(\delta_k) < \eta(\gamma_{k+1})$ for all k or $\eta(\delta_{k+1}) < \eta(\gamma_k) < \eta(\delta_k)$ for all k. In either case, since $c_k \to \gamma$ and $d_k \to \gamma$, we find that v has infinitely many sign changes on a compact interval. This contradiction proves (2.41).

As an immediate corollary, we see that if $t \in \mathbb{R}$, there exists $\delta > 0$ such that either $v(s) = 0$ for all $s \in [t, t + \delta]$ or $v(s) \neq 0$ for all $s \in (t, t + \delta]$. A similar statement holds for an interval $[t - \delta, t]$.

If $t_1, t_2 \in K = \{t : v(t) = 0\}$, we say that $t_1 \sim t_2$ or t_1 is equivalent to t_2, if there exists an interval I such that $t_1, t_2 \in I$ and $v(t) = 0$ for all $t \in I$. We allow I to consist of a single point. For $t \in K$, we define $\mu(t)$ to be the number of equivalence classes of K which have a nonempty intersection with the interval $(\eta(t), t]$. Our previous work shows that $\mu(t) < \infty$ for all $t \in K$, and we claim that $\mu \mid K$ is a decreasing function. A little thought shows that it suffices to prove that (a) if $[t_1, t_2] \subseteq K$ and $t_1 < t_2$, then $\mu(t_1) \geqq \mu(t_2)$ and (b) if $t_1, t_2 \in K$, $t_1 < t_2$, and $v(t) \neq 0$ for all $t \in (t_1, t_2)$, then $\mu(t_1) \geqq \mu(t_2)$. Assertion (a) is straightforward, because we are assuming that $\eta(t_1) < \eta(t_2)$. To prove (b), we assume for definiteness that $v(t) > 0$ on (t_1, t_2). If $v'(t_1) > 0$, then (2.40) implies that $v(\eta(t_1)) < 0$. If $v'(t_1) = 0$, then $v(\eta(t_1)) = 0$, and using (2.43) (with t_1 replacing c_k) we see that there are numbers s_1, $t_1 < s_1 < t_2$, s_1 arbitrarily close to t_1, such that $v(\eta(s_1)) < 0$. Obviously, we can also choose such an s_1 if $v(\eta(t_1)) < 0$. In fact, our previous remarks and the fact that η is strictly increasing imply that there exists $\delta > 0$ with $v(\sigma) < 0$ for all $\sigma \in (\eta(t_1), \eta(t_1) + \delta_1]$. If $v'(t_2) < 0$, then (2.40) implies that $v(\eta(t_2)) > 0$. If $v'(t_2) = 0$, then (2.40) implies $v(\eta(t_2)) = 0$, and (2.44) (with t_2 replacing d_k) shows that there are numbers s_2, $t_1 < s_2 < t_2$, s_2 arbitrarily close to t_2, with $v(\eta(s_2)) > 0$. Of course, such an s_2 also exists if $v(\eta(t_2)) > 0$. In fact, just as above there exists $\delta_2 > 0$ with $v(\sigma) > 0$ for all $\sigma \in [\eta(t_2) - \delta_2, \eta(t_2))$. We can assume that $t_1 < s_1 < s_2 < t_2$, and our assumptions about η imply that $\eta(t_1) < \eta(s_1) < \eta(s_2) < \eta(t_2)$.

It follows that an equivalence class of K is contained in $(\eta(s_1), \eta(s_2))$ and hence not contained in $(\eta(t_2), t_2]$. On the other hand, the only equivalence class of K which intersects $(\eta(t_2), t_2]$ and not $(\eta(t_1), t_1]$ is the equivalence class containing t_2. Thus, we conclude that $\mu(t_1) \geqq \mu(t_2)$. Since μ is decreasing on K and v is periodic, we conclude that μ is constant on K.

We now claim that if $v(\tau) = 0$, then $v'(\tau) \neq 0$. To see this, let $[t_1, t_2]$ be the largest interval containing τ such that $v(s) = 0$ for all $s \in [t_1, t_2]$. Possibly $t_1 = t_2 = \tau$. We assume that $v'(\tau) = 0$ and obtain a contradiction. We know that there exists $\delta > 0$ with $v(t) \neq 0$ for all $t \in [t_1 - \delta, t_1) \cup (t_2, t_2 + \delta]$. For definiteness, we assume that $v(t) > 0$ on $[t_1 - \delta, t_1)$, and we select $t_0 \in K$, $t_0 < t_1$, with $v(t) > 0$ for $t_0 < t < t_1$. We know that $v(t) = v'(t) = 0$ for $t_1 \leqq t \leqq t_2$, so the same argument used in the proof that μ is decreasing on K shows that there exists $\delta_1 > 0$ with $v(\sigma) < 0$ for $\eta(t_0) < \sigma \leqq \eta(t_0) + \delta_1$, $v(\sigma) > 0$ for $\eta(t_1) - \delta_1 \leqq \sigma < \eta(t_1)$, and $v(\sigma) \neq 0$ for $\eta(t_2) < \sigma \leqq \eta(t_2) + \delta_1$. It follows that there is an equivalence class L_1 of zeros of $v(t)$ which is contained in $(\eta(t_0) + \delta_1, \eta(t_1) - \delta)$ and that $[\eta(t_1), \eta(t_2)] = L_2$ is an equivalence class of zeros of v. Proposition 2.1 implies that $\eta(t_1) < t_1$, and we know that $v(\eta(t)) = 0$ for $t_1 \leqq t \leqq t_2$. Since $v(t) > 0$ for $t_0 < t < t_1$, it follows that $\eta(t_1) \leqq t_0$ and hence $\eta(t) \leqq t_0$ for $t_1 \leqq t \leqq t_2$. We also know that $\eta(t_0) < \eta(t_1) < \eta(t_2)$, so L_1 and L_2 are both contained in $(\eta(t_0), t_0]$ and are disjoint from $(\eta(t_2), t_2]$. On the other hand, $[t_1, t_2]$ is the only equivalence class of zeros of $v(t)$ which intersects $(\eta(t_2), t_2]$ but does not intersect $(\eta(t_0), t_0]$. We conclude that $\mu(t_0) \geqq \mu(t_2) + 1$, which contradicts the fact that μ is constant on K. □

Remark 2.2. It is tempting to try to prove a more general version of Lemma 2.5 by giving an analogue of Lemma 2.4 in which $w_\lambda(t)$ (w_λ as in Lemma 2.5) replaces $x'(t)$. Specifically, if $x(t)$ satisfies $(2.1)_\varepsilon$ for all $t \in \mathbb{R}$ and $x(t)$ is bounded, one would hope to prove, under appropriate assumptions on f and

r, that there does not exist t_0 with $w_\lambda(\eta^j(t_0)) = 0$ for all $j \geqq 0$ (unless $w_\lambda(t)$ is a constant). If $r(u)$ is constant and $f'(u) < 0$ for all u, an argument as in Lemma 2.4 can be given, but for general r, the argument in Lemma 2.4 fails. Of course, if one knew $x(t)$ to be real-analytic, there would be no problem, but even if f and r are real analytic, it is not known whether $x(t)$ is real-analytic. Results and techniques in [42] are not directly applicable.

With the aid of Lemma 2.5, we can extend to the case of state-dependent time lags some unpublished results which have been obtained by MALLET-PARET & SELL for the case of a constant time lag.

Theorem 2.6. *Assume that $\varepsilon > 0$ and that $x(t)$ is a nonconstant periodic solution of $(2.1)_\varepsilon$ with $-\beta < x(t) < \alpha$ for all t, where $\alpha > 0$ and $\beta > 0$. Assume that $f: (-\beta, \alpha) \to \mathbb{R}$ and $r: (-\beta, \alpha) \to \mathbb{R}$ are C^1 maps with $f(0) = 0$, $r(0) = 1$, $f'(u) < 0$ and $r(u) \geqq 0$ for all $u \in (-\beta, \alpha)$. If $\eta(t) = t - r(x(t))$, assume that $\eta'(t) > 0$ for all t. (Theorem 2.5 implies that if r is in C^2 and $r''(u) \leqq D(r'(u))^2$ for all $u \in [-\beta, \alpha)$ and $\varepsilon^{-1} > D$, then $\eta'(t) > 0$ for all t.) Let p denote the minimal period of $x(t)$ and (using Corollary 2.3) let $q_j, j \in \mathbb{Z}$, denote the zeros of $x(t)$, labelled consecutively. If m is a positive integer chosen so that $q_m - q_0 = p$, then $m = 2$. Furthermore, if q and $\bar{q}$ are consecutive zeros of $x(t)$, there exists exactly one t such that $q < t < \bar{q}$* and $x'(t) = 0$.

Proof. If q and $\bar{q}$ are consecutive zeros of $x(t)$, Corollary 2.3 implies that $p > \bar{q} - q$, $x'(q) \neq 0$ and $x'(\bar{q}) \neq 0$. For definiteness, we assume that $x(t) > 0$ for $q < t < \bar{q}$, so $x'(q) > 0$, and $x'(\bar{q}) < 0$. We claim that there exists a unique $t \in (q, \bar{q})$ such that $x'(t) = 0$. For suppose not. Corollary 2.2 implies that if $x'(t) = 0$, then $x''(t) \neq 0$, so $x'(t) = 0$ has finitely many solutions $t \in (q, \bar{q})$. Let t_j, $1 \leqq j \leqq n$, denote the complete set of solutions of $x'(t) = 0$ with $t \in (q, \bar{q})$; and for notational convenience write $t_0 = q$ and $t_{n+1} = \bar{q}$. We can assume that $t_j < t_{j+1}$ for $0 \leqq j \leqq n$; because $x''(t) \neq 0$ whenever $x'(t) = 0$, we know that $(-1)^j x'(t) > 0$ for $t \in (t_j, t_{j+1})$ and $0 \leqq j \leqq n$ and n is odd.

Because $\bar{q} - q < p$, Lemma 2.5 implies that the curve $t \to R(t) = (x(t), x'(t))$, $q \leqq t \leqq \bar{q}$, has no self-intersections. We now employ a simple geometrical argument to obtain a contradiction from this fact. The reader may find drawing a picture helpful at this point. Let Γ_j denote the arc given by $t \to R(t)$ for $t_j \leqq t \leqq t_{j+1}$, $0 \leqq j \leqq n$, and note that Γ_j lies in the first quadrant of the (x, x')-plane for j even and in the fourth quadrant for j odd. Because $x'(t) < 0$ for $t_1 \leqq t \leqq t_2$, it follows that $x(t_2) < x(t_1)$. Since Γ_2 does not intersect Γ_0 and both arcs lie in the first quadrant, it follows that $x(t_3) < x(t_1)$: otherwise Γ_2 would intersect Γ_0. We also know that $x(t_2) < x(t_3)$, because $x'(t) > 0$ for $t_2 < t < t_3$. We now repeat the argument but use the arcs Γ_1 and Γ_3. Γ_1 and Γ_3 lie in the fourth quadrant and $x(t_2) < x(t_3) < x(t_1)$, so Γ_3 must lie inside the region bounded by the x-axis and Γ_1: otherwise Γ_3 would intersect Γ_1. In particular, this implies that $x(t_2) < x(t_4) < x(t_3)$. Continuing in this way, we eventually find that $x(t_2) < x(t_n) < x(t_1)$. However, $x(t_{n+1}) = 0$ and $x'(t_{n+1}) < 0$, so the arc Γ_{n+1}, which lies in the fourth quadrant, necessarily intersects the arc Γ_1, a contradiction.

Let $\tau_j \in (q_j, q_{j+1})$ be the unique $t \in (q_j, q_{j+1})$ such that $x'(t) = 0$. For definiteness, assume that $x(t) > 0$ for $q_1 < t < q_2$. Because $x(t)$ is periodic of minimal period p, there exists $k \geqq 1$ such that $q_{2k+1} - q_1 = p$; to complete the proof, we must show that $k = 1$. We assume that $k > 1$ and obtain a contradiction. Note that Lemma 2.5 implies that $R(t) = (x(t), x'(t))$, $q_1 \leqq t \leqq q_{2k+1}$, gives a simple, closed Jordan curve in the (x, x')-plane. Furthermore (under the assumption that $x(t) > 0$ for $q_1 < t < q_2$), our results show that $R(t)$ lies in the first quadrant for $q_1 < t < \tau_1$, in the fourth quadrant for $\tau_1 < t < q_2$, in the third quadrant for $q_2 < t < \tau_2$, in the second quadrant for $\tau_2 < t < q_3$, and then cycles around again as t goes from q_3 to q_5, etc. If $|x'(q_3)| < |x'(q_1)|$, one can see (because $R(t)$ does not intersect itself for $q_1 \leqq t < q_{2k+1}$) that $|x'(q_{j+2})| < |x'(q_j)|$ and $|x(\tau_{j+2})| < |x(\tau_j)|$ for $1 \leqq j \leqq 2k-1$. This implies that $|x'(q_{2k+1})| < |x'(q_1)|$ and contradicts periodicity. A similar argument applies if $|x'(q_3)| > |x'(q_1)|$, so we obtain a contradiction if $k > 1$. □

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References

1. J. BELAIR & M. C. MACKEY, Consumer memory and price fluctuations on commodity markets: an integrodifferential model, *J. Dynamics and Diff. Eqs.* **1** (1989), 299–325.
2. ROBERT BROWN, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, Glenview, Illinois, 1971.
3. S.-N. CHOW & D. GREEN, JR., Some results on singular delay-differential equations, *Chaos, Fractals and Dynamics*, ed. by P. FISCHER & W. R. SMITH, Marcel Dekker, 1985, 161–182.
4. S.-N. CHOW & J. MALLET-PARET, Integral averaging and bifurcation, *J. Diff. Eqs.* **26** (1977), 112–159.
5. S.-N. CHOW & J. MALLET-PARET, Singularly perturbed delay-differential equations, *Coupled Nonlinear Oscillators*, ed. by J. CHANDRA & A. C. SCOTT, North Holland Math. Studies, **80**, 1983, 7–12.
6. B. D. COLEMAN & G. H. RENNINGER, Periodic solutions of certain non-linear integral equations with delay, *SIAM J. Appl. Math.* **31** (1976), 111–120.
7. M. W. DERSTINE, H. M. GIBBS, F. A. HOPF & D. L. KAPLAN, Bifurcation gap in a hybrid optically bistable system, *Phys. Rev. A* **26** (1982), 3720–3722.
8. M. W. DERSTINE, H. M. GIBBS, F. A. HOPF & D. L. KAPLAN, Alternate paths to chaos in optical bistability, *Phys. Rev. A* **27** (1983), 3200–3208.
9. R. D. DRIVER, Existence theory for a delay-differential system, *Contrib. Diff. Eq.* **1** (1963), 317–336.
10. R. D. DRIVER, A two-body problem of classical electrodynamics: the one-dimensional case, *Ann. Phys.* **21** (1963), 122–142.
11. R. D. DRIVER, A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics, *Internat. Symp. Nonlinear Differential Equations and Nonlinear Mechanics*, Academic Press, New York, 1963, 474–484.
12. R. D. DRIVER, The "backwards" problem for a delay-differential system arising in classical electrodynamics, *Proc. Fifth Internat. Conference on Nonlinear Oscillations*, Kiev, 1969, Volume 2: Qualitative methods in the theory of nonlinear oscillations (Russian) Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1970, 137–143.

13. R. D. Driver & M. J. Norris, Note on uniqueness for a one-dimentional two-body problem of classical electrodynamics, *Ann. Phys.* **42** (1967), 347–351.
14. B. Fiedler & J. Mallet-Paret, Connections between Morse sets for delay-differential equations, *J. Reine Angew Math.* **397** (1989), 23–41.
15. W. S. C. Gurney, S. P. Blythe & R. M. Nisbet, Nicholson's blowflies revisited, *Nature* **287** (1980), 17–21.
16. K. P. Hadeler & J. Tomiuk, Periodic solutions of difference-differential equations, *Arch. Rational Mech. Anal.* **65** (1977), 82–95.
17. F. A. Hopf, D. L. Kaplan, H. M. Gibbs & R. L. Shoemaker, Bifurcation to chaos in optical bistability, *Phys. Rev. A* **25** (1982), 2172–2182.
18. K. Ikeda, Multiple-valued stationary state and its instability of the transmitted light by a ring cavity system, *Opt. Comm.* **30** (1979), 257–261.
19. K. Ikeda, H. Daido & O. Akimoto, Optical Turbulence: chaotic behaviour of transmitted light from a ring cavity, *Phys. Rev. Lett.* **45** (1980), 709–712.
20. K. Ikeda, K. Kondo & O. Akimoto, Successive higher-harmonic bifurcations in systems with delayed feedback, *Phys. Rev. Lett.* **49** (1982), 1467–1470.
21. J. L. Kaplan & J. A. Yorke, On the stability of a periodic solution of a differential delay equation, *SIAM J. Math. Anal.* **6** (1975), 268–282.
22. J. L. Kaplan & J. A. Yorke, On the nonlinear differential-delay equation $x'(t) = -f(x(t), x(t-1))$, *J. Diff. Eqs.* **23** (1977), 293–314.
23. A. Longtin & J. Milton, Complex oscillations in the human pupil light reflex with mixed and delayed feedback, *Math. Biosciences* **90** (1988), 183–199.
24. A. Longtin & J. Milton, Modelling autonomous oscillations in the human pupil light reflex using nonlinear delay-differential equations, *Bull. Math. Biology* **51** (1989), 605–624.
25. A. Longtin & J. Milton, Insight into the transfer function, gain and oscillation onset for the pupil light reflex using non-linear delay-differential equations, *Biol. Cybern.* **61** (1989), 51–58.
26. A. Longtin, Nonlinear oscillations, noise and chaos in neural delayed feedback, Ph. D. Thesis, 1988, McGill University, Montreal, Canada.
27. M. C. Mackey, Commodity price fluctuations: price dependent delays and nonlinearities as explanatory factors, *J. of Economic Theory*, to appear.
28. M. C. Mackey & L. Glass, Oscillation and chaos in physiological control systems, *Science* **197** (1977), 287–289.
29. M. C. Mackey & U. an der Heiden, Dynamical diseases and bifurcations: understanding functional disorders in physiological systems, *Funkt. Biol. Med.* **156** (1983), 156–164.
30. M. C. Mackey & J. Milton, Feedback delays and the origin of blood cell dynamics, *Comments on Theoretical Biology* **1** (1990), 299–327.
31. J. Mallet-Paret, Morse decompositions and global continuation of periodic solutions for singulary perturbed delay equations, *Systems of Nonlinear Partial Differential Equations*, ed. by J. M. Ball, Reidel, Dordrecht, 1983, 351–365.
32. J. Mallet-Paret, Morse decompositions for delay-differential equations, *J. Diff. Eqs.* **72** (1988), 270–315.
33. J. Mallet-Paret & R. D. Nussbaum, Global continuation and complicated trajectories for periodic solutions of a differential-delay equations, *Proceedings of Symposia in Pure Mathematics* **45** (1986), 155–167.
34. J. Mallet-Paret & R. D. Nussbaum, Global continuation and asymptotic behavior for periodic solutions of a differential-delay equation, *Annali di Mat. Pura ed Appl.* **145** (1986), 33–128.

35. J. Mallet-Paret & R. D. Nussbaum, A bifurcation gap for a singularly perturbed delay equation, *Chaotic Dynamics and Fractals*, **2**, ed. by M. Barnsley & S. Demko, Academic Press, New York, 1986, 263–287.
36. J. Mallet-Paret & R. D. Nussbaum, A differential-delay equation arising in optics and physiology, *SIAM J. Math. Anal.* **20** (1989), 249–292.
37. J. Mallet-Paret & R. D. Nussbaum, Multiple transition layers in a singularly perturbed differential-delay equation, to appear in *Proc. Royal Soc. Edinburgh.*
38. J. Mallet-Paret & R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags, II, in preparation.
39. J. Mallet-Paret & H. Smith, The Poincaré-Bendixson theorem for monotone cyclic feedback systems, *J. Dynamics and Diff. Eqs.* **2** (1990), 367–421.
40. M. Martelli, K. Schmitt & H. Smith, Periodic solutions of some non-linear delay-differential equations, *J. Math. Anal. Appl.* **74** (1980), 494–503.
41. R. D. Nussbaum, Some asymptotic fixed point theorems, *Trans. Amer. Math. Soc.* **171** (1972), 349–375.
42. R. D. Nussbaum, Periodic solutions of analytic functional differential equations are analytic, *Michigan Math. J.* **20** (1973), 249–255.
43. R. D. Nussbaum, Periodic solutions of some nonlinear autonomous functional differential equations, *Ann. Mat. Pura Appl.* **101** (1974), 263–306.
44. R. D. Nussbaum, A global bifurcation theorem with applications to functional differential equations, *J. Func. Anal.* **19** (1975), 319–338.
45. R. D. Nussbaum, Periodic solutions of nonlinear autonomous functional differential equations, *Functional Differential Equations and Approximation of Fixed Points*, Springer-Verlag Lecture Notes in Math. **730**, ed. by H.-O. Peitgen & H.-O. Walther, 1978, 283–325.
46. R. D. Nussbaum, *The Fixed Point Index and Some Applications*, Seminaire de Mathématiques Supérieures, Les Presses de l'Univ. Montréal, 1985.
47. R. D. Nussbaum, Boundary layer phenomena for a differential-delay equation, *Oscillation, Bifurcation and Chaos*, *Canadian Math. Society Conference Proc.* **8**, ed. by F. V. Atkinson, W. F. Langford & A. B. Mingarelli, 1986, 579–599.
48. R. D. Nussbaum, Uniqueness and nonuniqueness for periodic solutions of $x'(t) = g(x(t-1))$, *J. Diff. Eqs.* **34** (1979), 25–54.
49. G. P. P'elyukh & A. N. Sharkovskii, *Introduction to the Theory of Functional Equations* (in Russian), Kiev, Naukova Dumka, 1974.
50. H. W. Stech, The Hopf bifurcation: a stability result and application, *J. Math. Anal. Appl.* **71** (1979), 525–546.
51. M. Wazewska-Czyzewska & A. Lasota, Mathematical models of the red cell system (in Polish), *Matematyka Stosowana* **6** (1976), 25–40.

Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

and

Mathematics Department
Rutgers University
New Brunswick, New Jersey 08903

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