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On the Classical Solutions of the Liouville Equation in a Four-Dimensional Euclidean Space

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Abstract. In terms of Fueter analytic functions, a classical solution of the Liouville equation in a four-dimensional Euclidean space depending on two arbitrary complex functions is obtained and discussed.

Functions with Fueter analyticity [1–4] may be used to solve various differential equations. In particular, as is well known, free Maxwell equations may be rewritten in a form coinciding with the Fueter analyticity condition, and every pure imaginary Fueter analytic function provides a solution to some of these equations [5]. The Laplace equation may be used as another example for obtaining solutions with the help of such functions [6].

On the other hand, these functions realize some transformations in four-dimensional space just as the usual analytic functions do in the complex plane. They act on and preserve the set of solutions of the above-mentioned differential equations. That is why we say that these equations are invariant with respect to the action of some operators forming an infinite-dimensional algebra, which can be obtained as a Lie algebra of these transformations.

In this Letter we show how to construct some classical solutions of the Liouville equation in a four-dimensional Euclidean space in terms of Fueter analytic functions.

Let us write the Liouville equation in our case in the form

$$\square \varphi(x) + \exp[\lambda\varphi(x)] = 0 \tag{1}$$

where

$$\square = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu} \quad (\mu = 0, 1, 2, 3)$$

is the four-dimensional Laplace operator and λ is a constant. The field $\varphi(x)$ is a complex field.

Then we can prove the following theorem:

THEOREM. *Let $A(x)$ and $B(x)$ be two arbitrary complex Fueter analytic functions. Then the function*

$$\varphi(x) = \frac{1}{\lambda} \ln \frac{8\, \partial_0 A(x)\, \partial_0 \bar{B}(x)}{\lambda[A(x) - \bar{B}(x)]^2}, \tag{2}$$

where $\bar{B}(x)$ denotes the complex conjugated function, is a solution of Equation (1).

The Fueter analyticity for the functions $A(x)$ and $B(x)$ means that they satisfy the following conditions:

$$\mathscr{D}A(x) = \mathscr{D}B(x) = 0, \tag{3}$$

where the operator $\mathscr{D}$ has the form

$$\mathscr{D} = \partial_0 + e_i \partial_i \quad (i = 1, 2, 3); \qquad \partial_\mu = \frac{\partial}{\partial x_\mu} \tag{4}$$

and e_i denotes the quaternion units

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k. \tag{5}$$

On the other hand, the theorem says that $A(x)$ and $B(x)$ are complex functions. To combine these two statements, we can consider the imaginary unit of the complex functions $A(x)$ and $B(x)$ as one of the quaternion units. This means that we assume $A(x)$ and $B(x)$ to be some quaternion functions but depending on one of the quaternion units only. In this case, the complex conjugation of the function $B(x)$ coincides with the quaternion conjugation of the same function.

To show that such functions exist, we use the method suggested in a previous paper [6]. There we can see that to every analytic function $w(z_1, z_2)$ of two complex variables corresponds a one-parameter family of Fueter analytic functions. To briefly describe this method, let us denote by

$$U = U(u_1, v_1; u_2, v_2), \qquad V = V(u_1, v_1; u_2, v_2)$$

the real and imaginary parts of $w(z_1, z_2)$ respectively, where

$$u_\alpha = \operatorname{Re} z_\alpha, \qquad v_\alpha = \operatorname{Im} z_\alpha \quad (\alpha = 1, 2).$$

If, moreover, we denote

$$\xi_\pm = x_0 \pm e_1 x_1; \qquad \eta_\pm = x_3 \pm e_1 x_2, \tag{6}$$

then the functions

$$f(x) = R(x) + e_3 J(x) \tag{7}$$

where

$$\begin{aligned} R(x) &= U(\xi_-, \eta_+; \xi_+, \eta_-) + cU(\xi_+, \eta_-; \xi_-, \eta_+), \\ J(x) &= V(\xi_-, \eta_+; \xi_+, \eta_-) + cV(\xi_+, \eta_-; \xi_-, \eta_+) \end{aligned} \tag{8}$$

(c is an arbitrary constant) have Fueter analyticity.

Because U is a real function, $U(\xi_-, \eta_+; \xi_+, \eta_-)$ and $U(\xi_+, \eta_-; \xi_-, \eta_+)$ are quaternions conjugated to each other. From (6), we can see that this function contains only one of the quaternion units $- e_1$. The same is valid for the function V.

Now if we choose

$$c = 1$$

then the functions $R(x)$ and $J(x)$ become real. In this case, $f(x)$ is a usual complex function and the imaginary unit coincides with e_3. After taking $A(x)$ and $B(x)$ as functions of the type described above, we can prove our theorem. For this purpose we must insert $\varphi(x)$ from Equation (2) into Equation (1). In the course of these calculations, one makes use of the relations

$$\Box = \mathscr{D}\tilde{\mathscr{D}} = \tilde{\mathscr{D}}\mathscr{D}$$

where $\tilde{\mathscr{D}} = \partial_0 - e_i \partial_i$ (the quaternion conjugate) and

$$\tilde{\mathscr{D}}\overline{B}(x) = \mathscr{D}A(x) = 0 ,$$

which lead to the following identities:

$$e_i \partial_i A(x) = - \partial_0 A(x) ,$$

$$e_i \partial_i \overline{B}(x) = \partial_0 \overline{B}(x) .$$

(Note that $A(x)$ and $\overline{B}(x)$ commute with each other.) Using direct calculations, we can obtain the relations

$$\Box \ln \frac{\partial_0 A \, \partial_0 \overline{B}}{(A - \overline{B})^2} = -8 \frac{\partial_0 A \, \partial_0 \overline{B}}{(A - \overline{B})^2} . \tag{9}$$

Using the last equality, it is easy to prove our theorem.

Finally, let us note that the obtained solution (2) has two different symmetries. The first one is connected with the group SL(2, R). Actually, the quantity

$$\frac{\partial_0 A(x) \, \partial_0 \overline{B}(x)}{[A(x) - \overline{B}(x)]^2}$$

is invariant under the following fractional linear transformation

$$A(x) \to \frac{\alpha A(x) + \beta}{\gamma A(x) + \delta} ; \qquad B(x) \to \frac{\alpha B(x) + \beta}{\gamma B(x) + \delta} , \tag{10}$$

where the 2×2 matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is an arbitrary element of the group SL(2, R). That is why solution (2) is also invariant under the transformations (10).

The second symmetry is connected with the group of diffeomorphisms of the four-dimensional real space as a manifold preserving the Fueter analyticity of functions $A(x)$ and $B(x)$. In our case, we can construct such diffeomorphisms on the basis of the correspondence between the analytic functions $w(z_1, z_2)$ of two complex variables and Fueter analytic functions, which we have considered above. Actually, to every diffeomorphism of the two-dimensional complex space as a manifold on which the analytic function $w(z_1, z_2)$ is defined, corresponds some diffeomorphism of the four-dimensional space on which the corresponding Fueter analytic function is defined. Then it is easy to see that, as a result of such transformation, the functions $A(x)$ and $B(x)$ preserve its Fueter analyticity and, on the other hand, they remain usual complex functions (we must keep in mind that $c = 1$). The group of these diffeomorphisms is an infinite-dimensional Lie group and its algebra has been described in [6] in this connection, see also [7]).

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