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Finite-Particle-Size Effects in Disperse Two-Phase Flows[1]

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Abstract. Three aspects of the finite radius of spherical particles in disperse two-phase flows are described. The first one is the relation between the exact volume fraction and the widely used approximation nv (n is the particle number density and v is the particle volume). The approximation affects the behavior of the effective equations at short wavelengths with possible consequences on stability and hyperbolicity. Secondly, the dilute theory of inviscid suspensions is corrected retaining the next leading order in the particle size and an application of this result to the linear problem is described. Thirdly, it is shown how several important properties of suspensions such as effective thermal conductivity and viscosity depend on the subtle effect of translation of the average fields over distances of the order of the particle size.

1. Introduction

The mathematical description of a suspension or other disperse two-phase flow by means of averaged equations presupposes that the particle size is much smaller than macroscopic length-scales. While this statement is obvious, it may be too superficial in certain circumstances. For example, the study of the well-posedness of the equations requires their analysis at vanishingly small scales. Even though it would be unreasonable to require realism of the modeling at these small scales, still a mathematically and physically consistent behavior should be expected. An analog may be found in ordinary gas dynamics where a shock wave may be thinner than the mean free path. The Navier–Stokes equations cannot describe the internal structure of the shock, but from their small-scale behavior it is still possible to infer its position and general properties correctly.

For many numerical integration schemes of the gas-dynamic equations, the presence of the higher-order derivatives due to viscosity—even if inaccurate at the smallest scales—is essential. Similarly, for the multiphase flow equations, a well-known relation exists between well-posedness and stability (see, e.g., Prosperetti and Jones, 1984) and a complete model failure at the small scales would adversely affect the possibility of accurate computation by nondissipative schemes.

As a final example, in dilute suspensions, situations in which the macroscopic scale is shorter than the mean particle distance can be envisaged. In this case, corrections to the lowest-order dilute limit model due to the finite particle size may be more important than those due to particle–particle interaction.

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These considerations suggest that a brief look at the issue of finite-size effects may be of some interest. Our attention was drawn to this problem by recent studies by Singh and Joseph (1993, 1995) presenting some intriguing finite-size effects. The direction of that work is however different from that of the present one. Here we do not propose giving an exhaustive treatment of the subject, but only pointing out with the aid of some examples that, far from being marginal, it lies at the very heart of the difficulties and physical content of the description of multiphase flows by means of averaged equations.

2. Volume Fraction

Consider an ensemble of realizations of a two-phase disperse flow characterized by a probability density $P(N;t)$. The notation implies that P is a function of all the variables necessary to specify uniquely the dynamical state—or *configuration* $\mathscr{C}^N$—of the systme of N particles constituting each realization of the suspension. In this paper we consider equal spherical particles with radius a and we think of P as depending on the position of the particle centers and their center-of-mass velocity. Depending on circumstances other variables, such as particle angular velocity and continuous-phase degrees of freedom, may be necessary. Conversely, in situations such as creeping flow, the particle velocities become dependent variables and should be dropped. Since the particles are indistinguishable, it is convenient to normalize P according to (see, e.g., Batchelor, 1972; Landau *et al.*, 1981)

$$\int d\mathscr{C}^N P(N;t)\chi_D(\mathbf{x};N) = N!, \tag{1}$$

where $d\mathscr{C}^N$ is short-hand for integration over all the degrees of freedom of the system.

Let $\chi_D(\mathbf{x};N)$ be the characteristic function of the disperse phase in the presence of the configuration $\mathscr{C}^N$ so that $\chi_D = 1$ when $\mathbf{x}$ is inside a particle and 0 otherwise. For spherical particles of equal radius a (Lundgren, 1972),

$$\chi_D(\mathbf{x};N) = \sum_{\alpha=1}^{N} H(a - |\mathbf{x} - \mathbf{y}^{(\alpha)}|), \tag{2}$$

where H is the Heaviside distribution and $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(N)}$ are the position vectors of the particle centers. The disperse-phase volume fraction β_D is the probability that, at time t, the point $\mathbf{x}$ is in the disperse phase and is therefore defined by

$$\beta_D(\mathbf{x},t) = \frac{1}{N!}\int d\mathscr{C}^N P(N;t)\chi_D(\mathbf{x};N). \tag{3}$$

By using the representation (2) and the identity of the particles it is easy to show that (Lundgren, 1972)

$$\beta_D(\mathbf{x},t) = \int_{|\mathbf{y}-\mathbf{x}|\le a} d^3y \int d^3w P(1;t), \tag{4}$$

where $\mathbf{w}$ is the center-of-mass velocity and $P(1;t) \equiv P(\mathbf{y},\mathbf{w};t)$ is the reduced one-particle distribution function obtained from $P(N;t)$ by integration over all the degrees of freedom of the system except the position and velocity of one particle,

$$P(\mathbf{y},\mathbf{w};t) = \frac{1}{(N-1)!}\int d\mathscr{C}^{N-1} P(N;t). \tag{5}$$

By definition the particle number density n is such that $n d^3y$ represents the number of particles with centers in the volume element d^3y irrespective of velocity and therefore

$$n(\mathbf{y},t) = \int d^3w P(1;t). \tag{6}$$

With this definition we then find from (4)

$$\beta_D(\mathbf{x},t) = \int_{|\mathbf{x}-\mathbf{y}|\le a} d^3y\, n(\mathbf{y},t). \tag{7}$$

This result may be compared with the relation frequently encountered in the multiphase literature:

$$\beta_D \simeq nv, \tag{8}$$

where $v = \frac{4}{3}\pi a^3$ is the particle volume. This expression may be obtained from (4) if the characteristic function (2) is approximated by

$$\chi_D \simeq v \sum_{\alpha=1}^{N} \delta(\mathbf{x} - \mathbf{y}^{(\alpha)}), \tag{9}$$

where δ indicates the delta distribution. The difference between (7) and (8) may therefore be considered a finite-size effect.

Upon expanding n in (7) in Taylor series around $\mathbf{x}$ we find

$$\beta_D = v\left[n + \tfrac{1}{10}a^2\nabla^2 n + O\left(\frac{a}{L}\right)^4\right], \tag{10}$$

where L is the macroscopic scale of variation of n. The difference between (7) and (8) therefore is of the order of $(a/L)^2$ so that (8) is in general a good approximation, as expected.

A striking difference between the two definitions is encountered however in a linear stability analysis at short wavelengths. Suppose that a uniform state is perturbed by a single Fourier mode

$$P(\mathbf{y}, \mathbf{w}, t) = P_0(\mathbf{w}) + P_1(\mathbf{w})\exp(ikx - i\omega t). \tag{11}$$

Then, from (6),

$$n(\mathbf{y}, t) = n_0 + n_1 \exp(ikx - i\omega t), \qquad \text{where} \quad n_{0,1} = \int d^3w P_{0,1}(\mathbf{w}), \tag{12}$$

and, from (7) (Singh and Joseph, 1993, 1995),

$$\beta_D = v\left[n_0 + 3n_1 \frac{j_1(ka)}{ka}\exp(ikx - i\omega t)\right]. \tag{13}$$

Clearly, the perturbation in $\beta_D \to 0$ as $k \to \infty$, as expected on physical grounds. On the other hand, the approximation (8) would only give

$$\beta_D \simeq v[n_0 + n_1 \exp(ikx - i\omega t)], \tag{14}$$

with a completely different behavior at short wavelengths. Thus, for $k \to \infty$, a term $\nabla\beta_D$ should tend to zero if computed exactly according to (13), but would be found to diverge according to (14). This major qualitative difference between the two expressions can have a significant impact on the mathematical structure of the theory such as hyperbolicity and well-posedness that depend on the response of the model to infinitesimally short wavelength perturbations.

3. Continuity Equation

The average velocity $\langle \mathbf{u}_D \rangle$ of the disperse phase material can be defined in terms of the probability density introduced in the previous section by

$$\langle \mathbf{u}_D \rangle(\mathbf{x}, t) = \frac{1}{N!\,\beta_D}\int d\mathscr{C}^N P(N; t)\chi_D(\mathbf{x}; N)\mathbf{u}_D(\mathbf{x}, t; N), \tag{15}$$

where the notation $\mathbf{u}_D(\mathbf{x}, t; N)$ stresses the dependence of the exact, microscopic variable $\mathbf{u}_D$ upon all the degrees of freedom of the system. Note that the quantity thus defined is the ensemble phase-average velocity in the sense that the presence of the characteristic function χ_D restricts the integral to only those members of the ensemble (i.e., those realizations of the flow) in which the point $\mathbf{x}$ is occupied by the disperse phase at time t. It is easy to show upon using the standard relation expressing the conservation of probability (Zhang and Prosperetti, 1994a) that, for constant a,

$$\frac{\partial \beta_D}{\partial t} + \nabla\cdot(\beta_D \langle \mathbf{u}_D \rangle) = 0, \tag{16}$$

which is a well-known form of the equation expressing the conservation of the disperse-phase volume.

Another important average velocity field is the center-of-mass velocity of the particles $\bar{\mathbf{w}}$ defined by

$$\bar{\mathbf{w}}(\mathbf{x}, t) = \frac{1}{n(\mathbf{x}, t)} \int d^3 w P(\mathbf{x}, \mathbf{w}, t)\mathbf{w}. \tag{17}$$

The difference between the two velocity fields (15) and (17) is that $\langle \mathbf{u}_D \rangle d\mathscr{V}$ is the volume flow rate of the particle phase due to the motion of the *particle material contained in the volume element* $d\mathscr{V}$, while $\bar{\mathbf{w}} d\mathscr{V}$ is the volume flow rate due to the motion of the particles with *center within* $d\mathscr{V}$. In other words, $\bar{\mathbf{w}} d\mathscr{V}$ includes in their entirety also the particles near the boundary of $d\mathscr{V}$ that lie in part outside the volume element, while $\langle \mathbf{u}_D \rangle d\mathscr{V}$ does not. Clearly, both $\langle \mathbf{u}_D \rangle$ and $\bar{\mathbf{w}}$ are well-defined, if different, velocity fields.

As shown in Zhang and Prosperetti (1994a), $\bar{\mathbf{w}}$ is the proper velocity field for the evolution of the particle number density in the sense that

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\bar{\mathbf{w}}) = 0. \tag{18}$$

For rigid particles v is a constant and (16) and (18) are one and the same if β_D is set equal to nv and $\langle \mathbf{u}_D \rangle$ is identified with $\bar{\mathbf{w}}$. Actually, by the same argument leading to (10), we find

$$\langle \mathbf{u}_D \rangle = \bar{\mathbf{w}} + \frac{a^2}{10}\left[\nabla^2 \bar{\mathbf{w}} - 2\nabla \times \bar{\Omega} + \frac{1}{n}(\nabla n \cdot \nabla \bar{\mathbf{w}} - 2\nabla n \times \bar{\Omega}) + O\left(\frac{a^2}{L^4}\right)\right], \tag{19}$$

where $\bar{\Omega}$ is the mean angular velocity of the particles. To the same approximation, (19) can also be written as

$$\bar{\mathbf{w}} \simeq \langle \mathbf{u}_D \rangle - \frac{a^2}{10}\left[\nabla^2 \langle \mathbf{u}_D \rangle - 2\nabla \times \bar{\Omega} + \frac{1}{n}(\nabla n \cdot \nabla \langle \mathbf{u}_D \rangle - 2\nabla n \times \bar{\Omega}) \right]. \tag{20}$$

Upon multiplying (18) by v and using this relation we then have

$$\frac{\partial}{\partial t}(nv) + \nabla \cdot (nv \langle \mathbf{u}_D \rangle) \simeq \frac{a^2}{10} \nabla \cdot [nv(\nabla^2 \langle \mathbf{u}_D \rangle - 2\nabla \times \bar{\Omega}) + \nabla(nv) \cdot \nabla \langle \mathbf{u}_D \rangle - 2\nabla(nv) \times \bar{\Omega}]. \tag{21}$$

In most models that neglect particle-size effects, the left-hand side of this equation is set to zero. The additional terms retained here would evidently strongly affect the dispersion properties of disturbances at the higher frequencies. In view of (21) and (10), it would seem that a description in terms of $\bar{\mathbf{w}}$ and n is more economical than one in terms of $\langle \mathbf{u}_D \rangle$ and β_D or nv.

4. "ZIP" Averaging

To proceed further we need to describe in broad outline the new "ZIP" averaging approach that we have developed to derive effective equations for disperse two-phase flows (Zhang and Prosperetti, 1994a, 1994b, 1995).

Average quantities for the disperse phase are defined by means of "particle averages" analogus to $\bar{\mathbf{w}}$ in (17). This choice is motivated primarily by the fact that, in so doing, the equation of motion of the particles $m\dot{\mathbf{w}} = \mathbf{F}$ can be averaged directly without consideration of the momentum equation for the particle material (e.g., the equations of elasticity). This possibility simplifies the treatment of rigid or nearly rigid particles or massless bubbles and responds better to physical intuition. Indeed, since the force $\mathbf{F}$ depends on continuous-phase quantities, the result of this procedure is an equation of motion expressing the fact that the disperse phase moves in response to the stresses exerted by the continuous phase rather than the stresses internal to the disperse phase itself. Furthermore, the use of particle averages enables the smallest number of degrees of freedom appropriate for the problem to be retained. To see the possible implications of this point consider the case of an inviscid fluid, for which the rotational degree of freedom of the particles cannot be dynamically relevant. If the momentum equation is cast in terms of the phase-average velocity $\langle \mathbf{u}_D \rangle$ rather than the particle-average velocity $\bar{\mathbf{w}}$, the effect of particle rotation evident in (19) must be canceled by other terms that necessarily render the equation more complicated. Here is another indication that the velocity field $\bar{\mathbf{w}}$ may be more useful than the phase average $\langle \mathbf{u}_D \rangle$.

It is shown in Zhang and Prosperetti (1995) that, upon averaging the particle momentum equation as described before, one finds

$$\rho_{\mathrm{D}}\left[\frac{\partial}{\partial t}(nv\bar{\mathbf{w}})+\nabla\cdot(nv\overline{\mathbf{ww}})\right]=nv\nabla\cdot\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle+nv\mathbf{A}_{\mathrm{D}}. \tag{22}$$

Here ρ_{D} is the disperse-phase density, $\boldsymbol{\sigma}_{\mathrm{C}}$ is the continuous-phase viscous stress tensor, and $\mathbf{A}_{\mathrm{D}}$ is defined by

$$nv\mathbf{A}_{\mathrm{D}}(\mathbf{x})=\int d^3wP(\mathbf{x},\mathbf{w},t)\int_{|\mathbf{z}-\mathbf{x}|=a}dS_z\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{z},t|\mathbf{x},\mathbf{w})\cdot\mathbf{n}-nv\nabla\cdot\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle. \tag{23}$$

Here the notation $\langle\cdots\rangle_1$ indicates the ensemble phase average conditional on the presence of a particle with center at $\mathbf{x}$ and velocity $\mathbf{w}$. Collisions and body forces have been neglected in writing (22). To proceed it is useful to rewrite $\mathbf{A}_{\mathrm{D}}$ identically as

$$\begin{aligned}nv\mathbf{A}_{\mathrm{D}}(\mathbf{x})=&\int d^3wP(\mathbf{x},\mathbf{w},t)\int_{|\mathbf{z}-\mathbf{x}|=a}dS_z[\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{z},t|\mathbf{x},\mathbf{w})-\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle(\mathbf{z},t)]\cdot\mathbf{n}\\&-\left[nv\nabla\cdot\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle-\int d^3wP(\mathbf{x},\mathbf{w},t)\int_{|\mathbf{z}-\mathbf{x}|=a}dS_z\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle(\mathbf{z},t)\cdot\mathbf{n}\right].\end{aligned} \tag{24}$$

By expanding the integrand in the last term around $\mathbf{x}$ and using (6) we find (Zhang and Prosperetti, 1994a)

$$\mathbf{A}_{\mathrm{D}}(\mathbf{x})=\frac{1}{nv}\int d^3wP(\mathbf{x},\mathbf{w},t)\int_{|\mathbf{z}-\mathbf{x}|=a}dS_z[\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{z},t|\mathbf{x},\mathbf{w})-\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle(\mathbf{z},t)]\cdot\mathbf{n}+\frac{a^2}{10}\nabla\cdot(\nabla^2\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle). \tag{25}$$

In the ZIP approach continuous-phase quantities are averaged according to the phase-averaging rule analogous to (15). For example,

$$\langle\mathbf{u}_{\mathrm{C}}\rangle=\frac{1}{N!\,\beta_{\mathrm{C}}}\int d\mathscr{C}^NP(N;t)\chi_{\mathrm{C}}(\mathbf{x};N)\mathbf{u}_{\mathrm{C}}(\mathbf{x},t;N), \tag{26}$$

where $\chi_{\mathrm{C}}=1-\chi_{\mathrm{D}}$ is the continuous-phase characteristic function and $\beta_{\mathrm{C}}=1-\beta_{\mathrm{D}}$ is the continuous-phase volume fraction. It is then possible to show (Zhang and Prosperetti, 1994a, 1995) that, upon averaging, the continuous-phase momentum equation takes the form

$$\rho_{\mathrm{C}}\left[\frac{\partial}{\partial t}(\beta_{\mathrm{C}}\langle\mathbf{u}_{\mathrm{C}}\rangle)+\nabla\cdot(\beta_{\mathrm{C}}\langle\mathbf{u}_{\mathrm{C}}\mathbf{u}_{\mathrm{C}}\rangle)\right]=\beta_{\mathrm{C}}\nabla\cdot\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle+\beta_{\mathrm{C}}\mathbf{A}_{\mathrm{C}}. \tag{27}$$

Here ρ_{C} is the continuous-phase density and

$$\beta_{\mathrm{C}}\mathbf{A}_{\mathrm{C}}=-\int_{|\mathbf{y}-\mathbf{x}|=a}dS_y\int d^3wP(\mathbf{y},\mathbf{w},t)[\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{x},t|\mathbf{y},\mathbf{w})-\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle(\mathbf{x},t)]\cdot\mathbf{n}. \tag{28}$$

Although similar, the two equations (22) and (27) differ in an essential respect as the integral defining $\mathbf{A}_{\mathrm{D}}$ is over the surface of a particle centered at $\mathbf{x}$, while that defining $\mathbf{A}_{\mathrm{C}}$ is over all the particles touching $\mathbf{x}$. As shown later, this difference is a subtle manifestation of finite-size effects.

The conditionally averaged fields appearing in the definitions (23) or (25) and (28) depend slowly on the position of the center of the particle. Hence a Taylor series expansion in this variable can be carried out similarly to Zhang and Prosperetti (1994a, 1995) to find

$$\beta_{\mathrm{C}}\mathbf{A}_{\mathrm{C}}=-nv\mathbf{A}_{\mathrm{D}}+\nabla\cdot[nv\mathbf{T}_{\mathrm{C}}+\nabla\cdot(nv\mathbf{S}_{\mathrm{C}})+\nabla\nabla:(nv\mathbf{R}_{\mathrm{C}})]+\frac{a^2}{10}nv\nabla^2(\nabla\cdot\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle)+O\left(\frac{a}{L}\right)^4. \tag{29}$$

The definitions of $\mathbf{T}_{\mathrm{C}}$, $\mathbf{S}_{\mathrm{C}}$, and $\mathbf{R}_{\mathrm{C}}$ are

$$nv\mathbf{T}_{\mathrm{C}}=a\int d^3wP(\mathbf{x},\mathbf{w},t)\int_{|\mathbf{z}-\mathbf{x}|=a}dS_z\mathbf{n}[\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{z},t|\mathbf{x},\mathbf{w})-\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle(\mathbf{z},t)]\cdot\mathbf{n}, \tag{30}$$

$$nv\mathbf{S}_{\mathrm{C}}=-\tfrac{1}{2}a^2\int d^3wP(\mathbf{x},\mathbf{w},t)\int_{|\mathbf{z}-\mathbf{x}|=a}dS_z\mathbf{nn}[\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{z},t|\mathbf{x},\mathbf{w})-\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle(\mathbf{z},t)]\cdot\mathbf{n}, \tag{31}$$

$$n v \mathbf{R}_{\mathrm{C}}=\tfrac{1}{6} a^{3} \int d^{3} w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{z}-\mathbf{x}|=a} d S_{z} \mathbf{n n n}\left[\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_{1}(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w})-\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle(\mathbf{z}, t)\right] \cdot \mathbf{n}. \tag{32}$$

We now illustrate the implications of these relations in a specific case.

5. Dilute Suspensions of Large Particles

As a further example of finite-size effects consider the case in which the correction to the lowest-order equations arising from the finite size of the particles is biggerl particles in potential flow, so that

$$\boldsymbol{\sigma}_{\mathrm{C}}=-p_{\mathrm{C}} \mathbf{I}, \tag{33}$$

with $\mathbf{I}$ the identity tensor. The quantities that need to be evaluated are then

$$\mathbf{A}_{\mathrm{D}}(\mathbf{x})+\frac{a^{2}}{10} \nabla \nabla^{2}\langle p_{\mathrm{C}}\rangle=-\frac{1}{n v} \int d^{3} w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{z}-\mathbf{x}|=a} d S_{z}\left[\langle p_{\mathrm{C}}\rangle_{1}(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w})-\langle p_{\mathrm{C}}\rangle(\mathbf{z}, t)\right] \mathbf{n}, \tag{34}$$

$$n v \mathbf{T}_{\mathrm{C}}=-a \int d^{3} w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{z}-\mathbf{x}|=a} d S_{z}\left[\langle p_{\mathrm{C}}\rangle_{1}(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w})-\langle p_{\mathrm{C}}\rangle(\mathbf{z}, t)\right] \mathbf{n n}, \tag{35}$$

$$n v \mathbf{S}_{\mathrm{C}}=\tfrac{1}{2} a^{2} \int d^{3} w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{z}-\mathbf{x}|=a} d S_{z}\left[\langle p_{\mathrm{C}}\rangle_{1}(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w})-\langle p_{\mathrm{C}}\rangle(\mathbf{z}, t)\right] \mathbf{n n n}, \tag{36}$$

$$n v \mathbf{R}_{\mathrm{C}}=-\tfrac{1}{6} a^{3} \int d^{3} w P(\mathbf{x}, \mathbf{w}, t) \int_{|\mathbf{z}-\mathbf{x}|=a} d S_{z}\left[\langle p_{\mathrm{C}}\rangle_{1}(\mathbf{z}, t \mid \mathbf{x}, \mathbf{w})-\langle p_{\mathrm{C}}\rangle(\mathbf{z}, t)\right] \mathbf{n n n n}. \tag{37}$$

In order to calculate these integrals, following Zhang and Prosperetti (1994a), we adopt a frame of reference translating without rotation with the particle velocity $\mathbf{w}$. In this frame introduce velocity potentials φ' and φ_1' for the unconditionally and conditionally averaged flows, respectively. We thus have

$$\frac{\langle p_{\mathrm{C}}\rangle_{1}-\langle p_{\mathrm{C}}\rangle}{\rho_{\mathrm{C}}}=\frac{\partial}{\partial t}(\varphi'-\varphi_{1}')+\tfrac{1}{2}(\langle\mathbf{u}_{\mathrm{C}}'\rangle \cdot\langle\mathbf{u}_{\mathrm{C}}'\rangle-\langle\mathbf{u}_{\mathrm{C}}'\rangle_{1} \cdot\langle\mathbf{u}_{\mathrm{C}}'\rangle_{1}), \tag{38}$$

where the prime refers to quantities evaluated in the moving frame. Then it is easy to show that

$$\varphi_{1}'=\sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left[1+\frac{\ell}{\ell+1}\left(\frac{a}{r}\right)^{2 \ell+1}\right](\mathbf{x}-\mathbf{y})^{(\ell)} \nabla^{(\ell)} \varphi'(\mathbf{y}), \tag{39}$$

where

$$(\mathbf{x}-\mathbf{y})^{(\ell)} \nabla^{(\ell)} \varphi'(\mathbf{y}) \equiv(\mathbf{x}-\mathbf{y})_{i}(\mathbf{x}-\mathbf{y})_{j} \cdots(\mathbf{x}-\mathbf{y})_{k}\, \partial_{i} \partial_{j} \cdots \partial_{k} \varphi'(\mathbf{y}). \tag{40}$$

In the remainder of this section we write $\mathbf{u}$, $\mathbf{u}'$ in place of $\langle\mathbf{u}_{\mathrm{C}}\rangle$, $\langle\mathbf{u}_{\mathrm{C}}\rangle'$ for convenience.

From (39) we find, for the unsteady contribution to $\mathbf{A}_{\mathrm{D}}$ (the first term in the right-hand side of (38)),

$$\int_{|\mathbf{z}-\mathbf{x}|=a} d S_{z} \mathbf{n} \frac{\partial}{\partial t}(\varphi'-\varphi_{1}')=-\tfrac{1}{2} v \frac{\partial \mathbf{u}'}{\partial t}, \tag{41}$$

as given in Zhang and Prosperetti (1994a). Similarly, the unsteady contribution to $\mathbf{T}_{\mathrm{C}}$ is

$$-a \int_{|\mathbf{z}-\mathbf{x}|=a} \mathbf{n n} \frac{\partial}{\partial t}(\varphi'-\varphi_{1}')\, d S_{z}=-\tfrac{2}{15} a^{2} v \frac{\partial \mathbf{E}_{\mathrm{C}}}{\partial t}, \tag{42}$$

where

$$\mathbf{E}_{\mathrm{C}}=\tfrac{1}{2}[\nabla \mathbf{u}+(\nabla \mathbf{u})^{\mathrm{T}}] \tag{43}$$

is the strain rate of the mean continuous-phase velocity field. The unsteady part of $\mathbf{S}_{\mathrm{C}}$ is

$$-\tfrac{1}{2} a^{2} \int_{|\mathbf{z}-\mathbf{x}|=a} d S_{z} n_{i} n_{j} n_{k} \frac{\partial}{\partial t}(\varphi'-\varphi_{1}')\, d S_{z}=-v \frac{a^{2}}{20}\left[\frac{\partial u_{k}'}{\partial t} \delta_{i j}+\frac{\partial u_{j}'}{\partial t} \delta_{i k}+\frac{\partial u_{i}'}{\partial t} \delta_{j k}\right], \tag{44}$$

while the unsteady contribution to $\mathbf{R}_{\mathrm{C}}$ vanishes.

The steady contribution to $\mathbf{A}_D$, obtained from the term in the right-hand side of (38), is

$$\int_{|\mathbf{z}-\mathbf{x}|=a} dS_z \mathbf{n}\tfrac{1}{2}(\mathbf{u}'\cdot\mathbf{u}' - \langle \mathbf{u}'_C\rangle_1\cdot\langle \mathbf{u}'_C\rangle_1) = -\tfrac{1}{2}\mathbf{u}'\cdot\nabla\mathbf{u}' + v\frac{a^2}{15}\nabla(\mathbf{E}_C:\mathbf{E}_C). \tag{45}$$

To simplify the evaluation of the similar contribution to $\mathbf{T}_C$ we note that

$$J_{ab} \equiv -\int_{|\mathbf{z}-\mathbf{x}|=a} dS_z n_a n_b \tfrac{1}{2}(\mathbf{u}'\cdot\mathbf{u}' - \langle \mathbf{u}'_C\rangle_1\cdot\langle \mathbf{u}'_C\rangle_1), \tag{46}$$

to $O(a^5)$, must depend bilinearly on

$$\mathbf{u}', \quad a\nabla\mathbf{u}', \quad a^2\nabla\nabla\mathbf{u}'. \tag{47}$$

The only possibility is therefore a structure of the form

$$J_{ab} = a^2\{\delta_{ab}[c_1(u')^2 + a^2c_2(\nabla\mathbf{u}'):(\nabla\mathbf{u}')] + c_3u'_a\bar{u}'_b + c_4a^2(\partial_i u'_a)(\partial_i u'_b) + c_5a^2u'_i\partial_a\partial_b u'_i\}, \tag{48}$$

where the c_i's are numerical coefficients, $\partial_i \equiv \partial/\partial x_i$, and summation over repeated indices is implied. This remark simplifes considerably the calculation. For example, to calculate c_3, we consider the special case in which velocity gradients vanish and the indices a and b are different. After having determined c_3 in this way, we repeat the calculation with $a = b$ to find c_1, and so on for the other coefficients. The final result is

$$c_1 = \frac{2v}{5a}, \quad c_2 = \frac{76\pi}{945}, \quad c_3 = -\frac{9v}{20a}, \quad c_4 = \frac{8\pi}{189}, \quad c_5 = \frac{6}{35}\pi. \tag{49}$$

Proceeding similarly the steady contribution to $\mathbf{S}_C$ is found to be

$$\begin{aligned} -a^2 &\int_{|\mathbf{z}-\mathbf{x}|=a} dS_z n_a n_b n_c \tfrac{1}{2}(\mathbf{u}'\cdot\mathbf{u}' - \langle \mathbf{u}'_C\rangle_1\cdot\langle \mathbf{u}'_C\rangle_1) \\ &= -\frac{a^2}{35}v(u'_lE_{lk}\delta_{ij} + u'_lE_{lj}\delta_{ik} + u'_lE_{li}\delta_{jk}) + \frac{a^2}{14}v(u'_iE_{jk} + u'_jE_{ik} + u'_kE_{ij}). \end{aligned} \tag{50}$$

Reverting to the original frame and carrying out the averaging over $\mathbf{w}$ as in Zhang and Prosperetti (1994a) we then have, writing $\mathbf{w}$ in place of $\bar{\mathbf{w}}$,

$$\mathbf{A}_D = \mathbf{V} + \rho_C\frac{a^2}{15}\nabla(\mathbf{E}_C:\mathbf{E}_C) + \frac{a^2}{10}\nabla(\nabla^2\langle p_C\rangle), \tag{51}$$

where

$$\mathbf{V} = \tfrac{1}{2}\rho_C\left[\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \frac{\partial\mathbf{w}}{\partial t} - \mathbf{w}\cdot\nabla\mathbf{w} + \frac{1}{nv}\nabla(nv\mathbf{M}_D)\right], \tag{52}$$

with

$$\mathbf{M}_D = \mathbf{w}\mathbf{w} - \overline{\mathbf{w}\mathbf{w}}, \tag{53}$$

the disperse-phase kinematic Reynolds stress. Similarly,

$$\begin{aligned} \mathbf{T}_C &= \tfrac{1}{5}\rho_C[2(\mathbf{u}-\mathbf{w})^2\mathbf{I} - \tfrac{9}{4}(\mathbf{u}-\mathbf{w})(\mathbf{u}-\mathbf{w})] - \tfrac{1}{5}\rho_C[2(\mathrm{Tr}\,\mathbf{M}_D)\mathbf{I} - \tfrac{9}{4}\mathbf{M}_D] \\ &\quad - \frac{a^2}{2}\rho_C\left[\tfrac{38}{315}(\mathbf{E}_C:\mathbf{E}_C)\mathbf{I} + \tfrac{4}{63}\mathbf{E}_C\cdot\mathbf{E}_C + \tfrac{4}{15}\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{E}_C + \tfrac{1}{105}(\mathbf{w}-\mathbf{u})\cdot\nabla\mathbf{E}_C\right], \end{aligned} \tag{54}$$

$$\begin{aligned} S_{Cijk} &= \frac{a^2}{10}[V_k\delta_{ij} + V_j\delta_{ik} + V_i\delta_{jk}] + \frac{a^2}{35}\rho_C(w_l - u_l)[E_{Clk}\delta_{ij} + E_{Clj}\delta_{ik} + E_{Cli}\delta_{jk}] \\ &\quad - \frac{a^2}{14}\rho_C[(w_i - u_i)E_{Cjk} + (w_j - u_j)E_{Cik} + (w_k - u_k)E_{Cij}]. \end{aligned} \tag{55}$$

The only contribution to $\mathbf{R}_C$ is from the steady term which gives

$$R_{Cablm} = -\frac{13a^2}{168}\rho_C(\delta_{ab}\delta_{lm}+\delta_{al}\delta_{bm}+\delta_{am}\delta_{bl})[(u_k-w_k)(u_k-w_k)-M_{Dkk}]$$
$$+\frac{3a^2}{280}\rho_C\{[(u_a-w_a)(u_b-w_b)-M_{Dab}]\delta_{lm}+[(u_a-w_a)(u_l-w_l)-M_{Dal}]\delta_{bm}$$
$$+[(u_a-w_a)(u_m-w_m)-M_{Dam}]\delta_{bl}+[(u_b-w_b)(u_l-w_l)-M_{Dbl}]\delta_{am}$$
$$+[(u_b-w_b)(u_m-w_m)-M_{Dbm}]\delta_{al}+[(u_l-w_l)(u_m-w_m)-M_{Dlm}]\delta_{ab}\}. \tag{56}$$

All the previous results are objective except for $\mathbf{V}$ that can be made objective by replacing the expression (52) by (Zhang and Prosperetti, 1994a)

$$\mathbf{V} = \tfrac{1}{2}\rho_C\left[\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}-\frac{\partial \mathbf{w}}{\partial t}-\mathbf{w}\cdot\nabla\mathbf{w}+\frac{1}{nv}\nabla(nv\mathbf{M}_D)\right]+\tfrac{1}{2}\rho_C(\nabla\times\mathbf{u})\times(\mathbf{w}-\mathbf{u}). \tag{57}$$

The last term represents a lift force (see, e.g., Auton *et al.*, 1988) that is seen to affect $\mathbf{S}_C$ in addition to $\mathbf{A}_D$. This result has not been noted before.

6. The Linear Problem

It may be of some interest to consider explicitly the simplest application of the results of the previous section, namely the linear case. As before, we omit the average symbols and write $\mathbf{u}$ for $\mathbf{u}_C$. For the sake of greater generality, we also consider particles with variable radius (Zhang and Prosperetti, 1994b). It is readily shown that, to the linear approximation, this additional feature leaves $\mathbf{A}_D$ and $\mathbf{S}_C$ unchanged, while $\mathbf{T}_C$ gets modified by the addition of

$$-\rho_C\beta_D a\ddot{a}\mathbf{I}, \tag{58}$$

where the dots denote partial derivatives with respect to time. Similarly, the unsteady part of $\mathbf{R}_C$ is no longer zero but is given by

$$R_{Cablm} = -\frac{a^2}{30}\rho_C(\delta_{ab}\delta_{lm}+\delta_{al}\delta_{bm}+\delta_{am}\delta_{bl})a\ddot{a}. \tag{59}$$

The disperse-phase momentum equation (22), upon substitution of (51) and linearization, is

$$nv\rho_D\frac{\partial \mathbf{w}}{\partial t} = -nv\nabla\left(1+\frac{a^2}{10}\nabla^2\right)p_C - \tfrac{1}{2}nv\rho_C\left(\frac{\partial \mathbf{w}}{\partial t}-\frac{\partial \mathbf{u}}{\partial t}\right). \tag{60}$$

The continuous-phase equation (27), after substitution of the expression (29) for $\mathbf{A}_C$ and linearization, becomes

$$\beta_C\rho_C\frac{\partial \mathbf{u}}{\partial t} = -\beta_C\nabla p_C+\rho_C\nabla\left(1+\frac{a^2}{10}\nabla^2\right)(nva\ddot{a})+\left[1+\frac{a^2}{10}\nabla\nabla\cdot+\frac{a^2}{20}\nabla^2\right]\left[\tfrac{1}{2}nv\rho_C\left(\frac{\partial \mathbf{w}}{\partial t}-\frac{\partial \mathbf{u}}{\partial t}\right)\right]. \tag{61}$$

It should be kept in mind that both these equations are affected by an error of order $o(nv)$.

Equation (61) can be solved for ∇p_C and substituted into (60) to eliminate the pressure gradient. In so doing however due attention must be paid to the fact that all terms of order $o(nv)^2$ should be dropped for consistency. The result is then

$$\frac{\partial \mathbf{w}}{\partial t} = \frac{3\rho_C}{2\rho_D+\rho_C}\left(1+\frac{a^2}{15}\nabla^2\right)\frac{\partial \mathbf{u}}{\partial t}. \tag{62}$$

The second term in parenthesis constitutes a correction to the standard result obtained in the limit $a/L\to 0$ (Landau and Lifshitz, 1959). In particular, for bubbles ($\rho_D = 0$) we have

$$\frac{\partial \mathbf{w}}{\partial t} = 3\left(1+\frac{a^2}{15}\nabla^2\right)\frac{\partial \mathbf{u}}{\partial t}, \tag{63}$$

which generalizes a well-known relation. These relations show that the usual added mass should be corrected by the addition of a differential operator to account for effects of order a^2/L^2.

To illustrate this point further we derive an equation for the mixture momentum, with $\rho_D = 0$ and a = constant for simplicity. By adding (60) and (61) and using (10) we have

$$\rho_C\beta_C\left(1+\frac{a^2}{10}\nabla^2\beta_D\right)\frac{\partial \mathbf{u}}{\partial t}-\frac{a^2}{10}\left[\beta_D\nabla^2\frac{\partial \mathbf{u}}{\partial t}+\left(\nabla\nabla\cdot+\tfrac{1}{2}\nabla^2\right)\left(\beta_D\frac{\partial \mathbf{u}}{\partial t}\right)\right]=-\nabla p_C. \tag{64}$$

If the left-hand side is considered in some sense as the inertia of the mixture, this relation may be interpreted by saying that, when finite-size effects are significant, the equivalent mixture density is not an intrinsic property of the mixture, but depends on the flow itself. In other words, the density becomes an operator. A similar conclusion has been reached for interacting point-like particles by Felderhof and Ooms (1989).

7. Effective Properties

Mathematically, transport processes in continua are described in terms of the divergence of suitable fluxes. At a particle interface these fluxes are discontinuous, which introduces a local singularity. After averaging upon the particle positions, the singularities get "smeared" into an effective source field distributed throughout the continuous phase. This fact has the consequence that spatial differentiation and averaging for a field f_C do not commute as the divergence of the average field cannot be directly sensitive to the particles, while the divergence of the microscopic field is. If the particles are considered as points, it would then be expected that the difference $[\nabla\langle f_C\rangle - \langle\nabla f_C\rangle](\mathbf{x})$ would equal a term giving the local effect of a particle at $\mathbf{x}$ weighted by the probability of finding a particle there. Actually, for equal spherical particles, the exact relation is (Zhang and Prosperetti, 1994a)

$$\nabla\langle f_C\rangle = \langle\nabla f_C\rangle + \frac{1}{\beta_C}\int d^3w\int_{|\mathbf{x}-\mathbf{y}|=a} dS_y P(\mathbf{y},\mathbf{w},t)[\langle f_C\rangle_1(\mathbf{x},t|\mathbf{y},\mathbf{w}) - \langle f_C\rangle(\mathbf{x},t)]\,\mathbf{n}. \tag{65}$$

As remarked previously, the integral here is over all the particles touching the field point $\mathbf{x}$ rather than over the surface of a particle centered at $\mathbf{x}$, as the point-particle model would lead one to expect. As in the step leading to (29), a Taylor series expansion in the position of the particle center can be carried out to find

$$\begin{aligned}\beta_C\nabla\langle f_C\rangle &= \beta_C\langle\nabla f_C\rangle + \beta_D\mathscr{A}[f_C] \\ &\quad -\nabla\cdot\{\beta_D\mathscr{T}[f_C] + \nabla\cdot(\beta_D\beta_D\mathscr{S}[f_C]) + \nabla\nabla:(\beta_D\beta_D\mathscr{R}[f_C])\} + O\left(\frac{a^3}{L^4}\beta_D\langle f_C\rangle_1\right),\end{aligned} \tag{66}$$

where (cf. (30))

$$\beta_D\mathscr{A}[f_C] = \int d^3w\int_{|\mathbf{x}-\mathbf{z}|=a} dS_z P(\mathbf{x},\mathbf{w},t)[\langle f_C\rangle_1(\mathbf{z},t|\mathbf{x},\mathbf{w}) - \langle f_C\rangle(\mathbf{z},t)]\mathbf{n}, \tag{67}$$

$$\beta_D\mathscr{T}[f_C] = a\int d^3w\int_{|\mathbf{x}-\mathbf{z}|=a} dS_z P(\mathbf{x},\mathbf{w},t)\,\mathbf{n}[\langle f_C\rangle_1(\mathbf{z},t|\mathbf{x},\mathbf{w}) - \langle f_C\rangle(\mathbf{z},t)]\mathbf{n}. \tag{68}$$

$\mathscr{S}[f_C]$ and $\mathscr{R}[f_C]$ are given by similar integrals with one and two more factors of $\mathbf{n}$, respectively (cf. (31) and (32)). The integrals are now over the surface of the particle centered at $\mathbf{x}$ and, to the extent that the first one mathematically would represent a source density (since it does not appear under a divergence sign in (66)), the second and higher ones may be considered finite-size corrections. This interpretation is not as clear-cut as in the previous section as the term $\mathscr{T}[f_C]$ can give leading-order contributions proportional to the same power of a as $\mathscr{A}[f_C]$. However, this very fact appears at first sight rather surprising as it might be expected that a shift by a distance equal to the particle radius would only introduce higher-order corrections. As we now show with some examples, a considerable amount of physics is actually contained in the term $\mathscr{T}[f_C]$ and similar ones in the right-hand side of (66). We only include terms up to $O(\beta_D)$ and, as before, we write $\mathbf{u}$, $\mathbf{w}$ in place of $\langle\mathbf{u}_C\rangle$, $\bar{\mathbf{w}}$.

As our first example we mention the case of spherical, variable radius bubbles (Zhang and Prosperetti, 1994b). It can be shown that, upon averaging the normal stress relation across the bubble surfaces and neglecting surface tension and viscosity for simplicity, one finds

$$\overline{p_{\mathrm{B}}} = \langle p_{\mathrm{C}} \rangle - \tfrac{1}{3}\mathrm{Tr}\{\mathscr{T}[p_{\mathrm{C}}]\}, \tag{69}$$

where p_{B} is the pressure on the liquid side of the bubble surface and Tr denotes the trace. The "shift" correction due to $\mathscr{T}[p_{\mathrm{C}}]$ is clearly seen here to augment the mean pressure in the liquid and thus to behave as an additional stress exerting a very real influence on the bubble dynamics. To $O(\beta_{\mathrm{D}})$ we have

$$\frac{1}{\rho_{\mathrm{C}}}\mathscr{T}[p_{\mathrm{C}}] = [\tfrac{2}{5}(\mathbf{u}-\mathbf{w})^2 - \overline{a\ddot{a}} - \tfrac{3}{2}\overline{\dot{a}^2} - \tfrac{2}{5}\mathrm{Tr}\,\mathbf{M}_{\mathrm{D}}]\mathbf{I} - \tfrac{9}{20}[(\mathbf{u}-\mathbf{w})(\mathbf{u}-\mathbf{w}) - \mathbf{M}_{\mathrm{D}}], \tag{70}$$

which, to this order, coincides with the previous result (54) for rigid particles if the radius derivatives are set to zero. Upon substitution into (69) we find the modified Rayleigh–Plesset equation

$$\overline{a\ddot{a}} + \tfrac{3}{2}\overline{\dot{a}^2} = \frac{\overline{p_{\mathrm{B}}} - \langle p_{\mathrm{C}} \rangle}{\rho_{\mathrm{C}}} + \tfrac{1}{4}(\mathbf{u}-\mathbf{w})^2 - \tfrac{1}{4}\mathrm{Tr}\,\mathbf{M}_{\mathrm{D}}, \tag{71}$$

as first shown by Biesheuvel and van Wijngaarden (1984) whose method however could not produce the contribution $\mathbf{M}_{\mathrm{D}}$ of the particle Reynolds stress. It is readily shown that inclusion of the next correction in the particle size only has the effect of replacing $\langle p_{\mathrm{C}} \rangle$ by $(1 + \frac{1}{6}a^2\nabla^2)\langle p_{\mathrm{C}} \rangle$.

As a second example we consider a viscous suspension at small particle Reynolds number (Zhang and Prosperetti, 1995). In this case the continuous-phase momentum equation is, aside from gravity and collisions,

$$\begin{aligned}\rho_{\mathrm{C}}\left[\frac{\partial}{\partial t}(\beta_{\mathrm{C}}\mathbf{u}) + \nabla\cdot(\beta_{\mathrm{C}}\langle \mathbf{uu} \rangle)\right]\\ = \beta_{\mathrm{C}}\nabla\cdot\langle \boldsymbol{\sigma}_{\mathrm{C}} \rangle - \beta_{\mathrm{D}}\mathscr{A}[\boldsymbol{\sigma}_{\mathrm{C}}] + \nabla\cdot\{\beta_{\mathrm{D}}\mathscr{T}[\boldsymbol{\sigma}_{\mathrm{C}}] + \nabla\cdot(\beta_{\mathrm{D}}\mathscr{S}[\boldsymbol{\sigma}_{\mathrm{C}}]) + \nabla\nabla:(\beta_{\mathrm{D}}\mathscr{R}[\boldsymbol{\sigma}_{\mathrm{C}}])\},\end{aligned} \tag{72}$$

where $\boldsymbol{\sigma}_{\mathrm{C}} = -p_{\mathrm{C}}\mathbf{I} + 2\mu_{\mathrm{C}}\mathbf{e}_{\mathrm{C}}$ is the microscopic stress tensor and $\mathbf{e}_{\mathrm{C}}$ is the microscopic rate of strain. Again applying the differentiation rule (66), we find

$$2\beta_{\mathrm{C}}\langle \mathbf{e}_{\mathrm{C}} \rangle = 2\beta_{\mathrm{C}}\mathbf{E}_{\mathrm{C}} - \{\beta_{\mathrm{D}}\mathscr{A}[\mathbf{u}] - \nabla\cdot(\beta_{\mathrm{D}}\mathscr{T}[\mathbf{u}]) - \nabla\nabla:(\beta_{\mathrm{D}}\mathscr{S}[\mathbf{u}])\} - \{\cdots\}^{\mathrm{T}}, \tag{73}$$

in which $\mathbf{E}_{\mathrm{C}}$ is the rate of strain of the average velocity field defined in (43) and $\{\cdots\}^{\mathrm{T}}$ denotes the transpose of the tensor in braces. The viscous stress tensor in the mixture $\boldsymbol{\Sigma}_v$ contains contributions not only from $\langle \mathbf{e}_{\mathrm{C}} \rangle$, but also from the terms in the second line of (72). All these terms contribute to the "particle stress" introduced by Batchelor (1970). Their explicit evaluation shows that, to first order in β_{D}, $\boldsymbol{\Sigma}_v$ is (Zhang and Prosperetti, 1995)

$$\boldsymbol{\Sigma}_v = 2\mu_{\mathrm{eff}}\mathbf{E}_m + \frac{\mu_{\mathrm{C}}^2\beta_{\mathrm{D}}}{2(\mu_{\mathrm{C}}+\mu_{\mathrm{D}})}\nabla\cdot[\beta_{\mathrm{D}}(\mathbf{w}-\mathbf{u})]\mathbf{I} + \frac{3}{4}\frac{\mu_{\mathrm{C}}\mu_{\mathrm{D}}}{\mu_{\mathrm{C}}+\mu_{\mathrm{D}}}\nabla[\beta_{\mathrm{D}}(\mathbf{w}-\mathbf{u})], \tag{74}$$

where μ_{C}, μ_{D} are the phase viscosities,

$$\mathbf{E}_m = \tfrac{1}{2}[\nabla\mathbf{u}_m + (\nabla\mathbf{u}_m)^{\mathrm{T}}], \qquad \mathbf{u}_m = \beta_{\mathrm{C}}\mathbf{u} + \beta_{\mathrm{D}}\mathbf{w}, \tag{75}$$

and

$$\frac{\mu_{\mathrm{eff}}}{\mu_{\mathrm{C}}} = 1 + \tfrac{5}{2}\beta_{\mathrm{D}}\frac{\beta_{\mathrm{D}} + \frac{2}{5}\mu_{\mathrm{C}}}{\mu_{\mathrm{D}} + \mu_{\mathrm{C}}} \tag{76}$$

is the effective viscosity (Landau and Lifshitz, 1959; Batchelor, 1970, 1974). It will be noticed that (74) expresses a non-Newtonian relation between the stress and the strain of the suspension. Thus, viewing the mixture as a single fluid with effective properties different from those of the pure solvent is an oversimplification.

These results are critically dependent on the shift correction due to the group of terms under the divergence operator in (66). Note that the effective viscosity correction $(\mu_{\mathrm{eff}} - \mu_{\mathrm{C}})\mathbf{E}_{\mathrm{C}}$ in $\boldsymbol{\Sigma}_v$ is of the same order of magnitude as the other terms. Previous investigators (see especially Batchelor, 1970) found the correction to μ_{C} because their procedure was equivalent to retaining $\mathscr{T}[\boldsymbol{\sigma}_{\mathrm{C}}]$. However, they missed the other, equally important, terms, as the other shift corrections were not properly accounted for.

We consider next steady heat conduction in a composite. Upon averaging, the steady continuous phase energy equation is (Zhang and Prosperetti, 1995)

$$-\beta_C \nabla \cdot \langle \mathbf{q}_C \rangle + \beta_D \mathscr{A}[\mathbf{q}_C] - \nabla \cdot \{\beta_D \mathscr{T}[\mathbf{q}_C] + \nabla \cdot (\beta_D \mathscr{S}[\mathbf{q}_C]) + \nabla\nabla : (\beta_D \mathscr{R}[\mathbf{q}_C])\} = 0, \tag{77}$$

where $\mathbf{q}_C$ is the heat flux for which, from Fourier's law and (66), we find

$$\langle \mathbf{q}_C \rangle = -K_C \nabla \langle T_C \rangle + \frac{K_C}{\beta_C} \{\beta_D \mathscr{A}[T_C] - \nabla \cdot (\beta_D \mathscr{T}[T_C]) - \nabla\nabla : \{(\beta_D \mathscr{S}[T_C])\}, \tag{78}$$

with K_C the thermal conductivity. To first order in β_D we have

$$\mathscr{A}[T_C] = -\frac{K_D - K_C}{K_D + 2K_C} \nabla \langle T_C \rangle, \tag{79}$$

$$\mathscr{T}[\mathbf{q}_C] = -2K_C \frac{K_D - K_C}{K_D + 2K_C} \nabla \langle T_C \rangle, \tag{80}$$

while $\mathscr{T}[T_C]$, $\mathscr{A}[\mathbf{q}_C]$, and the other terms give contributions of higher order in a/L. Substituting these results into (77) we find

$$\nabla \cdot (K_{\text{eff}} \nabla \langle T_C \rangle) = 0. \tag{81}$$

The effective thermal conductivity

$$\frac{K_{\text{eff}}}{K_C} = 1 + \frac{3\beta_D (K_D - K_C)}{K_D + 2K_C} \tag{82}$$

(Jeffrey, 1973; Batchelor, 1974) arises from the combination $K_C \nabla \langle T_C \rangle + (\beta_D/\beta_C) \mathscr{A}[T_C] + (1/\beta_C) \nabla \cdot (\beta_D \mathscr{T}[\mathbf{q}_C])$ for which the shift correction is determinant. The other shift terms happen to vanish due to the high symmetry of the spherical inclusions, but they would give nonzero contributions, e.g., for ellipsoidal particles.

8. Summary and Conclusions

We have considered some implications of the finite size of the particles on averaged-equations models for disperse two-phase flow. We have pointed out that the often-ignored difference between β_D and nv leads to a striking change in the mathematical structure of the equations at short scales with implications for stability and hyperbolicity. Next we have considered the case of a dilute suspension of "large" particles in a potential flow. The last section examined on some examples the consequences of a shift of the particle centers over distances equal to their radius.

In addition to presenting specific results, one of the aims of this paper is to demonstrate the flexibility and effectiveness of the new "ZIP" averaging technique that is briefly described in Section 4. For example, the large-particle results of Section 5 are readily and systematically derived by our method.

We have only considered the dilute limit. Finite-size effects may be expected to become even stronger at higher concentrations.

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It is a distinct honor and a great pleasure to dedicate this paper to Professor Daniel D. Joseph on the occasion of his 65th birthday. Since his early paper "Incompatibility of Beltami Flows with Viscous Adherence" (*Phys. Fluids* **7**, 648–651, 1964), for 30 years, Professor Joseph's intensely original work has been a source of inspiration, education, and information for a large community of fluid mechanicians and applied mathematicians. However, he cannot stop. All of his many admirers are eagerly waiting for the next gems issuing from his brilliantly creative mind and astonishing energy level. In wishing him many more happy birthdays, the senior author (AP) is grateful for the good fortune of having been able to enjoy Professor Joseph's intriguing comments, provocative advice, and warm friendship over the last decade.

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