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Review Paper

Flow of Viscoelastic Fluids Between Rotating Disks[1]

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Abstract. Few boundary-value problems in fluid mechanics can match the attention that has been accorded to the flow of fluids, Newtonian and non-Newtonian, between parallel rotating disks rotating about a common axis or about distinct axes. An interesting feature which has been recently observed is the existence of solutions that are not axially symmetric even in the case of flow due to the rotation of disks about a common axis. In this article we review the recent efforts that have been expended in the study of both symmetric and asymmetric solutions in the case of both the classical linearly viscous fluid and viscoelastic fluids.

1. Introduction

In 1921 von Karman [141] used a similarity transformation to study the steady axially symmetric swirling flow of the classical linearly viscous fluid, induced by the rotation of an infinite disk. Later, Batchelor [8] showed that such a similarity transformation would be appropriate for studying the flow of a linearly viscous fluid between two infinite parallel disks, rotating with constant but differing angular speeds, about a common axis (see Figure 1). These two works have been followed by extensive studies on the swirling flow of the classical linearly viscous fluid, the like of which has been accorded to few problems in fluid mechanics. These studies cover a broad spectrum ranging from those which are concerned with the physics and fluid mechanics of the problem to those which address rigorous mathematical questions regarding existence and uniqueness of solutions. The problems have also been used as test problems for numerical schemes and in the study of matched asymptotic expansions. Such intensive studies notwithstanding, several basic questions regarding the flows remain unanswered and the analysis of the problem is far from complete.

The recent works of Berker [12] and Parter and Rajagopal [93] have exacerbated the situation. Breaking away from the approaches of von Karman [141] and Batchelor [8] which assumed axial symmetry, Berker [12] considered the possibility of solutions that are not necessarily axially symmetric and established a one-parameter family of solutions for the flow of the classical linearly viscous fluid between two plane parallel disks rotating about a common axis with the *same* angular speed. The only axially symmetric solution in this family is the rigid-body motion; the only solution that would follow from the classical assumptions of von Karman. However, Berker [12] did not investi-

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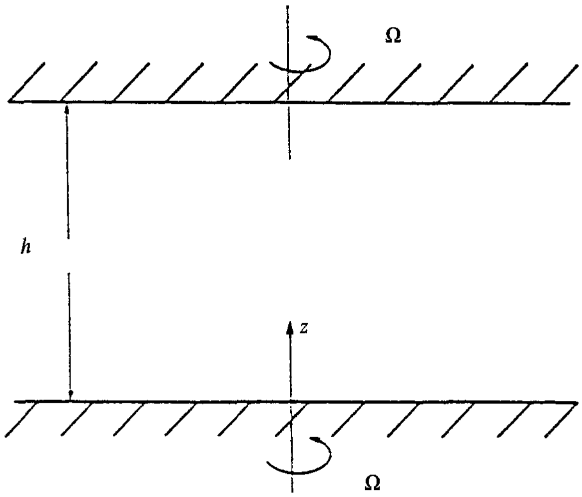

Figure 1. Flow domain—disks rotating about a common axis.

gate the implications of his study when the angular speed of the two disks are distinct in which case there is also a flow in the axial direction. More importantly, the boundary-value problem studied by Berker is linear while the problem governing the rotation of the two disks with distinct angular speeds is nonlinear.

In the light of Berker's work, Parter and Rajagopal [93] re-examined the problem of flow of the classical linearly viscous fluid between parallel disks rotating about a common axis with *differing* angular speeds. Parter and Rajagopal [93] rigorously proved that the problem admits solutions that lack axial symmetry and that the axially symmetric solutions are never isolated when considered within the full scope of the Navier–Stokes equation. Similar results apply to the case of the flow due to a single rotating disk and flow due to rotating disks subject to suction or injection at the disk. Based on the existence theorems of Parter and Rajagopal [93], extensive numerical computations have been carried out recently [79], [80].

An interesting related problem is the possibility of existence of such asymmetric solutions in the case of a viscoelastic fluid due to the rotation of a single disk or due to the rotation of two disks. As with the classical linearly viscous fluid flow problem, until recently the investigations have been concerned with the study of axially symmetric solutions. Motivated by the work of Berker [10], Rajagopal and Gupta [104] have examined the possibility of existence of asymmetric solutions for the flow of a special subclass of the fluids of the differential type between parallel plates rotating with the same angular velocity, about a common axis. More recently, Huilgol and Rajagopal [66] have established the possibility of asymmetric solutions for the flow of a popular class of viscoelastic fluids of the rate type, between parallel plates rotating with differing angular speeds. We discuss these and other more recent results on the steady asymmetric flow of viscoelastic fluids due to two rotating disks.

The results of Berker [12] have relevance to another very interesting application in fluid dynamics, the flow occurring in the orthogonal rheometer [84]. The apparatus consists of two parallel disks which rotate with the same constant angular speed about two parallel but different axes (see Figure 2). The fluid to be tested fills the space between the plates. If the fluid is non-Newtonian, then normal stress differences develop due to the flow and measuring these will help in characterizing the fluid that fills the apparatus. The motion occurring in such an instrument has been studied by several authors and all the early works ignored the inertial effects in the treatment of the problem [64]. Abbot and Walters [1] were the first to include inertial effects and they obtained an exact solution in the case of the Navier–Stokes fluid. They also studied the flow of a viscoelastic fluid in the same domain by means of a perturbation analysis by expanding in a power series in the distance between the axes of rotation.

Rajagopal[2] [101] recognized that a velocity field similar to that used by Berker [12] can be

[2] Goddard [51] later independently established results which are essentially the same.

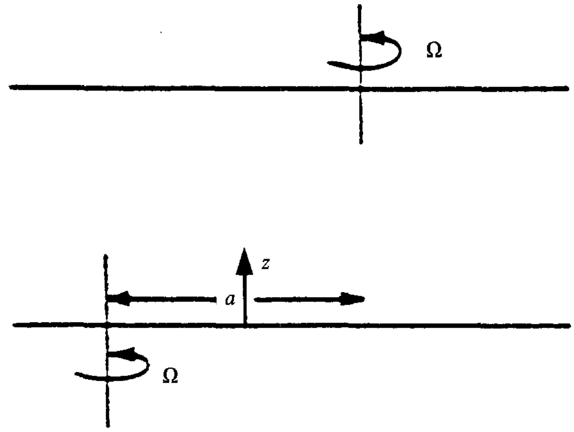

Figure 2. Flow domain—disks rotating about distinct axes.

employed in this problem and that the velocity field assumed by Berker [12] was a motion with constant principal relative stretch history. He used this fact to show that the flow of any homogeneous incompressible simple fluid in such a configuration is governed by a second-order partial differential equation (even in the case of fluids with finite memory wherein the Cauchy stress is given by an integral representation). Thus, unlike other boundary-value problems in which additional boundary conditions might be required for specific non-Newtonian fluid models of the differential type [102], [106], [107], [69], the adherence boundary condition is sufficient for determinancy. The problem being well posed, we can discuss issues concerning existence, uniqueness, and other related questions. Detailed numerical computations have been carried out recently for specific integral constitutive models and these results are also discussed in this article.

There has been such a great deal of work with regard to symmetric solutions for the linearly viscous fluid and for specific non-Newtonian fluid models due to a single rotating disk and due to two parallel disks rotating about a common axis or about noncoincident axes, that it would be impossible to discuss all of these here. In this review article we restrict our discussion to the flow between parallel disks,[3] present some of the recent results for the flow of both Newtonian and non-Newtonian fluids, and discuss questions that remain unanswered.

2. Flow Between Parallel Disks Rotating About a Common Axis

2a. Axially Symmetric Solutions

An up-to-date review of the numerical and mathematical work on the axially symmetric solutions for a Navier–Stokes fluid can be found in the review article by Parter [92]. Few boundary-value problems within the context of the Navier–Stokes theory have been the object of the kind of intense scrutiny that has been accorded to this problem and it has occupied a central place in the Navier–Stokes theory by continuing to attract interest to this day. Even early in the game, it became quite apparent that the problem would present the opportunity for the study of bifurcation and nonuniqueness of solutions, for Batchelor [8] predicted that at high Reynolds numbers in the case of flow between two disks rotating about a common axis, boundary layers would develop adjacent to the disks with the core rotating with constant angular speed, while Stewartson [130] reasoned that at high Reynolds numbers the flow in the core would be purely axial. This early disagreement set the stage for the intensive studies that have followed which show that not only are both Batchelor-type and Stewartson-type solutions possible, but also solutions that do not fall into either category. The analytical, numerical, and asymptotic studies (see [2], [9], [27], [35], [41], [56], [58], [60]–[63], [74], [77], [78], [81], [83], [85], [86], [88], [91], [94], [95], [116], [122], [124], [135], [145], [147], and [148]) within the context of the Navier–Stokes theory are too numerous to discuss in detail. After presenting the similarity transformation used by von Karman [141], which forms the backbone

[3] We comment on problems related to the flow due to a single disk, if relevant, but these comments are minimal.

for the problem under consideration, we turn our discussion to the flow of viscoelastic fluids between rotating plates.

Von Karman [141] assumed an axially symmetric velocity field of the form

$$v_r = rF', \qquad v_\theta = rG, \quad \text{and} \quad v_z = -2F. \tag{2.1}$$

Here, v_r, v_θ, and v_z denote the components of the velocity in the r, θ, and z directions, respectively. We notice that the velocity field (2.1) automatically satisfies the constraint of incompressibility.

Substituting (2.1) into the Navier–Stokes equation leads to the celebrated von Karman equations:

$$\varepsilon F^{\text{iv}} + 2FF''' + 2GG' = 0, \tag{2.2}$$

$$\varepsilon G'' + 2FG' - 2F'G = 0, \tag{2.3}$$

where $\varepsilon = (\mu/\rho)$.

If we are interested in the flow due to the two rotating disks at $z = h$ and $z = 0$, then (2.2) and (2.3) would be valid in the interval $0 \leq z \leq h$. The appropriate boundary conditions for the problem are

$$F(0, \varepsilon) = F(h, \varepsilon) = 0 \qquad \text{(no penetration)}, \tag{2.4}$$

$$F'(0, \varepsilon) = F'(h, \varepsilon) = 0 \qquad \text{(adherence)}, \tag{2.5$_1$}$$

$$G(0, \varepsilon) = \Omega_0, \quad G(h, \varepsilon) = \Omega_{+h} \qquad \text{(adherence)}, \tag{2.5$_2$}$$

where Ω_0 and Ω_{+h} are the angular speeds of the disks at $z = 0$ and $z = h$, respectively. References [2], [9], [27], [35], [41], [56], [58], [60]–[63], [74], [77], [78], [81], [83], [85], [86], [88], [91], [92], [94], [95], [116], [122], [124], [130], [135], [145], [147], and [148] are but a few of the many studies on the system of equations (2.2)–(2.5).

We now turn our attention to a discussion of the flow of non-Newtonian fluids due to rotating plates. Srivastava [129] studied the flow of a Reiner–Rivlin fluid between rotating parallel plates using a perturbation approach. Bhatnagar [17] studied the flow between two disks of a Reiner–Rivlin fluid in which one disk is stationary and the other rotating. This model was introduced by Reiner [117] to describe the behavior of wet sand but was at one time considered as a possible model for non-Newtonian fluid behavior. The model does not account for the possibility of both normal stress differences or shear-thinning or shear-thickening and is not currently considered as a viable model for viscoelastic fluids. Erdogan [43][4] seems to have been the first to study the flow of a fluid of second grade, a model [140] which allows for both the normal stress differences, due to a rotating disk, with the fluid at infinity also rotating with an angular speed. Later, Bhatnagar and Zago [21] studied the flow of a fluid of second grade due to two rotating disks, about a common axis.

The Cauchy stress $\mathbf{T}$ in an incompressible homogeneous fluid of second grade is given by

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.6}$$

where the spherical part of the stress $-p\mathbf{1}$ is due to the constraint of incompressibility, μ is the viscosity, α_1 and α_2 are the normal stress moduli, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematical tensors [121] given by

$$\mathbf{A}_1 = (\text{grad } \mathbf{v}) + (\text{grad } \mathbf{v})^{\mathrm{T}} \tag{2.7$_1$}$$

and

$$\mathbf{A}_2 = \frac{d}{dt}(\mathbf{A}_1) + \mathbf{A}_1(\text{grad } \mathbf{v}) + (\text{grad } \mathbf{v})^{\mathrm{T}}\mathbf{A}_1, \tag{2.7$_2$}$$

where d/dt denotes the usual material time derivative. A detailed study of the thermomechanics of the fluids modeled by (2.6) was carried out by Dunn and Fosdick [39]. The signs of the material moduli α_1 and α_2 are the subject of much controversy (see [48]–[50] and [40]), and we shall not discuss this here.

The equations governing the flow of fluids of second grade are of higher order than the Navier–Stokes equations because of the presence of the term $d\mathbf{A}_1/dt$ in the expression for the stress and, since

[4] Flow of non-Newtonian fluids due to the rotation of a single disk has also been studied by Balaram and Sastri [6] and Balaram and Luthra [5].

only the adherence boundary condition obtains, we do not have enough boundary conditions to make the problem determinate. To overcome this difficulty, Erdogan [43] perturbs in terms of the parameter which multiplies the highest-order term in the equation, thereby reducing the order of the problem but, however, treating a singular perturbation as though it were regular. Bhatnagar and Zago [21] use a numerical method which treats the higher-order terms in the equation as a lower iterate, essentially once again lowering the order of the equation. A plausible way out of the impasse of additional boundary conditions of fluids of the differential type of grade n, $n > 1$, is the presence of a thin layer adjacent to any solid surface wherein the fluid behaves as though it is a Navier–Stokes fluid. In this case there would be no problems with boundary conditions. This is, of course, a conjecture that may or may not be borne out physically.

Using the similarity transformation (2.1), Phan-Thien [97] studied the time-dependent flow of a Maxwell fluid between two disks rotating about a common axis. This was followed by a study by Bhatnagar and Parera [20] of an Oldroyd-B fluid [90] between two disks, one of which is rotating, the other being held fixed, and that of Phan-Thien [98]. Huilgol and Keller [65] set up the equations governing the flow of an Oldroyd-B fluid between two disks both rotating about a common axis and presented a numerical scheme for handling the problems. Walsh [143] used the formulation of Huilgol and Keller [65], and restricting his analysis primarily to a Maxwell model, studied the problem in great detail. He found subcritical bifurcation when the Weissenberg number W was 1.405. Walsh [143] also studied the nature of the bifurcation for various values of the Ekman and Weissenberg numbers.

Recently, Ji *et al.* [67] have carried out a detailed investigation of the flow an Oldroyd-B fluid in the above geometry, and as it includes the Maxwell model studied by Walsh [143] as a special case, we discuss this in some detail below. The Oldroyd-B fluid [90] is defined through

$$\mathbf{T} = -p\mathbf{1} + \mathbf{S}, \tag{2.8}$$

with

$$\mathbf{S} + \Lambda_1\left(\frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^T\right) = \mu\left[\mathbf{A}_1 + \Lambda_2\left(\frac{d\mathbf{A}_1}{dt} - \mathbf{LA}_1 - \mathbf{A}_1\mathbf{L}^T\right)\right], \tag{2.9}$$

where μ is the viscosity, and Λ_1 and Λ_2 are material constants which have the units of time and are called relaxation time and retardation time, respectively. We note that when $\Lambda_1 = \Lambda_2 = 0$, the model (2.8)–(2.9) reduces to the classical linearly viscous model, while when $\Lambda_2 = 0$, it reduces to the Maxwell model.

It follows from (2.1) and (2.9) that [67]

$$\begin{aligned} &S_{rr} + \Lambda_1[r(F'S_{rr,r} - 2F''S_{zr}) - 2F'S_{rr} - 2FS_{rr,z} + GS_{rr,\theta}] \\ &\quad = \mu\{2F' - 2\Lambda_2[r^2F''^2 - 2(F'^2 + FF'')]\}, \end{aligned} \tag{2.10$_1$}$$

$$\begin{aligned} &S_{r\theta} + \Lambda_1[r(F'S_{r\theta,r} - 2F''S_{z\theta} - G'S_{zr}) - 2F'S_{r\theta} - 2FS_{r\theta,z} + GS_{r\theta,\theta}] \\ &\quad = -2\mu\Lambda_2 r^2F''G', \end{aligned} \tag{2.10$_2$}$$

$$\begin{aligned} &S_{rz} + \Lambda_1[r(F'S_{rz,r} - F''S_{zz}) + F'S_{rz} - 2FS_{rz,z} + GS_{rz,\theta}] \\ &\quad = \mu r[F'' + 2\Lambda_2(3F'F'' - FF''')], \end{aligned} \tag{2.10$_3$}$$

$$\begin{aligned} &S_{\theta\theta} + \Lambda_1[r(F'S_{\theta\theta,r} - 2G'S_{z\theta}) - 2F'S_{\theta\theta} - 2FS_{\theta\theta,z} + GS_{\theta\theta,\theta}] \\ &\quad = \mu\{2F' - 2\Lambda_2[r^2G'^2 + 2(F'^2 + FF'')]\}, \end{aligned} \tag{2.10$_4$}$$

$$\begin{aligned} &S_{\theta z} + \Lambda_1[r(F'S_{\theta z,r} - G'S_{zz}) + F'S_{\theta z} - 2FS_{\theta z,z} + GS_{\theta z,\theta}] \\ &\quad = \mu r[G' + 2\Lambda_2(3F'G' - FG'')], \end{aligned} \tag{2.10$_5$}$$

$$\begin{aligned} &S_{zz} + \Lambda_1[rF'S_{zz,r} + 4F'S_{zz} - 2FS_{zz,z} + GS_{zz,\theta}] \\ &\quad = \eta_0[-4F' + 8\Lambda_2(FF'' - 2F'^2)]. \end{aligned} \tag{2.10$_6$}$$

It follows from the assumption of rotational symmetry that $\partial S/\partial\theta = 0$.

On substituting (2.1) and (2.9) into the balance of linear momentum

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt}, \tag{2.11}$$

we obtain

$$\rho r(F'^2 - 2FF'' - G^2) = -\frac{\partial p}{\partial r} + \frac{\partial S_{rr}}{\partial r} + \frac{\partial S_{rz}}{\partial z} + \frac{S_{rr} - S_{\theta\theta}}{r}, \qquad (2.12)_1$$

$$2\rho r(F'G - FG') = \frac{\partial S_{r\theta}}{\partial r} + \frac{\partial S_{\theta z}}{\partial z} + \frac{2}{r} S_{r\theta}, \qquad (2.12)_2$$

$$4\rho FF' = -\frac{\partial p}{\partial z} + \frac{\partial S_{rz}}{\partial r} + \frac{\partial S_{zz}}{\partial z} + \frac{S_{rz}}{r}. \qquad (2.12)_3$$

It is also easy to verify that

$$S = \sum_{n=0} r^n S_n(z), \tag{2.13}$$

where the $\{S_i\}$ have the following representations [67]:

$$S_0(z) = \begin{bmatrix} A(z) & 0 & 0 \\ 0 & B(z) & 0 \\ 0 & 0 & R(z) \end{bmatrix}, \tag{2.14}$$

$$S_1(z) = \begin{bmatrix} 0 & 0 & Z(z) \\ 0 & 0 & P(z) \\ Z(z) & P(z) & 0 \end{bmatrix}, \tag{2.15}$$

$$S_2(z) = \begin{bmatrix} X(z) & Y(z) & 0 \\ Y(z) & Q(z) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.16}$$

$$S_n(z) = 0, \qquad n > 2. \tag{2.17}$$

Let

$$p = p_1(z) + \frac{\rho K}{2} r^2. \tag{2.18}$$

We introduce the following nondimensional quantities:

$$z = d\bar{z}, \qquad F = \Omega_1 df(\bar{z}), \qquad G = \Omega_1 g(\bar{z}), \qquad E = \frac{\nu}{\Omega_1 d^2},$$

$$W = \Omega_1 \Lambda_1, \qquad \lambda_2 = \Omega_1 \Lambda_2, \qquad \beta = \frac{\Lambda_2}{\Lambda_1}, \qquad k = \frac{K}{\Omega_1^2},$$

$$Q = \frac{\mu \Omega_1}{d^2} \bar{Q}, \qquad P = \frac{\mu \Omega_1}{d} \bar{P}, \qquad R = \mu \Omega_1 R,$$

$$X = \frac{\mu \Omega_1}{d^2} \bar{X}, \qquad Y = \frac{\mu \Omega_1}{d^2} \bar{Y}, \qquad Z = \frac{\mu \Omega_1}{d} \bar{Z}, \qquad (2.19)_{1-14}$$

where E is the Ekman number, W is the Weissenberg number, and β is the ratio of retardation time relative to relaxation time.

After dropping the overscore bars, the above set of ordinary differential equations reads [67]:

$$Q - 2W(fQ' + g'P) = -2\beta W g'^2, \qquad (2.20)_1$$

$$P - W(2fP' - 2f'P + g'R) = g' + 2\beta W(3f'g' - fg''), \qquad (2.20)_2$$

$$R - 2W(fR' - 2f'R) = -4f' + 8\beta W(ff'' - 2f'^2), \qquad (2.20)_3$$

$$X - 2W(fX' + f''Z) = -2\beta W f_2''^2, \qquad (2.20)_4$$

$$Y - W(2fY' + g'Z + f''P) = -2\beta W f'' g', \qquad (2.20)_5$$

$$Z - W(2fZ' - 2f'Z + f''R) = f'' + 2\beta W(3f'f'' - ff'''), \qquad (2.20)_6$$

$$3X - Q + Z' = -\frac{2}{E}\left(ff'' - \frac{f'^2}{2} + \frac{g^2}{2} - \frac{k}{2}\right), \qquad (2.20)_7$$

$$4Y + P' = \frac{2}{E}(f'g - fg'). \qquad (2.20)_8$$

The boundary conditions appropriate for the flow under consideration are:

$$f(0) = 0, \quad f'(0) = 0, \quad g(0) = 1, \qquad (2.21)_1$$

$$f(1) = 0, \quad f'(1) = 0, \quad g(1) = s, \qquad (2.21)_2$$

where $s \equiv \Omega_2/\Omega_1$. We have eight differential equations governing the motion of the fluid and the above conditions yield but only six of the required twelve boundary conditions, and thus we have to augment $(2.21)_{1,2}$. We evaluate the stress components Z, P, and R at both $z = 0$ and $z = 1$. Thus,

$$Z(0) = f''(0), \quad Z(1) = f''(1), \qquad (2.22)_1$$

$$P(0) = g'(0), \quad P(1) = g'(1), \qquad (2.22)_2$$

$$R(0) = 0, \quad R(1) = 0. \qquad (2.22)_3$$

The system $(2.20)_{1-8}$ subject to the boundary conditions $(2.21)_{1,2}$ and $(2.22)_{1-3}$ was studied in detail by Ji *et al.* [67] using an analytic continuation method that is suited for nonlinear equations that exhibit turning points and bifurcation points [119], [120], [73]. There are four parameters s, E, W, and β which enter the problem. Thus, if one of them is held fixed, the corresponding solution manifold is three-dimensional. In order to make the problem manageable, Ji *et al.* [67] fix both the ratios of the speed s and the Ekman number E. They find that the solutions exhibit a turning point as the Weissenberg number W is increased for various values of β (see Figure 3). The value of the critical

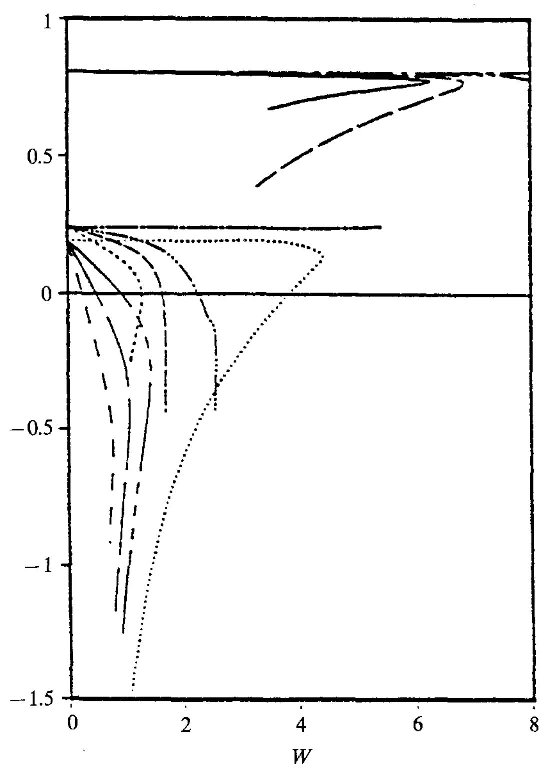

Figure 3. Variation of the Weissenberg number with k, for various values of β. ———, $\beta = 0$, $s = 0.8$; — — —, $\beta = 0.5$, $s = 0.8$; ----—, $\beta = 0.75$, $s = 0.8$; —·—, $\beta = 1.0$, $s = 0.8$; - - - - - -, $\beta = 0$, $s = 0$; ———-, $\beta = 0.5$, $s = 0$; —···—, $\beta = 0.75$, $s = 0$; —·—·—, $\beta = 1.0$, $s = 0$; ---, $\beta = 0$, $s = -1$; ———, $\beta = 0.5$, $s = -1$; ———, $\beta = 0.75$, $s = -1$; .... $\beta = 1$, $s = -1$.

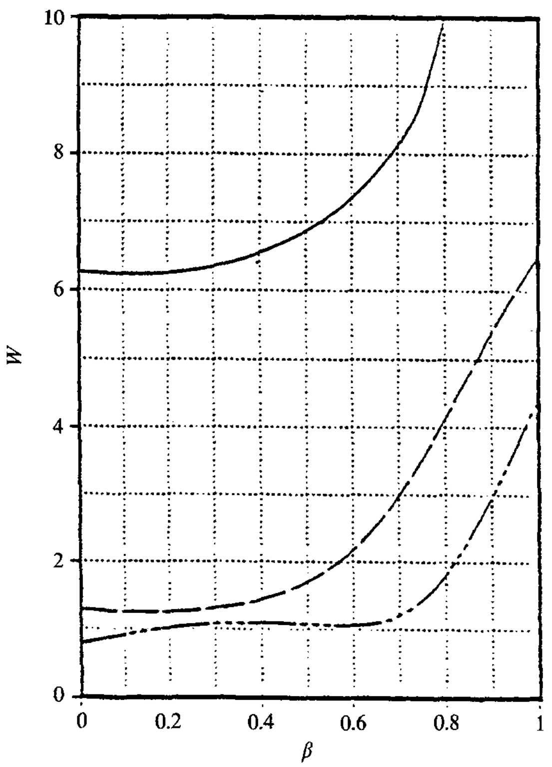

Figure 4. Fold curves on the W-β plane. ———, $s = 0.8$; — — —, $s = 0$; —·—·—, $s = -1$.

Weissenberg number is strongly dependent on β, and this can be seen from the projections on the W–β plane of the fold curves for various values of s, each point on the fold curve being a limit point, where two solution branches coincide (see Figure 4). Representative velocity profiles for the solutions on the two branches are shown in Figures 5 and 6. We note that the velocity profiles corresponding to the multiple solutions are quite different in structure. Ji *et al.* [67] also found that turning points persisted even at very low values of the Ekman number. The variation of the torque on the plates with the Weissenberg number is shown in Figure 7.

We next turn our attention to the recent exhaustive analysis by Crewther *et al.* [31] wherein they study steady and unsteady axisymmetric flows[5] of an Oldroyd-B fluid between rotating plates. Their results on steady flows are in keeping with work of Ji *et al.* [67] in that they also find the existence of

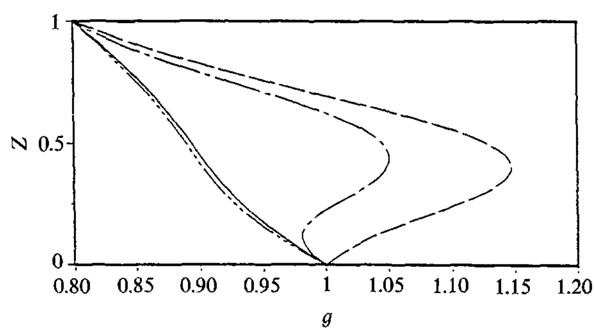

Figure 5. Nondimensional azimuthal velocity profiles. ———, Branch I, $\beta = 0.5$; —·—·—, branch I, $\beta = 0.0$; — — —, branch II, $\beta = 0.5$; —·—, branch II, $\beta = 0.0$.

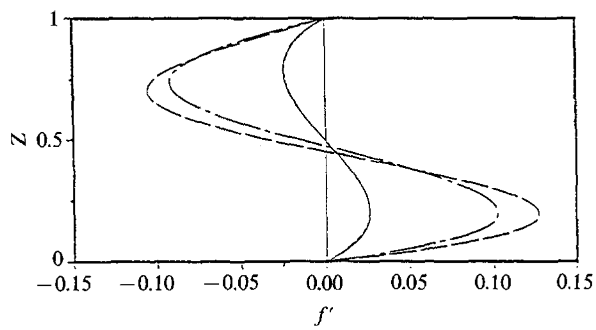

Figure 6. Nondimensional radial velocity profiles. ———, Branch I, $\beta = 0.5$; —·—·—, branch I, $\beta = 0.0$; — — —, branch II, $\beta = 0.5$; —·—, branch II, $\beta = 0.0$.

[5] Thornley [137] presented an exact solution for the unsteady rotating flow of the classical linearly viscous fluid and this and other related problems are discussed in Greenspan [57]. Recently, Rao and Kasivishwanathan [114] have studied unsteady asymmetric flows due to two disks rotating about a common axis within the context of the Navier–Stokes theory.

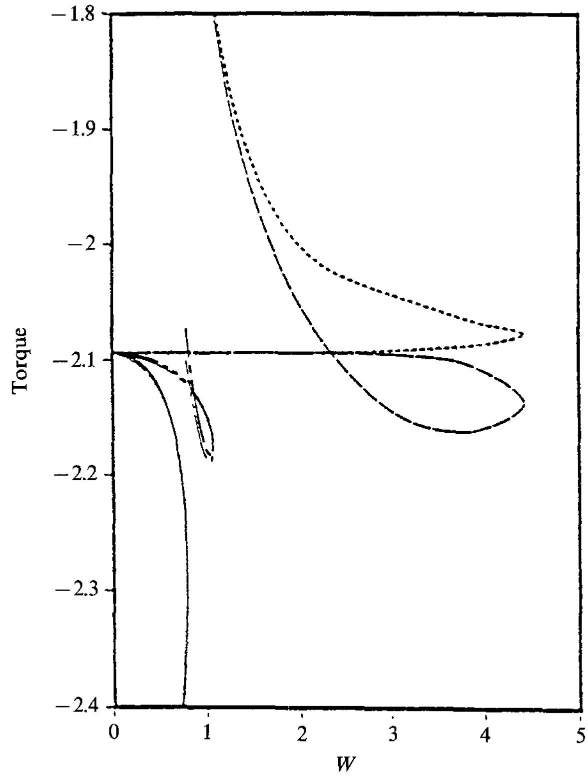

Figure 7. Variation of torque, with Weissenberg number. ———, Bottom, $\beta = 0$; — — —, upper, $\beta = 0$; — - - - —, bottom, $\beta = 0.5$; — - —, upper, $\beta = 0.5$; - - - - - -, bottom, $\beta = 1$, — — — —, upper, $\beta = 1$.

turning points. Crewther *et al.* [31] use a scaling lemma that allows them to extend their analysis to a larger domain in the parameter space once they are able to obtain solutions for a specific set of values for the parameter. They also carry out a linearized stability analysis of the problem.

Between the works of Ji *et al.* [67] and Crewther *et al.* [31] practically all the information that can be obtained from the equations governing the flow of an Oldroyd-B fluid between rotating plates have been extracted. The Oldroyd-B model is one of the several models that have been suggested to explain the behavior of non-Newtonian fluids, but it cannot adequately describe non-Newtonian fluids which exhibit shear thinning (or shear thickening) or those that have finite memory. In general, we need to use nonlinear integral models, and for such models even simpler kinematical situations have not been handled adequately. This open problem while daunting is definitely worth pursuing in view of the technological significance of the problem.

The earliest experiment on the flow between two rotating disks was carried out by Stewartson [130]. He studied the flow of air due to the rotation of two 6-in diameter cardboard disks to obtain some qualitative information regarding the flow. Experimental investigations have also been carried out by Mellor *et al.* [86], Picha and Eckert [99], and by Dijkstra and van Heijst [35]. Recently, Szeri *et al.* [134] have carried out a detailed experimental investigation of the flow between finite parallel rotating disks. They used Laser–Doppler velocity measurements and the fluid under consideration was water. They were able to duplicate the velocity field conjectured by Batchelor [8]. However, they were unable to observe the several other solutions predicted numerically or asymptotically. Experimental results regarding the onset of spiral vortices and instability waves in the Ekman layer have also been the object of considerable scrutiny (see [26], [59], [46], [47], [136], [146], [76], [132], [126], and [127]).

2b. Solutions that Are Not Axially Symmetric

Let us consider the case $\Omega_h = \Omega_0 = \Omega \neq 0$, within the context of the von Karman equations (2.2)–(2.6). Then the only solution to the Navier–Stokes equation is the rigid-body solution:

$$G \equiv \Omega \quad \text{and} \quad F \equiv 0, \tag{2.23}$$

which is isolated and stable. By isolated we mean that there is a neighborhood of this solution wherein there are no other solutions, and by stable we mean there is no bifurcation from this solution, in particular the linearized problem at this solution is nonsingular.

Recently, Berker [12] established a truly remarkable result for the problem of two infinite parallel disks rotating with the same angular velocity Ω, about a common axis. He sought solutions of the form

$$v_x = -\Omega[y - g(z)], \tag{2.24}$$

$$v_y = \Omega[x - f(z)], \tag{2.25}$$

$$v_z = 0, \tag{2.26}$$

where v_x, v_y, and v_z are the components of the velocity in the x, y, and z directions, respectively. Such a velocity field corresponds to a flow wherein streamlines in any z = constant plane are concentric circles, with no flow across any such plane; the locus of the centers of these circles as the z = constant plane shifts from $z = 0$ to $z = h$ being a curve in space described by $x = f(z)$ and $y = g(z)$. Notice that the velocity field (2.24)–(2.26) automatically satisfies the constraint of incompressibility.

The motion represented by (2.24)–(2.26) falls under the category of pseudoplane motions which have been studied extensively by Berker [10], [11].

The appropriate boundary conditions are

$$f(h) = 0, \qquad f(0) = 0, \qquad g(0) = 0, \qquad g(h) = 0. \tag{2.27}$$

Since the plane $z = h/2$ and the locus of the centers of the rotation intersect at some point $(l, 0, 0)$ (we can always pick such a Cartesian coordinate system)

$$f\left(\frac{h}{2}\right) = l \quad \text{and} \quad g\left(\frac{h}{2}\right) = 0. \tag{2.28}$$

When $l = 0$, Berker [12] showed that the rigid-body solution obtains. However, this is but one of an infinity of solutions that are possible. Moreover, when $l \neq 0$, the solutions are not axially symmetric. Thus, in the special case when $\Omega_h = \Omega_0$, the axially symmetric solution to the Karman equations is embedded in a much larger class of solutions. This naturally leads us to ask the question whether the axially symmetric solutions to the von Karman equations are embedded in a larger class of solutions when $\Omega_h \neq \Omega_0$? This question has been answered in the affirmative by Parter and Rajagopal [93].

Parter and Rajagopal [93] assumed a velocity field of the form

$$v_x = \frac{x}{2}H'(z) - \frac{y}{2}G(z) + g(z), \tag{2.29}$$

$$v_y = \frac{y}{2}H'(z) + \frac{x}{2}G(z) - f(z), \tag{2.30}$$

and

$$v_z = -H(z). \tag{2.31}$$

The above velocity field in cylindrical coordinates has the form

$$v_r = \frac{r}{2}H'(z) + g(z)\cos\theta - f(z)\sin\theta, \tag{2.32}$$

$$v_\theta = \frac{r}{2}G(z) - g(z)\sin\theta - f(z)\cos\theta, \tag{2.33}$$

$$v_z = -H(z). \tag{2.34}$$

Notice that when $f \equiv 0$, $g \equiv 0$, we recover the velocity field assumed by von Karman. When $H \equiv 0$ and $G \equiv 2\Omega$, we obtain the velocity field assumed by Berker. It should also be noted that the function G that appears in (2.29) is not the same as that defined in (2.1). We use the representation (2.29) to be consistent with that of Parter and Rajagopal [93].

Substituting (2.32)–(2.34) into the Navier–Stokes equations we obtain

$$\varepsilon H^{\text{iv}} + HH''' + GG' = 0, \tag{2.35}$$

$$\varepsilon G'' + HG' - H'G = 0, \tag{2.36}$$

$$\varepsilon f''' + Hf'' + \tfrac{1}{2}H'f' - \tfrac{1}{2}H''f + \tfrac{1}{2}(Gg)' = 0, \tag{2.37}$$

$$\varepsilon g''' + Hg'' + \tfrac{1}{2}H'g' - \tfrac{1}{2}H''g - \tfrac{1}{2}(Gf)' = 0. \tag{2.38}$$

The appropriate boundary conditions are

$$H(0, \varepsilon) = H(h, \varepsilon) = 0. \tag{2.39}$$

$$H'(0, \varepsilon) = H'(h, \varepsilon) = 0, \tag{2.40}$$

$$G(0, \varepsilon) = 2\Omega_0, \qquad G(h, \varepsilon) = 2\Omega_h, \tag{2.41}$$

$$f(0, \varepsilon) = f(h, \varepsilon) = 0, \tag{2.42}$$

$$g(0, \varepsilon) = 0, \qquad g(h, \varepsilon) = 0. \tag{2.43}$$

The system of equations (2.35)–(2.38) and boundary conditions (2.39)–(2.43) are underdetermined and as before we can augment the system by requiring

$$f\left(\frac{h}{2}, \varepsilon\right) = l_1, \qquad g\left(\frac{h}{2}, \varepsilon\right) = l_2. \tag{2.44}$$

The above system has a very interesting feature. Equations (2.35), (2.36), and the boundary conditions (2.39)–(2.43) are precisely the same as those that govern the axially symmetric problem. More importantly, these are the only equations which are nonlinear. The equations for f and g are linear with coefficients which are solutions to the nonlinear axially symmetric problem. We are now in a position to answer the following question: Whenever there is a solution $(F(z), G(z))$ to the system (2.35), (2.36), (2.39)–(2.41), can we find a one-parameter family of solutions (f, g) to the system (2.37), (2.38), (2.42)–(2.44)? The answer is in the affirmative.

In the special case $\Omega_h = \Omega_{-h}$, a one-parameter family of solutions that is not axially symmetric has been analytically established by Rajagopal and Gupta [104] in the case of the incompressible fluid of second grade,[6] and by Rajagopal and Wineman [109] in the case of a special subclass of the K-BKZ [72], [14] fluid.

When $\Omega_h \neq \Omega_0$, Huilgol and Rajagopal [66] show that a situation similar to that considered by Parter and Rajagopal [93] obtains in the case of an Oldroyd-B fluid [90]. Huilgol and Rajagopal [66] assume a velocity field of the form (2.32)–(2.34) and show that the problem is governed by four coupled equations for the four functions F, G, f, and g. A rigorous existence theorem has not yet been established for this problem. Similar to the situation in the case of the Navier–Stokes fluid the functions F and G are governed by two coupled nonlinear ordinary differential equations. The other two equations involve the four functions F, G, f, and g. Having established the existence of the solution (F, G) we can once again proceed to ask the question whether a one-parameter family of solutions (f, g) exists.

Using the formulation of Huilgol and Rajagopal [66] for flows of an Oldroyd-B fluid that lack axisymmetry, Crewther *et al.* [31] have recently carried out extensive numerical calculations. An interesting feature of their work is the existence of shear layers in the flow field. Such layers have also been observed experimentally by Sirivat *et al.* [128].

To this date there has been no attempt to study the flow between two disks rotating with distinct speeds within the context of nonlinear integral models which represent fluids with finite memory, when the angular speeds of the two disks are not the same, and this problem is well worth investigating.

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[6] Earlier, Drouot [38] extended Berker's analysis, and her work clearly implies the possibility of exact solutions in the case of an incompressible homogeneous fluid of second grade. However, she did not solve the specific boundary-value problem.

3. Flow Between Parallel Disks Rotating about Different Axes

3a. $\Omega_h = \Omega_0$

We first discuss the special case when $\Omega_h = \Omega_0 = \Omega \neq 0$. Let a denote the distance between the parallel but distinct axes (see Figure 2). In the case of Navier–Stokes fluid, Abbot and Walters [1] restricted themselves to solutions which possess midplane symmetry and exhibited an exact solution to that problem, the result being valid for arbitrary values of the offset. However, if we relax the requirement of midplane symmetry, it is easy to show that the problem under consideration possesses a one-parameter family of solutions [13]. As we mentioned earlier, the above flow has relevance to the flow occurring in an orthogonal rheometer. Unlike the preceding studies on the flows of fluid in such an instrument, Abbot and Walters [1] include inertial effects in their analysis.[7] In the case of a viscoelastic fluid they carried out a perturbation analysis in which they used the offset a as the small parameter. Goldstein and Schowalter [54] have also used a similar expansion to study the effect of inertial nonlinearities. While these studies solved the problem to various orders, no rigorous results regarding the convergence of such solutions have been established. The analysis of Drouot [38] once again implies the possibility of exact solutions, similar to Berker's, in the case of a fluid of second grade. Rajagopal and Gupta [105] and Rajagopal [100] have since established these exact solutions.

We now discuss in some detail the flow of an incompressible simple fluid in the orthogonal rheometer. We assume that the motion occurring in the orthogonal rheometer has the form (2.8)–(2.10). Let $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ denote the position occupied by a particle $\mathbf{X} = (X, Y, Z)$ at time τ. Let $\mathbf{x} = (x, y, z)$ denote the position occupied by the same particle $\mathbf{X}$ at time t. It follows from (2.24)–(2.26) that

$$\dot{\xi} = -\Omega(\eta - g(\varsigma)), \tag{3.1}$$

$$\dot{\eta} = -\Omega(\xi - f(\varsigma)), \tag{3.2}$$

$$\dot{\varsigma} = 0, \tag{3.3}$$

with

$$\xi(t) = x, \qquad \eta(t) = y, \quad \text{and} \quad \varsigma(t) = z.$$

Rajagopal [101] has shown that the motion (3.1)–(3.4) is a motion with constant principal relative stretch history [28], [29], [89], [138]. In such motion, the stress is determined by the first three Rivlin–Ericksen tensors $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ [144].

However, for the motion under consideration

$$\mathbf{A}_3 = -\Omega^2 \mathbf{A}_1. \tag{3.4}$$

Thus, the stress is given by

$$\mathbf{T} = -p\mathbf{1} + \hat{\mathbf{f}}(\mathbf{A}_1, \mathbf{A}_2). \tag{3.5}$$

The balance of linear momentum has the form [101]:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial \varphi}{\partial x} + \Omega^2[x - f] + \frac{1}{\rho}h_1(f', g', f'', g''), \tag{3.6}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{\partial \varphi}{\partial y} + \Omega^2[y - f] + \frac{1}{\rho}h_2(f', g', f'', g''), \tag{3.7}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{\partial \varphi}{\partial z} + \frac{1}{\rho}h_3(f', g', f'', g''). \tag{3.8}$$

The specific constitutive equations determine the functions h_1, h_2, and h_3. This can then be substituted into (3.6)–(3.8) and the appropriate partial differential equations analyzed. Notice that (3.6)–(3.8) are of second order and hence the no-slip boundary conditions are sufficient for determinacy.

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[7] The problem wherein the inertial effects are ignored, has been studied by numerous authors [84], [23], [64].

The appropriate boundary conditions for the velocity field are

$$u = \frac{\Omega a}{2} - \Omega y, \qquad v = \Omega x, \qquad w = 0 \quad \text{at} \quad z = h, \tag{3.9}$$

$$u = -\frac{\Omega a}{2} - \Omega y, \qquad v = \Omega x, \qquad w = 0 \quad \text{at} \quad z = 0, \tag{3.10}$$

and

$$u \to \mp\infty, \qquad v \to \pm\infty, \qquad \text{as} \quad x, y \to \pm\infty. \tag{3.11}$$

It follows from (3.9), (3.10), and (3.1)–(3.3) that

$$f(h) = f(0) = 0, \tag{3.12}$$

$$g(h) = \frac{a}{2}, \qquad g(0) = -\frac{a}{2}. \tag{3.13}$$

In eliminating the pressure field we have raised the order of the equations. Thus, the boundary conditions (3.12) and (3.13) are not sufficient to determine the solution to the system (3.6) and (3.8). As before, we augment the number of boundary conditions by recognizing that the locus of the centers of rotation cuts the plane $z = 0$ at some point, say (l_1, l_2). Thus

$$f\left(\frac{h}{2}\right) = l_1, \qquad g\left(\frac{h}{2}\right) = l_2. \tag{3.14}$$

However, if we restrict ourselves to solutions which have midplane symmetry, then

$$f\left(\frac{h}{2}\right) = 0, \qquad g\left(\frac{h}{2}\right) = 0. \tag{3.15}$$

For the rest of this section we restrict ourselves to a discussion of solutions which possess midplane symmetry.

Rajagopal [100] studied the flow of a fluid of second grade undergoing a motion of the form (3.1)–(3.3) subject to the boundary conditions (3.12) and (3.13), and the augmented condition (3.14). He also obtained expressions for the traction and moments acting on the plates. Recently, Kaloni [68] extended the analysis of Rajagopal to incompressible homogeneous fluids of third grade.

We now discuss a nontrivial example, wherein no approximations with regard to the Reynolds number or the Weissenberg number are made and the field equations are solved numerically, namely the flow of the K-BKZ fluid. The problem is also interesting because there are conflicting claims about the nature of the solution and additional work needs to be done to resolve the apparent numerical discrepancies.

The Cauchy stress $\mathbf{T}$ in the K-BKZ fluid has the structure

$$\mathbf{T} = -p\mathbf{1} + 2\int_{-\infty}^{t} \{U_1 \mathbf{C}_t^{-1}(\tau) - U_2 \mathbf{C}_t(\tau)\}\, dt, \tag{3.16}$$

where

$$\mathbf{C}_t(\tau) = \mathbf{F}_t^T(\tau)\mathbf{F}_t(\tau). \tag{3.17}$$

In (3.16) U denotes the strain energy function for the viscoelastic fluid and is a function of the principal invariants of $\mathbf{C}_t(\tau)$ and $\mathbf{C}_t^{-1}(\tau)$:

$$U = U(I_1, I_2, t - \tau), \tag{3.18}$$

$$I_1 = \operatorname{tr} \mathbf{C}_t^{-1}(\tau), \qquad I_2 = \operatorname{tr} \mathbf{C}_t(\tau), \tag{3.19}$$

and

$$U_i = \frac{\partial U}{\partial I_i}, \qquad i = 1, 2. \tag{3.20}$$

For the motion under consideration, a lengthy but straightforward computation yields

$$\mathbf{C}_t(\tau) = \mathbf{1} - \frac{s}{\Omega}\mathbf{A}_1 + \frac{(1-c)}{\Omega^2}\mathbf{A}_2 \tag{3.21}$$

and

$$\mathbf{C}_t^{-1}(\tau) = 1 + \frac{s}{\Omega}[1 + 2(1-c)(f'^2 + g'^2)]\mathbf{A}_1 - \frac{(1-c)}{\Omega^2}[1 + 2(1-c)(f'^2 + g'^2)]\mathbf{A}_2 + \frac{s^2}{\Omega^2}\mathbf{A}_1^2$$
$$+ \frac{(1-c)}{\Omega^4}\mathbf{A}_2^2 + \frac{s(1-c)}{\Omega^3}(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1), \tag{3.22}$$

where

$$s \equiv \sin \Omega(t-\tau), \qquad c \equiv \cos \Omega(t-\tau). \tag{3.23}$$

Also, notice that

$$I_1(t,\tau) = I_2(t,\tau) = 3 + 2(1-c)(f'^2 + g'^2) \equiv I(\Omega(t-\tau), z). \tag{3.24}$$

It follows from (3.6)–(3.8), (3.16), (3.21), and (3.22) that

$$\frac{d}{dz}\{f'B(\kappa) + g'A(\kappa)\} = \rho\Omega^2 f, \tag{3.25}$$

$$\frac{d}{dz}\{-f'A(\kappa) + g'B(\kappa)\} = \rho\Omega^2 g, \tag{3.26}$$

where

$$\kappa \equiv (f'^2 + g'^2)^{1/2} \tag{3.27}$$

and

$$A(\kappa) = 2\int_0^\infty \tilde{U}[3 + 2(1-\cos\Omega\alpha)\kappa^2, \alpha]\sin\Omega\alpha\, d\alpha, \tag{3.28}$$

$$B(\kappa) = 2\int_0^\infty \tilde{U}[3 + 2(1-\cos\Omega\alpha)\kappa^2, \alpha](1-\cos\Omega\alpha)\, d\alpha, \tag{3.29}$$

$$\tilde{U}(I,\alpha) \equiv U_1(I,I,\alpha) + U_2(I,I,\alpha). \qquad (3.30)^8$$

Let t_x and t_y denote the x and y components of the traction on the plates. It follows that

$$t_x(h) = +B(\kappa,\Omega)f'(h) + A(\kappa,\Omega)g'(h), \tag{3.31}$$

$$t_y(h) = -A(\kappa,\Omega)f'(h) + B(\kappa,\Omega)g'(h). \tag{3.32}$$

Thus the material parameters $A(\kappa, \Omega)$ and $B(\kappa, \Omega)$ can be expressed in terms of t_x and t_y as [25]

$$A(\kappa,\Omega) = \frac{-1}{\kappa^2}[t_y f' + t_x g'], \tag{3.33}$$

$$B(\kappa,\Omega) = \frac{1}{\kappa^2}[t_x f' + t_y g']. \tag{3.34}$$

Let $\bar{A}$ and $\bar{B}$ denote the value of the material properties for the inertialess case. In the inertialess case

[8] Rajagopal and Wineman [110] have found an exact solution for the flow of a special subclass of models of the K-BKZ type.

the solutions for f and g are given by

$$f(z) \equiv 0, \qquad g(z) = \frac{a}{h}z \tag{3.35}$$

$$\bar{A} = \frac{h}{a}t_x, \tag{3.36}$$

$$\bar{B} = \frac{h}{a}t_y. \tag{3.37}$$

The relative error made in evaluating the material properties A and B, by neglecting inertia, can be defined as [25]

$$\frac{\bar{A} - A}{A} = \frac{h}{a}f'(h)\frac{B}{A} + \left(\frac{h}{a}g'(h) - 1\right), \tag{3.38}$$

$$\frac{\bar{B} - B}{B} = -\frac{h}{a}f'(h)\frac{A}{B} + \left(\frac{h}{a}g'(h) - 1\right), \tag{3.39}$$

where f and g are the solutions in which inertia is included. Computations indicate that f' is very small and the first term in (3.38) and (3.39) can be neglected. However, g' is not small and, for flows with high Reynolds number, the error can be significant. Bower *et al.* [25] discuss in detail the error made in neglecting the inertial effects in this problem.

Rajagopal *et al.* [108] have studied the flow of fluids of the Wagner [142] and Currie [32] types in an orthogonal rheometer. In the case of the Currie model, the strain energy function U has the structure

$$U(I_1, I_2, s) = -\dot{G}(s)[5\ln(J - 1) - 9.73], \tag{3.40}$$

where

$$J = I_1 + 2(I_2 + 3.25)^{1/2}. \tag{3.41}$$

When

$$\dot{G}(s) = -Ce^{-\lambda s}, \tag{3.42}$$

the material functions $A(\kappa, \Omega)$ and $B(\kappa, \Omega)$ are given by

$$A = \frac{2C}{\Omega[1 - \exp(-2\pi\lambda/\Omega)]}\int_0^{2\pi} e^{-\lambda\alpha}\tilde{U}[3 + 2(1 - \cos\Omega\alpha)\kappa]\sin\Omega\alpha\, d\alpha, \tag{3.43}$$

$$B = \frac{2C}{\Omega[1 - \exp(-2\pi\lambda/\Omega)]}\int_0^{2\pi} e^{-\lambda\alpha}\tilde{U}[3 + 2(1 - \cos\Omega\alpha)\kappa](1 - \cos\Omega\alpha)\, d\alpha. \tag{3.44}$$

The equations can be nondimensionalized appropriately [108] to yield

$$\frac{d}{d\bar{z}}\left(\bar{B}\frac{d\bar{f}}{d\bar{z}} + \bar{A}\frac{d\bar{g}}{d\bar{z}}\right) = \frac{1}{E}We\bar{f} + \bar{q}_2, \tag{3.45}$$

$$\frac{d}{d\bar{z}}\left(-\bar{A}\frac{d\bar{f}}{d\bar{z}} + \bar{B}\frac{d\bar{g}}{d\bar{z}}\right) = \frac{1}{E}We\bar{g} + \bar{q}_2, \tag{3.46}$$

where

$$\bar{z} = \frac{z}{h}, \qquad \bar{f} = \frac{f}{a}, \qquad \bar{g} = \frac{g}{a}, \qquad \bar{A} = \frac{A\lambda}{C}, \qquad \bar{B} = \frac{B\lambda}{C}, \tag{3.47}$$

and

$$W = \frac{\Omega}{\lambda}, \qquad E = \frac{(C/\lambda^2)}{\rho h^2\Omega}, \tag{3.48}$$

where W is the Weissenberg number and E is the Ekman number. The Ekman number is the inverse of a Reynolds number and is a ratio of the Coriolis forces to the viscous forces. We first discuss the

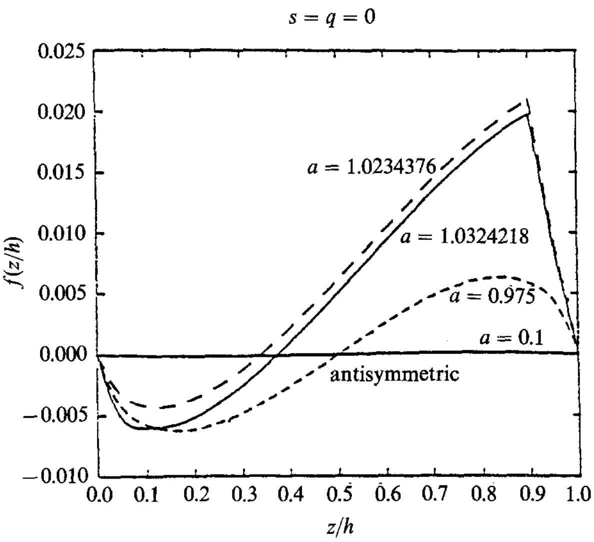

Figure 8. Locus of centers of rotation with discontinuous derivative.

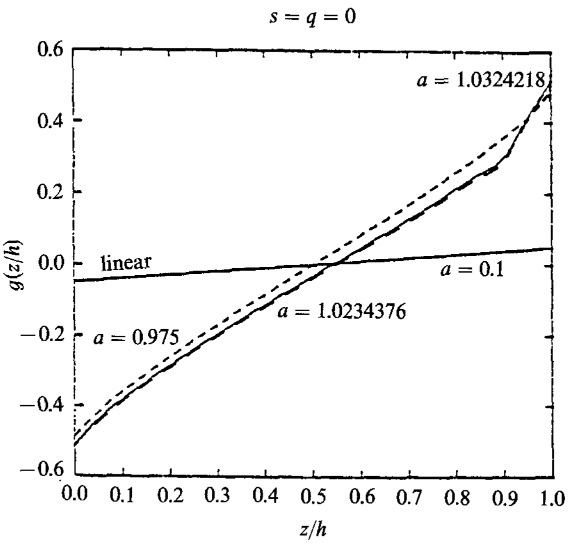

Figure 9. Locus of centers indicating boundary layers.

situation when the Ekman number is large and the Weissenberg number is large. When $q_1 = q_2 = 0$, we find that the numerical solution converges to a solution which is symmetric about the midplane with $f(z)$ and $g(z)$ being approximately linear up to some critical value of a/h. For values of a/h larger than this critical value, the solutions lose their symmetry and the locus of the centers of rotation possesses discontinuous derivatives (see Figures 8 and 9). When $s \neq 0$, $q \neq 0$, the solutions are not symmetric. Above a critical value of a/h the numerical scheme fails, no range of values of a/h for which solutions with discontinuous slopes exist. In the case of small Weissenberg numbers and large Reynolds numbers, as would be expected there is a boundary-layer structure to the solution (see Figures 10 and 11).

Zhang and Goddard [150] reinvestigated the problem studied by Rajagopal *et al.* [108]. They indicated that they were able to find smooth solutions for the range of parameters for which Rajagopal *et al.* [108] found discontinuous solutions. Their numerical results are nearly two orders of magnitude higher than those obtained by Rajagopal *et al.* [108] for the same values of the parameters. Moreover, Zhang and Goddard [150] remark that they encountered convergence problems with their numerical scheme for larger values of the parameter and allude to the possibility of bifuration of solutions. However, their numerical method is not suitable for the study of bifurcation, and they do

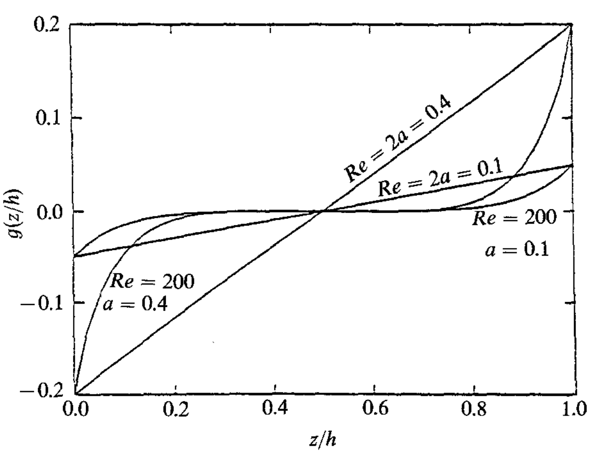

Figure 10. Locus of centers indicating boundary layers.

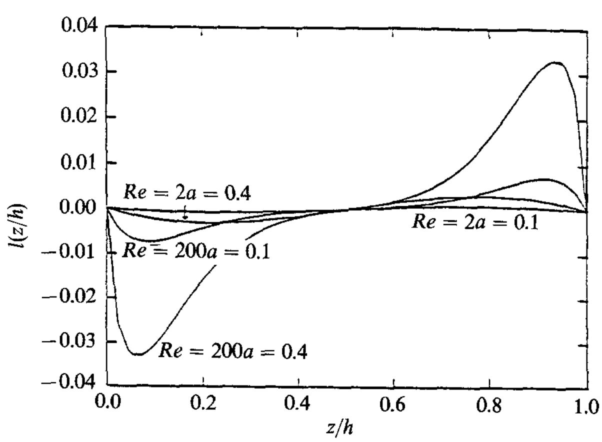

Figure 11. Locus of stagnation points.

not exhibit bifurcated solutions. Since Zhang and Goddard [150] do not provide details regarding their numerical work (and more importantly their error estimates), Dai *et al.* [33] have recently restudied the problem in detail using a parameter continuation method which is specifically used to study bifurcation problems numerically. In contrast to the work of Zhang and Goddard [150], but in keeping with the results of Rajagopal *et al.* [108], they have no difficulty with convergence, and more importantly their results agree perfectly with those of Rajagopal *et al.* [108]. Dai *et al.* [33] were able to push their study to much larger values of the Weissenberg and Reynolds numbers. They were able to go as high as 10,000 in the Reynolds number, and as is to be expected the solutions exhibit a strong boundary-layer structure. At the value $\gamma = 0.9750$, where γ is the separation between the axes normalized with the distance between the plates, Rajagopal *et al.* [108] obtained solutions with discontinuous velocity gradients. Dai *et al.* [33] were able to go well past this value of the parameter using the continuation method. Also, they found that the discontinuity occurs immediately adjacent to the disks so that it suggests that there might be a singularity at the boundary. Also, since the equations are highly nonlinear, it is possible that there are smooth solutions to the problem which have not been discovered.

When $\Omega_h \neq \Omega_0$, the flow of a Navier–Stokes fluid between plates rotating about different axes is governed by the same system of differential equations (2.24)–(2.27). The only difference in the boundary-value problem from that governing the flow about a common axis occurs in the specification of boundary conditions. The boundary conditions $(2.32)_1$ and $(2.32)_2$ are replaced by

$$g(-h, \varepsilon) = -\frac{a\Omega_0}{2}, \qquad g(h, \varepsilon) = -\frac{a\Omega_{+h}}{2}. \tag{3.49}$$

Approximate analytical and numerical solutions to the above problem were first obtained by Knight [75].

Parter and Rajagopal [93] have discussed questions regarding the existence of solutions to the system (2.24)–(2.27), (2.28)–(2.31), (2.33), and (3.43). A detailed numerical study has been carried out by Lai *et al.* [79].[9] In this case the locus of the stagnation points is far from simple and once again does not possess any midplane symmetry (see Figure 12).

Not much has been done with regard to non-Newtonian fluids and as it stands is an important open problem, both with regard to analytical and numerical results.

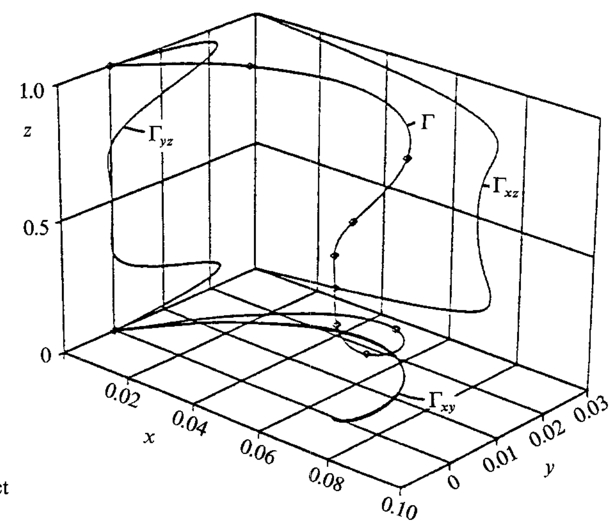

Figure 12. Locus of stagnation points for flow about distinct axes.

[9] The corresponding problem of solutions that lack symmetry due to the rotation of a single disk has been studied by Lai *et al.* [80] in the case of the Navier–Stokes fluid.

4. Related Problems

There are many problems of practical importance wherein swirling flows that are turbulent are encountered. Some details regarding such flows can be found in Schlichting [123].[10]

Thus far we have restricted our discussions to the rotating flow of a single fluid. There are however several practical applications where the study of the flow of a stratified fluid between rotating plates is relevant. One important example is the study of reaction kinetics at the fluid–solid interface due to a rotating disk, in electrochemistry. Such flows can be used to generate surfaces across which the transfer of heat and mass are uniform. Goddard *et al.* [52] have studied the flow of a system of two homogeneous fluids confined between infinite porous plates rotating with different angular velocities, about a common axis. They find that an axisymmetric similarity solution of the form suggested by von Karman is possible and a flat interface exists for a range of rotation speeds and injection rates. The structure of these solutions are most interesting as the numerical simulations indicate a very wide range of characteristics, based on the rotation rates, the rate of injection, etc. Questions regarding the existence and uniqueness of these solutions are interesting open problems.

Peucheux and Boutin [96] have studied the steady flow of two immiscible vertically stratified fluids between two plates rotating about a common axis. Recently, Gogus [53] has studied the flow of a binary mixture of fluids between plates rotating about distinct axes, within the context of the theory of interacting continua [4], [24], [139] using a model due to Craine [30].

Early work incorporating heat transfer in the classical linearly viscous fluid between two disks rotating about a common axis was carried out by Reshotko and Rosenthal [118] and this has been followed by several other studies (see [3], [125], [36], [37], and [87]), Banerjee and Borkakati [7] studied the heat transfer characteristics of the flow of a linearly viscous fluid between disks rotating with different speeds about distinct axes. Recently, Kasivishwanathan and Rao [71] have considered the unsteady flow due to two disks rotating about noncoincident axes with heat transfer taking place.

Stuart [131] studied the effects of suction on the flow of the classical linearly viscous fluid due to a rotating disk in the case of axially symmetric flows, and this work was extended to flows which lack axial symmetry due to streaming by Szeri *et al.* [133]. Szeri *et al.* [133] found bifurcation of solutions as the parameter s, which is the ratio of the angular velocity of the fluid at infinity to the angular velocity of the disk, is varied. Rajagopal [103] obtained a class of exact solutions, that lack symmetry, for the flow due to the rotations of two porous coaxial disks. As nothing in the way of rigorous mathematical analysis has been carried out for these flows, except for the work of Elcrat [42] in the case of one disk being porous and the rotating disk being impervious, such work remains an open area meriting the interest of mathematical analysts.

Flows of the linearly viscous fluid due to a single rotating disk or two rotating disks under the application of a magnetic field has also received considerable attention because of its relevance to astrophysical and geophysical problems. Devanathan [34] has studied rotating flows when the fluid is assumed to be electrically conducting, in the presence of an applied magnetic field. Extensions to the case of specific non-Newtonian fluid models have been carried out by Bhatnagar [18], [19]. Unsteady flow due to the oscillations of a disk in the presence of a magnetic field normal to the disk, in the case of a Reiner–Rivlin fluid, was studied by Bhatnagar [15]. Murthy and Ram [87] have studied the flow due to a porous rotating disk under the action of a magnetic field allowing for heat transfer when there are also rotations at infinity. Rao and Rao [115] have studied the heat transfer between disks rotating about distinct axes under the influence of an applied magnetic field for the linearly viscous fluid.

There has also been some work on the swirling flows of polar and micropolar fluids. Bhatnagar [16] studied the flow of a dipolar fluid [22] between two plates rotating about a common axis. Rao and Kasivishwanathan [113] established a class of exact solutions for the flow of a micropolar fluid [44], [45] between disks rotating about noncoincident axes, but with the same angular speed. Recently, Kasivishwanathan and Gandhi [70] have investigated the flow of micropolar fluids in the previous geometry, subject to a magnetic field.

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[10] Of particular interest is the effect of the roughness of surfaces on the flow characteristics [55].

The review article by Zandbergen and Dijkstra [149] discusses several issues related to that covered here. It would be apropos to conclude by remarking that there are several analogous problems within the context of finite elasticity which have been studied recently (see [111], [112], and [82]).

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