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Some Remarks on the Transverse Poisson Structures of Coadjoint Orbits

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Abstract. In this paper, we describe how to compute the transverse Poisson structures of coadjoint orbits using Dirac's constraint bracket formula, and we prove that if the isotropy algebra admits a complementary subalgebra, then the transverse structure is, at most, quadratic.

1. Introduction

It is well known that there is a natural Poisson structure on the dual $\mathfrak{G}^*$ of Lie algebra $\mathfrak{G}$, the so-called 'Lie–Poisson structure' on $\mathfrak{G}^*$. Many completely integrable systems turn out to be closely related to this special Poisson structure. In this regard, the 'Kostant–Symes theorem' is an important result. (See [1, 3, 4].) One version of this theorem says that if $\mathfrak{G} = \mathfrak{N} \oplus \mathfrak{M}$, where $\mathfrak{N}$ and $\mathfrak{M}$ are subalgebras, then the restrictions to $\mathfrak{N}^*$ (or $\mathfrak{M}^*$) of any two coadjoint invariant functions on $\mathfrak{G}^*$ commute with respect to the Lie–Poisson structure on $\mathfrak{N}^*$ (or $\mathfrak{M}^*$).

We prove that if the isotropy algebra $\mathfrak{G}_\mu$ of $\mu \in \mathfrak{G}^*$ admits a complementary subalgebra, then the transverse Poisson structure at μ is, at most, quadratic. Notice that the hypothesis in our result is incidentally(?) the same as that of Kostant–Symes. This leads us to suspect that there may be some interesting connection between our result and the Kostant–Symes. Another interesting aspect of our result is that it explains how a quadratic Poisson structure naturally arises from a geometric structure like our hypothesis. Recall that any linear Poisson structure is a Lie–Poisson structure, i.e., has its origin in a Lie algebra structure. Therefore, our theorem indicates that it might be interesting to classify the quadratic Poisson structures, which are not linearizable, with respect to some hidden geometry behind them.

2. Statement and Proof

In his paper [5], A. Weinstein gave the following 'splitting theorem' for Poisson manifolds.

PROPOSITION 1 (A. Weinstein [5]). *Let x_0 be any point in a Poisson manifold P. Then there is a neighborhood U of x_0 in P an isomorphism $\phi_S \times \phi_N$ from U to a product $S \times N$*

such that S is symplectic and the rank of N at $\phi_N(x_0)$ is zero. The factors S and N are unique up to local isomorphism.

Weinstein called the Poisson structure on N the 'transverse Poisson structure' at x_0. Although the isomorphism class of the structure is well defined, there is no natural representative, and so it was not so easy to compute the transverse structure. Meanwhile, T. Courant and R. Montgomery observed that Dirac's constraint bracket formula is useful for explicit computation and formulated the following proposition.

PROPOSITION 2. *Let x_0, U be as in Proposition* 1. *And let the ϕ_α's $= 1, \ldots, 2k$, be functions on U such that*

$$N = \{x \in U \mid \phi_\alpha(x) = \text{const. } \alpha = 1, \ldots, 2k\}.$$

$(C_{\alpha\beta}) = \{\phi_\alpha, \phi_\beta\}$, *and* $(C^{\alpha\beta})$ *be the inverse of* $(C_{\alpha\beta})$. *Then the bracket formula for the transverse Poisson structure in N is given as follows;*

$$\{F, H\}_N(x) = \{\tilde{F}, \tilde{H}\}_P(x) - \sum_{\alpha,\beta}^{2k} \{\tilde{F}, \phi_\alpha\}_P(x) C^{\alpha\beta}(x) \{\phi_\beta, \tilde{H}\}_P(x) \qquad (*)$$

for all $x \in N$, where F, H are functions on N, and $\tilde{F}$, $\tilde{H}$ are extensions of F, H to a neighborhood of N.

Proof. See the appendix. □

Notice that the formula (*) does not depend on the extensions of F, H but depends only on F, H themselves, and that the X_{ϕ_α}'s which are the Hamiltonian vector fields generated by the ϕ_α's, span the tangent space of the symplectic leaf at x_0.

Now, let us restrict our attention to the case of the coadjoint orbits. Let $\mu \in \mathfrak{G}^*$ and $\mathcal{O}_\mu$ be the coadjoint orbit of μ. To compute the transverse structure of $\mathcal{O}_\mu$ using the formula (*), we need to choose a transverse manifold N and corresponding ϕ_α's. Our main observation is that there is a kind of 'natural procedure' for choosing the ϕ_α's and N.

Recall that the tangent space of the symplectic leaf at a point in that symplectic leaf is generated by Hamiltonian vector fields. (See Lemma 1.1 in [5].) Thus, the tangent space of $\mathcal{O}_\mu$ at μ can be identified with $\mu + \mathfrak{G}_\mu^\perp$ using elementary linear algebra, where $\mathfrak{G}_\mu = \{y \in \mathfrak{G} \mid \mathrm{ad}_y^* \mu = 0\}$ and $\mathfrak{G}_\mu^\perp$ is the annihilator of $\mathfrak{G}_\mu$ in $\mathfrak{G}^*$. So we can choose N to be $\mu + \mathfrak{N}^\perp$ where $\mathfrak{N}$ is a complementary subspace of $\mathfrak{G}_\mu$, i.e., $\mathfrak{G} = \mathfrak{G}_\mu \oplus \mathfrak{N}$, and $\{\phi_\alpha\}_{\alpha=1,\ldots,2k}$ to be a basis of $\mathfrak{N}$ since $\mu + \mathfrak{N}^\perp = \{\phi_\alpha(x) = \text{const.} \ (= \phi_\alpha(\mu)), \alpha = 1, \ldots, 2k\}$. With these observations, we are almost ready to state the main theorem.

DEFINITION. We say that a Poisson structure is quadratic if we can choose a local coordinate system $\{\psi_i\}_{i=1,\ldots,p}$ in which $\{\psi_i, \psi_j\}$ are polynomials of a degree $\leqslant 2$ in the ψ_i's on a coordinate neighborhood U.

THEOREM. *If $\mathfrak{G}_\mu$ is a 'Lie summand', i.e., it has a complementary subalgebra $\mathfrak{N}$ on $\mathfrak{G}$, then the transverse structure of $\mathcal{O}_\mu$ is quadratic.*

Proof. One of the key observations is that if we choose $\{\phi_\alpha\}$ and N as above so that $\mathfrak{N}$ may be a subalgebra which exists by the hypothesis, then $C_{\alpha\beta} = \{\phi_\alpha, \phi_\beta\}$ and so the $C^{\alpha\beta}$ are constants in N, since $\mathfrak{N}$ is a subalgebra; $\{\phi_\alpha, \phi_\beta\}$ is a linear combination of the ϕ_α's and so $\{\phi_\alpha, \phi_\beta\}(x) = \{\phi_\alpha, \phi_\beta\}(\mu)$, constant for all $x \in \mu + \mathfrak{N}^\perp$.

Another observation is that we can choose, as a coordinate system on N, the linear functions on $\mathfrak{G}^*$. Furthermore, we can choose a basis of $\mathfrak{G}_\mu$ as the coordinate system since they are independent on $\mathfrak{N}^\perp$. Let's write a basis of $\mathfrak{G}_\mu$ by $\{\psi_i\}$, $i = 1, \ldots, p$ $(p + 2k = \dim \mathfrak{G})$. Then,

$$\{\psi_i, \psi_j\}_N(x) = \{\psi_i, \psi_j\}_{\mathfrak{G}*}(x) - \sum_{\alpha,\beta}^{2k} \{\psi_i, \phi_\alpha\}_{\mathfrak{G}*} C^{\alpha\beta}(x)\{\phi_\beta, \psi_j\}_{\mathfrak{G}*}(x), \quad 1 \leqslant i,\ j \leqslant p. \tag{**}$$

Here, $\{\psi_i, \phi_\alpha\}$, $\{\psi_i, \psi_j\}$ are, at most, linear functions of ψ_l in N because the ϕ_α's and ψ_i's together form a basis of $\mathfrak{G}$ and $\phi_\alpha \equiv$ const. in N. So the first term in (**) is linear and the second sum is, at most, quadratic for all i, j. So the theorem is proved. □

REMARK. It is known that the transverse structure of $\mathcal{O}_\mu$ is isomorphic to $\mathfrak{G}_\mu^*$ when $\mathfrak{G}_\mu$ has a complementary subspace such that $[\mathfrak{G}_\mu, \mathfrak{N}] \subset \mathfrak{N}$ (See [6].) This can be easily proved by our 'natural procedure'.

In Section 4 of [6], Weinstein discussed two examples whose transverse Poisson structures are not linearizable. One is the adjoint orbit of

$$\mu = \begin{pmatrix} 001 \\ 000 \\ 000 \end{pmatrix} \text{ in } \mathfrak{Sl}(3:\mathbb{R})$$

which is the example given by Givental. (Notice that $\mathfrak{Sl}(3:\mathbb{R})$ is semisimple and so $\mathfrak{Sl}^*(3:\mathbb{R})$ can be identified with $\mathfrak{Sl}(3:\mathbb{R})$ by using the Killing form.) The other is the singular orbit of the points $z = 0$, $x^2 + y^2 > 0$ in $\mathbb{R}^4$ with the structure

$$\{x, y\} = z, \qquad \{t, x\} = x, \qquad \{t, y\} = y \quad \text{and} \quad \{t, z\} = 2z.$$

By our 'natural procedure' of choosing N and the ϕ_α's, we can easily show that these two examples satisfy the hypothesis of our theorem and so they are quadratic. Somewhat long but trivial calculation (see [2]) shows that the explicit structure of Givental's example is given as follows:

$$(W_{ij}) = (\{\psi_i, \psi_j\}) = \begin{pmatrix} 0 & -\frac{14}{5}\psi_1\psi_4 & -\psi_2 + 6\psi_4^2 & \frac{3}{5}\psi_1 \\ \frac{14}{5}\psi_1\psi_4 & 0 & -\frac{14}{5}\psi_1\psi_4 & 0 \\ \psi_2 - 6\psi_4^2 & \frac{14}{5}\psi_1\psi_4 & 0 & -\frac{3}{5}\psi_3 \\ -\frac{3}{5}\psi_1 & 0 & \frac{3}{5}\psi_3 & 0 \end{pmatrix}$$

Appendix: Proof of Proposition 2

The induced structure on N is given by the composed map

$$T^*N \xrightarrow{\pi^*} T^*_N P \xrightarrow{B} T_N P \xrightarrow{\pi} TN$$

(See Prop. 1.4 of [5].) where π is the bundle projection along the symplectic leaves in U onto TN and π^* is its adjoint. In other words,

$$\{F, H\}(x_0) = W_P(\pi^* \mathrm{d}F(x_0), \pi^* \mathrm{d}H(x_0))$$

where W_P is the Poisson tensor of P. Let $\tilde{F}$, $\tilde{H}$ be any extensions of F, H, respectively. Then,

$$\pi^* \mathrm{d}F(x_0) - \mathrm{d}\tilde{F}(x_0) = \sum_\alpha a^\alpha_F(x_0)\, \mathrm{d}\phi_\alpha(x_0)$$

because

$$(\pi^* \mathrm{d}F(x_0) - \mathrm{d}\tilde{F}(x_0)(T_{x_0}N) = 0,$$

i.e., $\pi^* \mathrm{d}F(x_0) - \mathrm{d}\tilde{F}(x_0) \in (T_{x_0}N)^\perp$ in $T^*_{x_0}P$ which is exactly the span of $\{\mathrm{d}\phi_\alpha(x_0)\}$, $\alpha = 1, \ldots, 2k$. Now, extend $\{\phi_\alpha\}_{\alpha=1,\ldots,2k}$ to a coordinate system in a neighborhood of x_0 in P so that

$$\pi\left(\frac{\partial}{\partial\phi_\alpha}(x_0)\right) = 0, \quad \alpha = 1, \ldots, 2k.$$

Then,

$$a^\alpha_F(x_0) = (\pi^* \mathrm{d}F(x_0) - \mathrm{d}\tilde{F}(x_0))\left(\frac{\partial}{\partial\phi_\alpha}(x_0)\right) = -\mathrm{d}\tilde{F}(x_0)\left(\frac{\partial}{\partial\phi_\alpha}(x_0)\right)$$

because

$$\pi\left(\frac{\partial}{\partial\phi_\alpha}(x_0)\right) = 0$$

in this coordinate system.

Now,

$$\frac{\partial}{\partial\phi_\alpha}(x_0) = \sum_\gamma C^{\alpha\gamma}(x_0) X_{\phi_\gamma}(x_0).$$

Thus,

$$-\mathrm{d}\tilde{F}(x_0)\left(\frac{\partial}{\partial\phi_\alpha}(x_0)\right) = -\sum_\gamma C^{\alpha\gamma}(x_0)\, \mathrm{d}\tilde{F}(x_0) X_{\phi_\gamma}(x_0) = \sum_\gamma C^{\alpha\gamma}(x_0)\{\tilde{F}, \phi_\gamma\}(x_0)$$

therefore

$$\pi^*\mathrm{d}F(x_0) = \mathrm{d}\tilde{F}(x_0) + \sum_{\alpha,\gamma} C^{\alpha\gamma}(x_0)\, \{\tilde{F}, \phi_\gamma\}(x_0)\, \mathrm{d}\phi_\alpha(x_0).$$

Similarly,

$$\pi^* \mathrm{d}H(x_0) = \mathrm{d}\tilde{H}(x_0) + \sum_{\beta,\delta} C^{\beta\delta}(x_0)\{\tilde{H}, \phi_\delta\}(x_0)\,\mathrm{d}\phi_\beta(x_0).$$

Therefore,

$$\begin{aligned}
\{F, H\}(x_0) &= W_P(\mathrm{d}\tilde{F}(x_0) + \sum_{\alpha,\gamma} C^{\alpha\gamma}(x_0)\{\tilde{F}, \phi_\gamma\}(x_0)\,\mathrm{d}\phi_\gamma(x_0),\\
&\mathrm{d}\tilde{H}(x_0) + \sum_{\beta,\delta} C^{\beta\delta}(x_0)\{\tilde{H}, \phi_\delta\}(x_0)\,\mathrm{d}\phi_\delta(x_0))\\
&= W_P(\mathrm{d}\tilde{F}(x_0), \mathrm{d}\tilde{H}(x_0)) +\\
&\quad + \sum_{\alpha,\beta,\gamma,\delta} C^{\alpha\gamma}(x_0)\,C^{\beta\delta}(x_0)\{\tilde{F}, \phi_\gamma\}(x_0)\{\tilde{H}, \phi_\delta\}(x_0)\times\\
&\quad \times W_P\{\mathrm{d}\phi_\gamma(x_0), \mathrm{d}\phi_\delta(x_0)\}\\
&= \{\tilde{F}, \tilde{H}\}(x_0) + \sum_{\alpha,\beta,\gamma,\delta} C^{\alpha\gamma}(x_0)\,C^{\beta\delta}(x_0)\,C_{\gamma\delta}(x_0)\times\\
&\quad \times \{\tilde{F}, \phi_\gamma\}(x_0)\{\tilde{H}, \phi_\delta\}(x_0)\\
&= \{\tilde{F}, \tilde{H}\}(x_0) + \sum_{\alpha,\beta} C^{\alpha\beta}(x_0)\{\tilde{F}, \phi_\gamma\}(x_0)\{\tilde{H}, \phi_\beta\}(x_0)\\
&= \{\tilde{F}, \tilde{H}\}(x_0) - \sum_{\alpha,\beta}\{\tilde{F}, \phi_\alpha\}(x_0)\,C^{\alpha\beta}(x_0)\{\phi_\beta, \tilde{H}\}(x_0)
\end{aligned}$$

□

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