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Fusion of the Eight Vertex SOS Model

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Abstract. A higher spin analogue is presented of the eight vertex-SOS correspondence in the sense of Andrews–Baxter–Forrester. The resulting hierarchy of solvable models contain the hard hexagon model and its recent multi-state generalizations.

1. Introduction

Among exactly solved models in two-dimensional lattice statistics, the eight vertex model and the hard hexagon model stand as the prototype [1]. In these models, the fluctuation variables are placed either on each edge or on each site of the lattice, and are allowed to take two possible states. The Boltzmann weights satisfy the star-triangle relation (STR), which ensures solvability of the model.

It has been noted earlier by Baxter [2] that one can associate to the eight vertex model a series of multi-state SOS models in which neighbouring heights must differ by unity (see Section 4 below). Subsequently, the hard hexagon model was identified as the simplest nontrivial one in this SOS hierarchy [3].

The algebraic structure that underlies the eight vertex model is a deformation of the universal enveloping algebra of $\mathfrak{sl}(2)$ as formulated by Sklyanin [4]. From this point of view, the original eight vertex model corresponds to the spin $\frac{1}{2}$ representation. In [5] was developed a method to generate vertex models corresponding to higher spin representations, sometimes referred to in the literature as the fusion procedure. One is then tempted to examine SOS hierarchies associated with these 'higher spin' models. The aim of the present article is to execute this program. In so doing we find that the recent $(N+1)$-state generalization of the hard hexagon model of Kuniba–Akutsu–Wadati [6] $(N = 2, 3, 4)$ and Baxter–Andrews [7] $(N = 2)$ is actually contained in the spin $N/2$ SOS hierarchy.

In Section 2 we reformulate the fusion procedure in terms of the eight vertex SOS model. For each N we obtain a series of models in which the height variables l_i run over integers with the restriction $l_i - l_j = N, N-2, \ldots, -N$ for adjacent sites i, j. The Boltzmann weights are Nth order polynomials of elliptic theta functions of u depending on arbitrary parameters λ and ξ. An explicit formula is given in the appendix. In Section 3, we describe a reduction of these models by specializing ξ to 0 and λ to a fraction $2K/L$ of the period, thereby getting a model with finitely many states. In Section 4, we discuss the relation between the fusion procedures for the vertex and SOS models.

Throughout the paper we use the following notations for the Jacobian theta functions of half periods K, iK' [8]:

$$[u] = H(\lambda u)\Theta(\lambda u) \quad (\lambda\text{: a constant}),$$

$$[u]_m = [u][u-1]\dots[u-m+1], \quad \begin{bmatrix} u \\ m \end{bmatrix} = [u]_m/[m]_m .$$

2. Fusion of the Eight Vertex SOS Model

Consider a weight $W_{pq}(a, b, c, d \mid u)$ where $p, q \in \mathbb{N}$, $a, b, c, d \in \mathbb{Z}$ and $u \in \mathbb{C}$. Here, a, b, c, d represent the 'heights' of four surrounding sites of a face at the SW, SE, NE, NW corners, respectively. We require that the heights l_i, l_j of two adjacent sites i, j are subject to the following condition

$$(l_i - l_j + N)/2 \in \{0, 1, \dots, N\}, \tag{1}$$

where $N = p$ for a horizontal pair (i, j), $N = q$ for a vertical pair (i, j). Sometimes we denote the (p, q)-weight $W_{pq}(a, b, c, d \mid u)$ by

$$\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)_{pq} .$$

By the STR of type (p, q, r) we mean

$$\sum_g W_{pq}(a, b, g, f \mid u) W_{pr}(f, g, d, e \mid u + v) W_{qr}(g, b, c, d \mid v)$$
$$= \sum_g W_{qr}(f, a, g, e \mid v) W_{pr}(a, b, c, g \mid u + v) W_{pq}(g, c, d, e \mid u) . \tag{2}$$

Let us construct such a solution to (2) that the (1, 1)-weight is given by the weight of the eight vertex SOS model [2]:

$$\left(\begin{matrix} l & l \pm 1 \\ l \pm 1 & l \pm 2 \end{matrix}\middle| u\right)_{11} = [u + 1]/[1], \qquad \left(\begin{matrix} l & l \pm 1 \\ l \pm 1 & l \end{matrix}\middle| u\right)_{11} = [\xi + l \mp u]/[\xi + l],$$

$$\left(\begin{matrix} l & l \pm 1 \\ l \mp 1 & l \end{matrix}\middle| u\right)_{11} = [\xi + l \pm 1][u]/[1][\xi + l], \tag{3}$$

where ξ is a constant.

The $(p, 1)$-weight is obtained by 'fusing' the (1, 1)-weight:

$$W_{p1}(a, b, c, d \mid u)$$
$$= \left(\sum \prod_{i=1}^{p} W_{11}(b_{i-1}, b_i, c_i, c_{i-1} \mid u + p - i)\right) [1]^{p-1}/[u + p - 1]_{p-1}, \tag{4}$$

where $b_0 = a$, $b_p = b$, $c_0 = d$, $c_p = c$, the sum is over $b_1, \ldots, b_{p-1}$ such that $|b_i - b_{i+1}| = 1$ and $c_1, \ldots, c_{p-1}$ are arbitrary under $|c_i - c_{i+1}| = 1$. By using the STR(2) with $u = -1$ for the (1, 1)-weight, one can show that (4) is independent of the choice of $c_1, \ldots, c_{p-1}$. In fact, we have

$$\begin{aligned}
\begin{pmatrix} l & l' \\ l+1 & l'+1 \end{pmatrix}\!\!\left|u\right)_{p1} &= \left[\xi + \frac{l+l'-p}{2}\right]\left[u + \frac{l'-l+p}{2}\right]\Big/[1]\,[\xi + l]\,, \\
\begin{pmatrix} l & l' \\ l+1 & l'-1 \end{pmatrix}\!\!\left|u\right)_{p1} &= \left[\xi - u + \frac{l+l'-p}{2}\right]\left[\frac{l'-l+p}{2}\right]\Big/[1]\,[\xi + l]\,, \\
\begin{pmatrix} l & l' \\ l-1 & l'+1 \end{pmatrix}\!\!\left|u\right)_{p1} &= \left[\xi + u + \frac{l+l'+p}{2}\right]\left[\frac{l-l'+p}{2}\right]\Big/[1]\,[\xi + l]\,, \\
\begin{pmatrix} l & l' \\ l-1 & l'-1 \end{pmatrix}\!\!\left|u\right)_{p1} &= \left[\xi + \frac{l+l'+p}{2}\right]\left[u + \frac{l-l'+p}{2}\right]\Big/[1]\,[\xi + l]\,.
\end{aligned} \tag{5}$$

For general (p, q), the weight is given by (see the appendix)

$$\begin{aligned}
&W_{pq}(a, b, c, d \mid u) \\
&\quad = W_{qp}(a, d, c, b \mid u + p - q) \\
&\quad = \left(\sum \prod_{i=1}^{q} W_{p1}(a_i, b_i, b_{i-1}, a_{i-1} \mid u - q + i)\right)[1]^q/[q]_q \quad (q \leqslant p)\,,
\end{aligned} \tag{6}$$

where $a_0 = d$, $a_q = a$, $b_0 = c$, $b_q = b$, and the sum is over $a_1, \ldots, a_{q-1}$ such that $|a_i - a_{i+1}| = 1$. Again, (6) is independent of the choice of $b_1, \ldots, b_{q-1}$ under $|b_i - b_{i+1}| = 1$.

The following is a direct consequence of the fact that the (1, 1)-weight satisfies the STR.

THEOREM 1. *The system of weights* $W_{pq}(a, b, c, d \mid u)$ $(p, q \in \mathbb{N})$ *given by* (3)–(6) *satisfies the STR* (2).

Define

$$S_{pq}(a, b, c, d \mid u) = \left(\frac{(a, b)_p (d, a)_q}{(d, c)_p (c, b)_q}\right)^{1/2} W_{pq}(a, b, c, d \mid u)\,,$$

$$\begin{aligned}
&(l, l')_p^2 = (l', l)_p^2 \\
&\quad = \begin{bmatrix} \xi + \dfrac{l+l'+p}{2} \\ \dfrac{l-l'+p}{2} \end{bmatrix}\begin{bmatrix} \xi + \dfrac{l+l'+p}{2} \\ \dfrac{l'-l+p}{2} \end{bmatrix}\begin{bmatrix} \xi + l - 1 \\ \dfrac{l-l'+p}{2} \end{bmatrix}\begin{bmatrix} \xi + l' - 1 \\ \dfrac{l'-l+p}{2} \end{bmatrix}\Bigg/\begin{bmatrix} p \\ \dfrac{l-l'+p}{2} \end{bmatrix}^4.
\end{aligned}$$

The STR (2) is equally valid for S_{pq}. Using the explicit formulas in the appendix, we have

THEOREM 2.

$$S_{pq}(a, b, c, d \mid u) = S_{pq}(c, d, a, b \mid u),$$
$$= (g_a g_c / g_b g_d) S_{pq}(b, a, d, c \quad -p + q - 1 - u),$$

where

$$g_l = \varepsilon_l \sqrt{[\xi + l]}, \qquad \varepsilon_l^2 = 1 \quad \text{and} \quad \varepsilon_l \varepsilon_{l+1} = (-)^l.$$

3. Restricted Model

In this section, we will treat the case $p = q = N$ only. So far the parameters λ, ξ were arbitrary. We now specialize them to $\lambda = 2K/L$ (L: a positive integer $> N$), $\xi = 0$, and impose besides (1) the following restrictions to the height variables l_i:

$$l_i \in \{1, 2, \ldots, L - 1\}, \qquad N < l_i + l_j < 2L - N, \tag{7}$$

where i and j are adjacent sites. This provides us with finitely many Boltzmann weights. In [3] the case $N = 1$ was considered and the model obtained was called the restricted SOS model. We have in general

THEOREM 3. *For an arbitrary N, the weights of the restricted SOS model satisfy the STR (2) with $p = q = r = N$ among themselves.*

Sketch of proof. Because of the lemma in [9], it is sufficient to show the unitarity

$$A_{N,k,l}(u) A_{N,k,l}(-u) = \begin{bmatrix} N + u \\ N \end{bmatrix} \begin{bmatrix} N - u \\ N \end{bmatrix} I_M, \tag{8}$$

where

$$A_{N,k,l}(u)$$
$$= (S_{NN}(l + N - 2r, l + 2(N - k), l + N - 2s, l \mid u))_{m_1 \leqslant r, s \leqslant m_2},$$

$$M = m_2 - m_1 + 1, \quad m_1 = \max(0, 2N - k + l - L + 1) \quad \text{and} \quad m_2 = \min(l - 1, k).$$

If $m_1 = 0$ and $m_2 = k$, then (8) follows from Theorem 1. Other cases reduce to this case by the following identities.

$$S_{NN}(l + N - 2r, l + 2(N - k), l + N - 2s, l \mid u)$$
$$= ([N - u]_{k-l+1} / [N]_{k-l+1}) S_{\overline{N}\overline{N}}(\bar{l} + \overline{N} - 2r, \bar{l} + 2(\overline{N} - \bar{k}), \bar{l} + \overline{N} - 2s, \bar{l} \mid u)$$
$$(\overline{N} = N - k + l - 1, \quad \bar{k} = l - 1, \quad \bar{l} = k + 1, \quad m_1 \leqslant r, s \leqslant \bar{k} < k),$$
$$S_{NN}(L - a, L - b, L - c, L - d \mid u) = S_{NN}(a, b, c, d \mid u).$$

The resulting models are grouped together by parity of N, L, and l_i. For even N, it suffices to consider the three cases (a) L even, l_i even (b) L even, l_i odd (c) L odd, l_i odd. For odd N, the lattice is divided into two sublattices X, Y in such a way that l_i is odd for $i \in X$ and l_i is even for $i \in Y$. There are two cases (d) L odd (e) L even. In the case (e), the number of states is $L/2$ for $i \in X$ and $L/2 - 1$ for $i \in Y$.

These hierarchies contain several known models. For $N = 2$, the cases (a) and (b) are obtained in [10] and [11], respectively (communicated by A. Kuniba). The two cases $N = k - 1$, $L = 2k + 1$, and $N = k$, $L = 2k + 1$ of (c) and (d) lead to the same model. In terms of the variable σ_i = the integer part of $|l_i - L/2|$, the restrictions (1), (7) then read as $\sigma_i = 0, 1, \ldots, k - 1$, $\sigma_i + \sigma_j \leqslant k - 1$. For small k this model coincides with the hard hexagon model [1] ($k = 2$) and its generalizations obtained in [6] ($k = 3, 4, 5$) and [7] ($k = 3$).

4. Vertex-SOS Correspondence

Originally the fusion procedure was developed for vertex models [5]. Here we relate the construction therein to our SOS model.

The fusion procedure for the eight vertex model goes as follows [12]. Let

$$R(u) = \sum_{a=0}^{3} w_a(u)\sigma_a \otimes \sigma_a ,$$

$$w_{0,3}(u) = \Theta(\lambda)\,(\Theta(\lambda u)H(\lambda(u+1)) \pm H(\lambda u)\Theta(\lambda(u+1)))\,,$$

$$w_{1,2}(u) = H(\lambda)\,(\Theta(\lambda u)\Theta(\lambda(u+1)) \pm H(\lambda u)H(\lambda(u+1)))\,,$$

be the matrix of vertex weights of the eight vertex model, where σ_0 = id. and σ_a are the Pauli matrices. We regard $R(u)$ as an operator acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Let p, q be positive integers and let $V_i = V'_j = \mathbb{C}^2\,(i = 1, \ldots, p, j = 1, \ldots, q)$. By $R^{ij}(u)$ we mean the operator $R(u)$ acting on $V_i \otimes V'_j$. We define an operator acting on $(V_1 \otimes \cdots \otimes V_p) \otimes V'_j$ by

$$\begin{aligned} &R_{(1\ldots p)j}(u) \\ &\quad = P_{(1\ldots p)}R^{1j}(u+p-1)\cdots R^{pj}(u)P_{(1\ldots p)}/[u+p-1]_{p-1}\,, \end{aligned}$$

where $P_{(1\ldots p)}$ denotes the projector on the space of symmetric tensors in $V_1 \otimes \cdots \otimes V_p$. We also define an operator acting on $V_1 \otimes \cdots \otimes V_p \otimes V'_q \otimes \cdots \otimes V'_1$ by

$$R_{pq}(u) = P'_{(q\ldots 1)}R_{(1\ldots p)1}(u)\cdots R_{(1\ldots p)q}(u-q+1)P'_{(q\ldots 1)}\,.$$

The matrix elements of $R_{pq}(u)$ are the vertex weights of the spin$(p/2, q/2)$ representation. Define vectors

$$\phi_{ab}(u) = {}^t(H(\lambda(s_a^\pm \mp u)), \Theta(\lambda(s_a^\pm \mp u)))$$

if $b = a \pm 1$, $\phi_{ab}(u) = 0$ otherwise. Here $s_a^\pm = s^\pm + a$ and $s^\pm$ are constants related to

ξ by $\xi = (s^+ + s^-)/2 - K/\lambda$. Put

$$\phi_{p,ab}(u) = P_{(1\dots p)}(\phi_{a_0a_1}(u+p-1) \otimes \cdots \otimes \phi_{a_{p-1}a_p}(u))) \quad (a_0 = a, a_p = b),$$

$$\phi'_{q,ab}(u) = P'_{(q\dots 1)}(\phi_{a_0a_1}(u+q-1) \otimes \cdots \otimes \phi_{a_{q-1}a_q}(u)) \quad (a_0 = a, a_q = b),$$

where a_i is a sequence of integers satisfying $|a_i - a_{i+1}| = 1$. These are independent of the choice of a_i. We have then

$$R_{pq}(u)\,(\phi_{p,dc}(u) \otimes \phi'_{q,cb}(0)) = [q]_q \sum_a W_{pq}(a,b,c,d\,|\,u)\phi_{p,ab}(u) \otimes \phi'_{q,da}(0) \quad (p \geqslant q).$$

This type of relation was first established by Baxter [2] for the case $p = q = 1$. We have checked that the vertex weights corresponding to the spin(1, 1) representation are identical with those of the model of Fateev [13], as pointed out in [5]. We note also that Theorem 3 of [12] fails except in the case $p = 1$ or $q = 1$, an error due to overcounting common zeros of the weights.

Appendix

We give below formulas for the unsymmetrized Boltzmann weights $W_{pq}(a, b, c, d\,|\,u)$ (6). In either of the four cases $|a - b| = p$ or $|b - c| = q$ or $|c - d| = p$ or $|d - a| = q$ they have the factorized form:

$$\begin{pmatrix} l & l+2m-p \\ l+2r-q & l+2m-p+q \end{pmatrix}\!\!\bigg|\, u \bigg)_{pq} = \frac{\begin{bmatrix} p-m \\ q-r \end{bmatrix}\begin{bmatrix} \xi+l+m+r-p-1 \\ r \end{bmatrix}\begin{bmatrix} m+u \\ r \end{bmatrix}\begin{bmatrix} \xi+l+m+u \\ q-r \end{bmatrix}}{\begin{bmatrix} \xi+l+r \\ q-r \end{bmatrix}\begin{bmatrix} \xi+l+2r-q-1 \\ r \end{bmatrix}},$$

$$\begin{pmatrix} l & l+2m-p \\ l+2r-q & l+2m-p-q \end{pmatrix}\!\!\bigg|\, u \bigg)_{pq} = \frac{\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} \xi+l+m \\ q-r \end{bmatrix}\begin{bmatrix} p-m+u \\ q-r \end{bmatrix}\begin{bmatrix} \xi+l+m-p+r-1-u \\ r \end{bmatrix}}{\begin{bmatrix} \xi+l+r \\ q-r \end{bmatrix}\begin{bmatrix} \xi+l+2r-q-1 \\ r \end{bmatrix}}.$$

Using these weights we have in general

$$\left(\begin{matrix} l & l' \\ l+2r-q & l'+2s-q \end{matrix}\middle| u\right)_{pq} \begin{bmatrix} q \\ s \end{bmatrix}$$

$$= \sum_{j=\max(0,r+s-q)}^{\min(r,s)} \left(\begin{matrix} l & l' \\ l+2j-s & l'+s \end{matrix}\middle| u-q+s\right)_{ps} \times$$

$$\times \left(\begin{matrix} l+2j-s & l'+s \\ l+2r-q & l'+2s-q \end{matrix}\middle| u\right)_{p,q-s}$$

$$= \sum_{k=\max(0,r-s)}^{\min(q-s,r)} \left(\begin{matrix} l & l' \\ l+2k-q+s & l'-q+s \end{matrix}\middle| u-s\right)_{p,q-s} \times$$

$$\times \left(\begin{matrix} l+2k-q+s & l'-q+s \\ l+2r-q & l'+2s-q \end{matrix}\middle| u\right)_{ps}.$$

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