ON SCATTERING OF DIFFUSION PROCESS GENERATORS

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ABSTRACT. If a selfadjoint generator of a diffusion process is perturbed by nonnegative potentials different on a compact region of non-zero measure the corresponding wave operators exist and are asymptotically complete even if one potential is singular on the region considered. That includes the hardcore potential scattering problem for second-order elliptic differential operators with variable coefficients.

1. INTRODUCTION

Let K be a generator of a diffusion process defined in $L^2(\mathbb{R}^n)$. K is perturbed by a nonnegative potential $V(x) = V_R(x) + V_S(x)$. Let $V_R(x) \in L_\infty(\mathbb{R}^n)$ and V_s, the singular part, be given by $V_S(x) = \infty$ on a compact region G, $G \subset \mathbb{R}^n$, and zero otherwise. Let $V_M(x) = V_R(x) + M\chi_G(x)$, $M \geqslant 0$, χ_G indicator function of G. Moreover, set $K_M = K + V_M$ and define a truncation projection P by $(Pf)(x) = \chi_{\mathbb{R}^n \setminus G}(x) f(x)$, $f \in L^2(\mathbb{R}^n)$. Then the semigroup e^{-tK_M} tends in $L^2(\mathbb{R}^n \setminus G)$ to a contractive C_0-semigroup e^{-tK_∞} as $M \to \infty$. By means of Markov process properties the differences

$$e^{-tK_M}P - P\,e^{-tK_\infty}, \qquad e^{-tK_\infty}P - P\,e^{-tK_M} \tag{1}$$

are shown to be trace class operators for each $M \geqslant 0$ and $t \geqslant 0$, and to tend to zero in trace norm as $M \to \infty$. Hence, for selfadjoint K_M and K_∞ such a result implies with respect to the scattering theory the existence and completeness of the corresponding wave operators. That means the wave operator existence for perturbed diffusion process generators is independent of changing the potential arbitrarily on compact regions.

Generators of diffusion processes can be obtained by certain second-order differential operators of the form

$$L = \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}. \tag{2}$$

Consequently the present paper connects hard-core potential problems and wave operator existence for diffusion process generators including scattering problems for second-order

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elliptic differential operators by means of the stochastic process theory and stochastic differential equation solutions. More details of this subject can be found in [1] and [2].

Wave operator existence for pairs of selfadjoint differential operators with variable coefficients was proved by means of several methods by Birman [3], Kako [4] or Hörmander [5]. Scattering problems in the exterior of bounded domains are studied for the Laplacian by Kupsch, Sandhas [6] and Ikebe [7] in $\mathbb{R}^3$, by Hunziger [8], Arsenev [9], Deift, Simon [10], Baumgärtel, Demuth [11], and Semenov [12] in $\mathbb{R}^n$. Birman [13] studied the same for elliptic differential operators with variable coefficients.

In Section 2 the results of this paper are concentrated. All proofs are given in Section 3. They are reduced to the bare essentials.

2. RESULTS

2.1. *Unperturbed Diffusion Process Generators*

ASSUMPTION 1. *Let* $(\Omega_x, \mathfrak{M}_x, \omega(t), P_x)$ *be a time-homogeneous diffusion process. Here* Ω_x *denotes the set of all continuous function* $\omega(\cdot)$ *mapping* $[0, \infty]$ *into* $\mathbb{R}^n$ *with* $\omega(0) = x$, $s \geqslant 0$, *B any Borel set of* $\mathbb{R}^n$. $P_x\{\cdot\}$ *is a probability measure on* $\mathfrak{M}_x$. *Let* $b(x) = (b_1(x), \ldots, b_n(x))$ *be the drift vector and* $a(x) = \{a_{ij}(x)\}$, $i, j = 1, \ldots, n$, *the diffusion matrix of the process considered.*

ASSUMPTION 2. *The transition probability function* $P(t, x, B)$ *of the process considered has to possess a density* $p(t, x, y)$ *which can be estimated by*

$$p(t, x, y) \leqslant \frac{\alpha}{(2\pi t)^{n/2}} \exp \frac{-\beta |x - y|^2}{2t}, \qquad t > 0, \tag{3}$$

α, β *are some positive constants.*

DEFINITION. Because of (3) the function $P(t, x, B)$ allows us to define (see [1]) a contractive C_0-semigroup T_t in $L^2(\mathbb{R}^n)$ given by $(T_t f)(x) = \int_{\mathbb{R}^n} f(y) p(t, x, y)\ \mathrm{d}y$, $t > 0$, and $T_0 f = f$. The generator K of T_t with domain $\vartheta(K)$ is defined by

$$Kf = \lim_{t \to 0} \frac{1}{t}(I - T_t) f \tag{4}$$

$$\vartheta(K) = \left\{ f : f \in L^2(\mathbb{R}^n),\ \lim_{t \to 0} \frac{1}{t}(I - T_t) f \text{ exists} \right\}. \tag{5}$$

Now, let us specify the generator K in more detail.

THEOREM 1. *Let assumptions 1 and 2 be satisfied.*

(a) *Let $a(x)$ be nonnegative definite, symmetric, and twice continuously differentiable with bounded second-order derivatives. Let* $|b(x) - b(y)| \leqslant C|x - y|$, $x \in \mathbb{R}^n$, *C some constant. Then the diffusion process generator K is an extension of $-L$ (see* (2)) *with domain*

$$\vartheta(L) = \{f: f \in W_2^2(\mathbb{R}^n), \quad Lf \in L^2(\mathbb{R}^n), \quad (1 + x_i^2)^{1/2} f_{x_i}(x) \in L^2(\mathbb{R}^n)\}. \quad (6)$$

$W_2^2(\mathbb{R}^n)$ *denotes the Sobolev-space of all L^2-functions with derivatives up to second order in* $L^2(\mathbb{R}^n)$. f_{x_i} *is the partial derivative of $f(x)$ with respect to x_i.*

(b) *If L is symmetric and strongly elliptic K is a selfadjoint semibounded extension of $-L$ with $\vartheta(L)$ given by* (6).

REMARKS. The simplest example is the usual Wiener process where $\alpha = \beta = 1$ and $Kf = -\Delta f$ for $f \in W_2^2(\mathbb{R}^n) = \vartheta(K)$. Estimate (3) is satisfied also for canonical diffusion processes (see Dynkin [14]) or for generalized diffusion processes (see Portenko [15]).

Assumption 2 implies $|e^{-tK} f| = \alpha\beta^{-n/2} e^{t/\beta^4} |f|$. Such an estimate is also satisfied in the physically more interesting case of a Schrödinger operator with magnetic field (see Simon [16]) where K is given by $\frac{1}{2}(i\nabla - A)^2$ with some vector potential $A(x)$.

2.2. *Perturbed Diffusion Process Generators*

ASSUMPTION 3. *Let V be a nonnegative multiplication operator given by* $V(x) = V_R(x) + V_S(x)$ *with* $V_R(x) \in L_\infty(\mathbb{R}^n)$ *and* $V_S(x) = \infty$, $x \in G$, *and* $V_S(x) = 0$, $x \in \mathbb{R}^n \backslash G$, *where G is some compact region in* $\mathbb{R}^n$. *Let $\chi_G(\cdot)$ be the indicator function of G. We set* $V_M(x) = V_R(x) + M\chi_G(x)$, $M \geqslant 0$. *Finally, we assume a truncation projection P given by* $(Pf)(x) = \chi_{\mathbb{R}^n \backslash G}(x) f(x)$, $f \in L^2(\mathbb{R}^n)$.

In the following theorems all three assumptions are supposed.

THEOREM 2. *Set* $K_M = K + V_M$. *Then K_M generates a contractive C_0-semi group which can be represented by*

$$(\exp(-tK_M)f)(x) = \int_{\Omega_x} \exp\left\{-\int_0^t V_M(\omega(s))\, ds\right\} f(\omega(t)) P_x(d\omega), \quad t \geqslant 0, \quad (7)$$

for all $f \in L^2(\mathbb{R}^n)$ *and* a.a. $x \in \mathbb{R}^n$.

THEOREM 3. *The semigroup* e^{-tK_M} *has a strong limit as* $M \to \infty$ *for each* $t \geqslant 0$. *Calling this limit $U(t)$ it holds for all* $f \in L^2(\mathbb{R}^n)$ *and* a.a. $x \in \mathbb{R}^n$

$$(U(t)f)(x) = \int_{\Omega_x} \exp\left\{-\int_0^t V_R(\omega(s))\, ds\right\} f(\omega(t)) \chi(\omega) P_x(d\omega), \quad (8)$$

with $\chi(\omega) = \chi\{\omega\colon \omega(s) \notin G,\ s \in [0, t],\ \omega(0) = x\}$. *In particular,* $U(t)P = PU(t)$. $U(t)$ *is a contractive* C_0*-semigroup in* $L^2(\mathbb{R}^n \backslash G)$. *Denoting its generator by* K_∞ *we get* $e^{-tK_\infty}P = U(t)P = U(t)$.

THEOREM 4. *Let* $n > 2$

(a) K_∞ *is an extension of* $K + V_R$ *restricted to* $L_0^2(\mathbb{R}^n \backslash G) \cap \vartheta(K)$ *where* $L_0(\mathbb{R}^n \backslash G)$ *is the space of all* L^2*-functions with compact support in* $\mathbb{R}^n \backslash G$.

(b) *If K is selfadjoint and semibounded then* K_∞ *is the Friedrichs extension of* $K + V_R$ *restricted to* $L^2(\mathbb{R}^n \backslash G) \cap \vartheta(K)$.

THEOREM 5. *Let* $n > 2$.

(a) *The difference* $e^{-tK_M} - e^{-tK_\infty}P$, $t > 0$ *is a trace class operator for each* $M \geqslant 0$. *It tends to zero in trace norm as* $M \to \infty$.

(b) *If K and* K_∞ *are selfadjoint the wave operators* $W_\pm(K_\infty, P, K_M) = s - \lim_{t \to \pm\infty} \times\ e^{itK_\infty}P\, e^{-itK_M} P_{ac}^M$ *exist and are complete.* P_{ac}^M *is the projection onto the absolutely continuous subspace of* K_M.

In proving the last two theorems we need the following Lemma.

LEMMA: *Let* $T_t(\omega)$ *be the penetration time of* $\omega(s)$ *in G during* $[0, t]$. *Let* $x \in \mathbb{R}^n \backslash G$ *and the distance between x and G* $\rho(x, G) > 0$. *Let* $n > 2$. *Then the measure of all paths with values in G can be estimated by*

$$P_x\{\omega\colon T_t(\omega) > 0,\ \omega(0) = x\} \leqslant C_1 \exp\left(-C_2 \frac{\rho^2(x, G)}{t}\right) \tag{9}$$

where C_1, C_2 *are some constants depending on* α, β *and n.*

REMARKS. The Equations (7) and (8) are the diffusion process form of the Feynman–Kac formula or a special form of Ito's formula. They can be extended to more general $V_R(x)$. But the aim here is to investigate the singular potential influence.

The most interesting case of Theorem 4 with respect to the scattering theory are essentially. selfadjoint elliptic differential operators. In that case K_∞ is the selfadjoint operator in $L^2(\mathbb{R}^n \backslash G)$ generated by $\Sigma_{i,j=1}^n\, \partial/\partial x_i(a_{ij}(x)\partial/\partial x_j) + V_R(x)$ with Dirichlet boundary condition with respect to G.

3. PROOFS

As mentioned, the main parts of this paper are the results and remarks of Section 2. The proofs here are reduced very strongly or only sketched. More details are given in [1] and [2].

Proof to Theorem 1: (a) See [2]. Let $f \in \vartheta(L)$ given by (6). It is to prove $\|t^{-1}(I - e^{-tK})f + Lf\|$ tends to zero as $t \to 0$. Expanding $f(y) - f(x)$ up to second order it suffices to show

$$\lim_{t\to\infty} \Big\| \frac{1}{t} \int_{\mathbb{R}^n} (y_i - x_i)(y_j - x_j)p(t, x, y)\, \mathrm{d}y f_{x_i x_j}(\hat{x}) - $$

$$- a_{ij}(x) f_{x_i x_j}(x) \Big\| = 0, \qquad (10)$$

$$\lim_{t\to\infty} \Big\| \frac{1}{t} \int_{\mathbb{R}^n} (y_i - x_i)p(t, x, y)\, \mathrm{d}y f_{x_i}(x) - b_i(x) f_{x_i}(x) \Big\| = 0, \qquad (11)$$

with $\hat{x} = (x_1, \ldots, x_i + \theta |x_i - y_i|, \ldots, x_j + \theta |x_j - y_j|, \ldots, x_n)$, $0 \leqslant \theta \leqslant 1$.
(10) holds using the fact that $a(x)$ is the diffusion matrix of the process considered, using the definition (6) of $\vartheta(L)$ and estimation (3). For (11) we need that $b(x)$ is the drift vector of the process and estimation (3) such that (11) is satisfied if $1/t \int_{|x-y|<\epsilon} (y_i - x_i) p(t,x,y)\, \mathrm{d}y f_{x_i}(x)$ is in $L^2(\mathbb{R}^n)$. This holds because by Assumptions 1 and the assumptions in the theorem the process considered is a unique solution of the stochastic differential equation corresponding to $b(x)$ and $\sigma(x)$, where $\sigma(x)$ is given by $a(x) = \sigma(x)\sigma(x)$. Hence (see e.g. Friedman [17])

$$\Big|\frac{1}{t} \int_{|x-y|<\epsilon} (y_i - x_i)p(t, x, y)\, \mathrm{d}y \Big| \leqslant \Big| \int_{\Omega_x} \frac{1}{t} \int_0^t b_i(\omega(s))\, \mathrm{d}s P_x(\mathrm{d}\omega) \Big| = (C_1 + C_2 x_i^2)^{1/2}$$

with some positive constants C_1, C_2. By (6) $(1 + x_i^2)^{1/2} f_{x_i}(x) \in L^2(\mathbb{R}^n)$.

(b) If L is symmetric and semibounded on $C_0^\infty(\mathbb{R}^n) \subset \vartheta(K)$ the same holds for K because C_0^∞ is dense in $\vartheta(K)$. Then the Hille–Yosida theorem provides the selfadjointness of K.

Proof of Theorem 2. For K and V_M the Trotter product formula holds. Then the representation (7) of the semigroup $\mathrm{e}^{-t(K+V_M)}$ follows by the definition of the measure $P_x\{.\}$. The C_0-property is a consequence of the C_0-property of e^{-tK}.

Proof of Theorem 3. By the dominated convergence theorem it suffices to prove the pointwise convergence of $(\mathrm{e}^{-tK_M} f)(x)$. Using (7) we have for each $f \in L^2(\mathbb{R}^n)$ and for $N > M$

$$(\mathrm{e}^{-tK_M} f)(x) - (\mathrm{e}^{-tK_N} f)(x) = \int_{\Omega_x} f(\omega(t))(\mathrm{e}^{-T_t(\omega)M} - \mathrm{e}^{-T_t(\omega)N}) P_x(\mathrm{d}\omega).$$

$T_t(\omega)$ is the penetration time of $\omega(s)$ in G during $[0, t]$. Because the paths are continuous the integral over $\{\omega\colon T_t(\omega) < \epsilon\}$ tends to zero as $\epsilon \to 0$. And the integral over $\{\omega\colon T_t(\omega) \geqslant \epsilon\}$ is smaller than $\mathrm{e}^{-\epsilon M} \|f\|$ tending to zero as $M \to \infty$. Calling the limit $U(t)$ (8) follows on account of the definition of $P_x\{.\}$. By (8) $U(t) = U(t)P = PU(t)$ follows immediately. The C_0-semigroup property of $U(t)$ comes from that of e^{-tK_M}.

Proof of the Lemma. Let $\{t_j\}$ be a countable dense set in $[0, t]$. Let $x \in \mathbb{R}^n \backslash G$, $\rho(x, G) > 0$.

On account of the continuity of $\omega(t)$ we obtain $P_x\{\omega: T_t(\omega)>0\} \leqslant P_x\{\cup_j \{\omega: |\omega(t_j)-x| \geqslant \frac{1}{2}\rho(x, G)\}$. Using the Markov process theory (see Dynkin [18] p. 128) this is smaller than $P_x\{\omega: |\omega(t)-x| \geqslant \frac{1}{8}\rho(x, G)\} + P_x\{\omega: |\omega(s)-\omega(t)| \geqslant \frac{1}{8}\rho(x, G)\}$ with some $s(\omega) \in [0, t]$. By means of (3) it follows

$$P_x\{\omega: T_t(\omega)>0\} \leqslant C_1 \int\limits_{C_2(\rho^2(x,G)/t)}^{\infty} \exp\left(-\frac{u}{4}\frac{n-2}{2}\right) du.$$

That implies estimate (9) if $n>2$.

Proof of Theorem 4. (a) Let $f \in L_0^2(\mathbb{R}^n \backslash G)$ with support S. Let G_1 be a region, such that $G \subset G_1 \subset \mathbb{R}^n \backslash S$ with $\rho(G, \mathbb{R}^n \backslash G_1) \geqslant d$, $\rho(G_1, S) \geqslant d$ with some $d>0$. Then

$$\begin{aligned} \|t^{-1}(f - e^{-tK_\infty}f) - (K+V_R)f\|^2_{L^2(\mathbb{R}^n)} &\leqslant \|t^{-1} e^{-tK_\infty}f\|_{L^2(G_1)} + \\ &+ \|t^{-1}(e^{-tK_\infty}f - e^{-t(K+V_R)}f)\|^2_{L^2(\mathbb{R}^n\backslash G_1)} + \\ &+ \|t^{-1}(f - e^{-t(K+V_R)}f) - (K+V_R)f\|^2_{L^2(\mathbb{R}^n\backslash G_1)}. \end{aligned}$$

The first term tends to zero as $t \to 0$ because of (3) and $\rho(G_1, S) \geqslant d$. The second term vanishes by use of the Lemma. The third term is zero as $t \to 0$, obviously.

(b) If K is selfadjoint, semibounded the same holds for K_∞ in the same manner as in Theorem 1(b). The proof that K is exactly the Friedrichs extension of $K+V_R$ restricted to $L^2(\mathbb{R}^n \backslash G) \cap \vartheta(K)$ is given in [10].

Proof of Theorem 5. (a) Let $\Omega_x^{y,t}$ be the set of all continuous functions from $[0, t]$ with $\omega(0)=x$ and $\omega(t)=y$. On $\Omega_x^{y,t}$ there is a conditional measure $P_x^{y,t}(\cdot)$ corresponding to $P_x(\cdot)$ such that for $x \in \mathbb{R}^n \backslash G$.

$$P_x^{y,t}\{\omega: T_t(\omega)>0\} = p(t, x, y) P_x\{\omega: T_t(\omega)>0\}.$$

The advantage of $P_x^{y,t}$ is that the semigroups e^{-tK_M} and e^{-tK_∞} turn out to be integral operators with kernels

$$\int\limits_{\Omega_x^{y,t}} \exp\left\{-\int\limits_0^t V_M(\omega(s))\, ds\right\} P_x^{y,t}(d\omega); \quad \int\limits_{\Omega_x^{y,t}} \chi(\omega) \exp\left\{-\int\limits_0^t V_R(\omega(s))\, ds\right\} P_x^{y,t}(d\omega).$$

Let the operator D be given by $(Df)(x) = (1+|x|^2)^{n/2} f(x)$. The trace norm, if it exists, can be estimated by

$$\|e^{-2tK_M} - e^{-2tK_\infty}P\|_1 \leqslant \|e^{-tK_M}D^{-1}\|_2 \, \|D(e^{-tK_M} - e^{-tK_\infty}P)\|_2 +$$

$$+ \|D^{-1} e^{-tK_\infty}P\|_2 \, \|(e^{-tK_M} - e^{-tK_\infty}P)D\|_2 ,$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm. The kernels of $e^{-tK_M}D^{-1}$ and $D^{-1}e^{-tK_\infty}P$ are in $L^2(\mathbb{R}^n \cdot \mathbb{R}^n)$, trivially. $\|D(e^{-tK_M} - e^{-tK_\infty}P)\|_2$ is smaller than

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} dx \, dy (1 + |x|^2)^n \times$$

$$\times \left\{ \int_{\Omega_x^{y,t}} \left[\exp\left(-\int_0^t V_M(\omega(s)) \, ds \right) - \chi(\omega) \exp\left(-\int_0^t V_R(\omega(s)) \, ds \right) \right] P_x^{y,t}(d\omega) \right\}^2 .$$

Because of Theorem3 the integrand tends to zero as $M \to \infty$. Its square root is smaller than $(1 + |x|^2)^{n/2} p(t, x, y) P_x \{\omega: T_t(\omega) > 0\}$, which is in $L^2(\mathbb{R}^n \cdot \mathbb{R}^n)$ by the lemma.

(b) Because $e^{-K_\infty}P - P\,e^{-K_M}$ and $e^{-K_M}P - P\,e^{-K_\infty}$ are trace class operators the two-space wave operators $W_\pm(e^{-K_\infty}, P, e^{-K_M})$ exist and are complete. Hence the invariance principle for wave operators (see Pearson [19]) provides the existence of $W_\pm(K_\infty, P, K_M)$ and $W_\pm(K_M, P, K_\infty)$.

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