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QUANTUM AND CLASSICAL LOGIC: THEIR RESPECTIVE ROLES*

I. CLASSICAL OR QUANTUM LOGIC?

That quantum mechanics is different from classical mechanics, in what it says about the physical world and how it says it, needs no proof. How precisely to describe and explain these differences, and what significance to attach to them is being continually discussed.

One of the claims that is being made is that the most significant difference between classical mechanics and quantum mechanics is that the latter uses or needs to use a non-classical kind of propositional logic, a logic that has been called a 'quantum logic'. The classical logic is often described as Aristotelian. More accurately, it is the propositional logic of two-valued truth-functional propositions, the logic of classes and the logic of quantification as, for example, these are developed in Russell and Whitehead's *Principia Mathematica* (*PM*).

G. Birkhoff and J. von Neumann were the first in 1936 to put forward the view that the 'physical quantities' of quantum mechanics constitute an orthocomplemented non-distributive lattice, and not the Boolean algebra of classical *PM* logic.[1] Other proposals were made about the same time and later by Reichenbach, von Weizsäcker, Heisenberg and others in favor of multi-valued logics in which the classical principle *tertium non datur* is violated. Bas C. van Fraasen has given a brief and systematic survey to the various quantum logics in his paper 'The Labyrinth of Quantum Logics'. I shall not be concerned in this paper with multi-valued quantum logics, but only with the view common to Birkhoff and von Neumann, Segal, Mackey, Finkelstein, Jauch, Putnam[2] and others, that quantum logic is a non-distributive lattice.

This theory says that the basic empirical propositions of quantum mechanics obey, not the axioms and rules of classical *PM* logic, but those of a logic obtained from classical logic by dropping the distributive laws for 'and' and 'or' and replacing them by what looks like a weaker form of connection. In Jauch's version, the basic empirical propositions

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of quantum mechanics are those which express the outcome of Yes-no tests "[Yes-no tests] are observations which permit only *one* of two alternatives as an answer (hence the name Yes-no experiment). Such experiments are part of the daily routine of every experimental physicist: for example, a counter which registers the presence of a particle within a certain region of space. ... Every measurement on a physical system can be reduced at least in principle to measurements with a certain number of Yes-no experiments. ... Each measurable quantity has a certain range of values which may be indicated as a subset of the real line (or perhaps a Euclidean space). A determination of this quantity is obtained by dividing the real line into smaller intervals and then deciding whether the measured value falls within any one of the intervals. By making the intervals sufficiently small, one can determine the value of the quantity to any desired accuracy."[3] It is claimed that the kind of logic which these propositions obey is an orthocomplemented non-distributive lattice of weak modularity.[4]

In favor of this proposal are two sets of arguments, one positive the other negative. The negative argument is twofold: it is an attack on a priori intuitionistic arguments used to justify the retention of classical logic and it is an attempt to show that the retention of classical logic involves the rejection of quantum mechanics in its present form. The positive argument is the claim that logic, like physical geometry, is subject to empirical confirmation and disconfirmation procedures, and that quantum mechanics has disconfirmed classical logic in the micro-domain. It is consistent with Putnam's position and, I think, also with Finkelstein's, that as a consequence of the alleged breakdown of classical logic in the quantum domain, the distributive law of classical logic is to be held, at least suspect, in all domains (except in the postulational one of mathematics). By a kind of correspondence principle, it can be treated as a rough and ready approximate rule in everyday affairs, much as Euclidean geometry remains the rough and ready approximate geometry of the everyday world even after the general theory of relativity has shown that the physical world is in fact non-Euclidean.

II. THE CASE FOR A QUANTUM LOGIC IN QUANTUM MECHANICS

The role which a quantum logic would play in resolving some of the seemingly paradoxical aspects of quantum mechanics can be illustrated with reference to the two-slit experiment (see Figure 1). Let a_1, a_2 and b be the following sentences:

a_1: Electrons of the beam pass through slit 1
a_2: Electrons of the beam pass through slit 2
b: The distribution of electrons at the screen is the arithmetic sum of the distributions obtained when only one slit at a time is open.

It is an experimental fact that b is false.

Consider now the following argument:

Major[5]: $(a_1 \vee a_2) \supset b$
Minor: $\sim b$
Conclusion: $\sim(a_1 \vee a_2)$.

The conclusion says that it is not true that electrons of the beam pass through one or both of the slits. Let us suppose this conclusion is unacceptable, since we can suppose that the screen is shielded in such a way that electrons arriving at it could only have reached it via the slits in the diaphragm. We now ask ourselves how one might avoid the conclusion of the argument. One might avoid it by denying the major, or the minor, or the scheme of inference (the *modus tollens*), or by making distinctions in one or both of the premises. Putting distinctions aside, let us consider the three principal ways of voiding the argument.

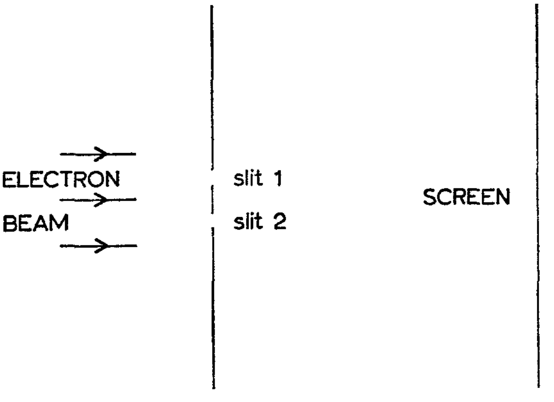

Fig. 1. Diagram of the two-slit experiment.

The minor is the statement of an experimental fact and let us suppose it is untouchable.

One might consider adopting a new logic in which the *modus tollens* was an invalid deductive scheme. None of the writers, however, seems to have considered this possibility. It happens that the *modus tollens* is a valid scheme within the quantum logic of Birkhoff and von Neumann, Finkelstein, Jauch and others. The quantum logic referred to, however, does exclude some important classical schemes, e.g., the scheme,

$$\begin{array}{l} p \vee q \\ \sim q \\ \hline p \end{array}$$

The third possibility consists in declaring the major premise false. Now the major premise involves three sentences, a_1, a_2 and b, each formulated within the language of classical physics and supposing the truth of the basic conceptual framework of classical physics. These sentences are joined together by two logical functors '$\vee$' ('or') and '$\supset$' ('implies').

Now the most natural solution which suggests itself is that the language framework and concepts of classical physics do not apply to electrons in the two slit experiment. Underlying the belief that $(a_1 \vee a_2) \supset b$ is the set of classical assumptions that each particle is an independent kinematical unit, that each travels along a uniquely determined trajectory from the source to the screen, that each particle consequently passes through one and only one slit. In this model, the set of particles arriving at any area of the screen would be the set theoretic union of two sets of particles, of those that pass through slit 1 and of those that pass through slit 2. In rejecting the validity of classical descriptive assumptions for the 2-slit experiment, one can falsify the major without impugning the validity of classical logic. On the other hand, one might claim that, whether or not classical assumptions are correct, the classical logic of classes which is used in the deriviation of b, is invalid. This latter position seems to be the residue of the claim that classical logic – or the part of it which is the logic of classes – is not valid in the kinematical description of quantum mechanical systems.

The position just outlined is not a very revolutionary one, even supposing it were unexceptional. Some of the more articulate and public spokesmen on the side of quantum logic, Putnam for example and Finkelstein, have made broader claims for the validity of quantum logic, ones which *are* quite revolutionary. They would argue that the discovery of quantum logic dethrones classical logic in all empirical domains as the discovery of the general theory of relativity dethroned Euclidean geometry in the domain of physics. The argument goes: Just as the general theory of relativity has shown that what one was accustomed to call a 'straight line' is not an Euclidean object, so what one was accustomed to call a 'proposition' is not a *PM*-classical object, but an object subject to the weaker logic of a non-distributive lattice.

III. ABOUT THE ALLEGED UNIVERSALITY OF QUANTUM LOGIC

It is not my purpose, however, to add my endorsement to the view that quantum logic *in the form proposed* say, by Jauch, Finkelstein and Putnam, is correct but rather to criticize this view and to transform it into what I believe is a better account of the role of various logics in quantum mechanics.

My first move is against the more extreme thesis that the truth of quantum logic in the quantum domain calls into question the truth of classical logic in every empirical domain. The main argument is a reasoning by analogy with the fate of Euclidean geometry in general relativistic physics. The principal author of this argument is Putnam, although it is supported by Heisenberg, von Weizsäcker, Finkelstein and others. It is based upon the analogy between physical geometry and logic. Putnam puts down the proportional equation:

$$\frac{\text{Euclidean geometry}}{\text{General theory of relativity}} = \frac{\text{Classical logic}}{\text{Quantum mechanics}}$$

In his paper 'Is Logic Empirical?' given to this colloquium, Putnam wrote: "I regard the analogy between the epistemological situation in logic and the epistemological situation in geometry as a perfect one."[6] A propos of classical logic, he writes: "Quantum mechanics explains the approximate validity of classical logic 'in the large' just as non-Euclidean geometry explains the approximate validity of Euclidean geometry 'in the small'."

To explore the analogy between geometry and logic, it is necessary to state Putnam's views about geometry. They are summarized in the following points. (i) Arguments in favor of Euclidean geometry based upon intuitionistic a priori reasons are unacceptable and false. (ii) Without empirical evidence there is no truth. The one physically correct geometry is to be determined by objective, empirical, scientific criteria. (iii) Conventionalism, or the view that a variety of geometries is admissible depending upon the rule of congruence conventionally chosen, is false to the extent that it allows human arbitrariness and subjective bias and prejudice to enter where objective criteria are or should be the sole relevant criteria.

One can have a great deal of sympathy with the thrust of these arguments, without agreeing with Putnam's interpretation of conventionalism and the consequence he draws from his rejection of it, namely, the belief that there is a unique objective physical geometry, one which is altogether removed from free conventions. I prefer to think that his views about conventionalism spring from certain options as to what he believes knowledge and objectivity imply or should imply. Putnam's belief in a unique empirically determined geometry is carried over into his belief that there is a unique kind of empirically determined logic. Since there are already a sufficient number of Putnam's adversaries in the field of geometry, I shall accept a challenge made by Putnam to present an elegant refutation of Putnam by Putnam.[7]

Putnam is committed to the defence of the position that only those schemes of inference are valid which are derivable from quantum logic. Among the invalid schemes of inference is one I have already noted:

$$\frac{\begin{array}{l} p \vee q \\ \sim q \end{array}}{p}$$

(To illustrate in what sense this scheme is held by quantum logicians to be invalid in quantum mechanics, one might consider the following example[8]:

$p \vee q$: The electron has spin up or spin horizontally to the left
$\sim q$: The electron has not spin horizontally to the left
p: The electron has spin up

The major is taken to mean that the spin state of the electron is some linear superposition of spin up and spin horizontally to the left. Such a set of spin states spans the entire spin space. Hence, the only legitimate conclusion from the minor is that the spin of the electron is horizontally to the right. This solution satisfies the major and the minor, but it is different from p, the only conclusion authorized by classical logic. The classical scheme breaks down whenever it is a question of complementary propositons p and q.)

Now it is interesting to note that the principal argument of Putnam's paper 'Is Logic Empirical?' is one whose natural expression falls into the scheme I have just referred to. After an exposition of quantum logic[9], the structure of the argument in support of quantum logic goes this way: Either quantum logic is correct or classical logic is correct; but it is not true that classical logic is correct (since this would involve the use of hidden variables, the observer-observed cut, a perturbation theory of measurement, etc.); hence quantum logic is correct. The scheme of inference for this argument is the one I have just mentioned and it is formally invalid in quantum logic if the two members of the disjunction are complementary. Putnam does not discuss whether or not his disjuncts are complementary: he seems to be unaware he is using a scheme which is no longer universally valid. I believe the propositions are complementary, but the proof depends on accepting the conclusions of this paper – namely, that one can have a quantum logic on the level of a meta-context-language of conditions together with a classical logic on the level of quantum event language.

What I have done in this section is to overturn the argument used by Putnam to support that aspect of his universal thesis which says that quantum logic is the kind of logic to be used on all occasions. I am not taking issue with that other aspect of his thesis which says that empirical evidence is relevant to the kind of propositional logic one endorses. This part of the thesis, I think, is true. My complaint, however, is that Putnam draws unwarranted conclusions and that he fails to be sufficiently attentive to the quality of the empirical evidence available.

IV. QUANTUM LOGIC IN QUANTUM MECHANICS

My next move is to examine the substance of the more modest claim that in the domain of quantum physics, quantum mechanical propositions obey a special empirical logic, the so-called quantum logic, which is an orthocomplemented non-distributive lattice with weak modularity.

In all the recent discussions of quantum logic, the argument has taken the following form: If the received version of quantum mechanics entails the use of quantum logic, then the rejection of quantum logic entails the rejection of the received version of quantum mechanics, and this consequence is further taken to entail the belief that the true theory of microphysical phenomena will be one of a classical kind not yet formulated, with 'hidden variables'.

When one turns to the recent works on quantum logic, one naturally looks for the sense in which it is novel, exciting, significant and controversial to say that quantum mechanics entails the use of a special quantum logic. I do not find a clear answer in the literature. For in the development of the argument, the antecedent is usually taken trivially to do nothing more than to label a well-known structural property of quantum mechanics related to its Hilbert space formulation. The real thrust of the argument is in the consequent, which is not really about logic, but about quantum mechanics and the alternatives to quantum mechanics.

Now the question of hidden variables or the observer-observed cut or the perturbation theory of measurement are not new problems and merely to express them in new terminology is neither novel, exciting, significant nor controversial. It is not my intention in this paper to speak about the alternatives to the present version of quantum mechanics. I am not challenging quantum mechanics in its present form, but I am concerned with the kinds of logic compatible with or required by the present form of quantum mechanics.

I am claiming that the recent defenders of quantum logic are quite confused about some fundamental things – in what they conceive to be the objects of a quantum logic within quantum physics, and in what the true significance of the new quantum logical operations consists.

Most of the current writers on quantum mechanics give as its domain the domain of those categorical propositions expressing the outcome of Yes-no tests. I say this claim is simply false. I do not deny, however, that a special quantum logic has a place within quantum physics but I put it on the level of a meta-context-language about the conditions under which particular quantum event-languages are applicable, and not, as the writers on quantum logic are accustomed to put it, on the level of quantum event-language itself.

The accounts writers give of what constitutes the class of quantum mechanical propositions which is subsumed under quantum logic are subject to at least one systematic ambiguity which conceals the real significance and true role of quantum logic. Let me first of all try to make clear the different types of propositions involved in this discussion before taking on the role of critic.

V. VARIETIES OF QUANTUM MECHANICAL PROPOSITIONS

Let me start with a class of propositons each of which states the outcome of a Yes-no experiment.[10] I shall call the categorical statement asserting or denying the outcome of a Yes-no experiment, a *simple empirical proposition.* Let me list the necessary and sufficient conditions which warrant a simple empirical proposition.

A simple empirical proposition presupposes a fully specified kind of measurement situation, and one in which only a definite part of the range of the observables is considered. If the quantum system relative to the observables in question falls into the part of the range being considered, a positive symbol '1', standing for Yes! is produced; if the quantum system does not fall within the range considered, a negative '0' standing for No! is produced. The sentence token standing for the Yes-propositions will have some form like: 'The observables X, Y for the electron now in the apparatus fall within the range $\Delta_{xy} \in B(R_x \otimes R_y)$', where $B(R_x \otimes R_y)$ is the set of Borel sets in the direct product of the ranges R_x and R_y of the observables X and Y. Let us call this proposition a. The No-proposition will have the form: 'The observables X, Y for the electron now in the apparatus do not fall within the range $\Delta_{xy} \in B(R_x \otimes R_y)$'. Since in both cases, the measurement situation is the same, the latter No-proposition is equivalent to the following Yes-proposition: 'The observables X, Y for the electron now in the apparatus fall within the complement of the range Δ_{xy} in $B(R_x \otimes R_y)$'. Let us call this proposition $\sim a$. The kind of negation in question is called by van Fraasen 'choice negation', as distinct from what he calls 'exclusion negation'. In order to warrant a or $\sim a$, it is necessary and sufficient that the physical measurement situation be fully specified in kind.

Let us suppose, what I presume Jauch and other writers on quantum logic supposed, *that the observables X and Y are compatible observables,* represented by commuting self-adjoint operators. We shall return to criticize this assumption later on in the paper.

There is also a class of more abstract propositions of the following kind: 'The projection operator $P(S)$ on the subspace S of the Hilbert space H preserves the state vector ψ; i.e., $P(S)\psi=\psi$'. Let us call this a 'simple theoretical proposition', and denote it by ϕ. Then the space of ϕ's obey a non-classical logic which is isomorphic with the orthocomplemented non-distributive lattice of subspaces of a Hilbert space. I doubt that it is necessary to prove this assertion. It has been proved by Birkhoff and von Neumann and every writer on two-valued quantum logic has used this theorem since.

Now let us suppose that the simple empirical proposition a implies ϕ ($=_{\text{def.}}$'$P(S)\psi=\psi$') through the semantical rules or uniformities, where S is the subspace of H spanned by the eigenvectors of X and Y whose eigenvalues lie in the range Δ_{xy}. Then $\sim a$ implies ϕ'($=_{\text{def.}}$'$P(S^{\perp})\psi=\psi$') where $S^{\perp}$ is the orthogonal complement of S in H. In other words,

$$a \supset \phi \tag{1}$$

and

$$\sim a \supset \phi' \tag{2}$$

Since both a and $\sim a$ presuppose a fully specified kind of measurement situation, and ϕ does not; neither (1) nor (2) are in general biconditionals. That is, it is not true in general that

$$\phi \supset a$$
$$\phi' \supset \sim a$$

The reason there is no biconditional relation between a and ϕ in general is that, as long as X and Y are supposed to be compatible observables, the subspace S might possibly be spanned by other observables, say, U and V, one or both of which are incompatible with X and Y. For example, if the proposition a is 'The spin of the electron is $\frac{1}{2}$ and polarized vertically up or down', the subspace S corresponding to this proposition is the entire spin space. The physical situation, however, also includes an apparatus for measuring vertically polarized spin. Consider, now, the proposition b 'The spin of the electron is $\frac{1}{2}$ and polarized horizontally to right or left.' The subspace corresponding to b is also S. However, in this case the physical situation includes an apparatus for measuring horizontally polarized spin. Both a and b imply ϕ, but ϕ does not imply a or b separately, although it does imply $a \vee b \vee \cdots$ where the disjunctions include all possible simple empirical propositions which imply ϕ.

I do not know to what to attribute the oversight in recent literature of the problem that maybe there is not in general a biconditional relation between a Yes-no test and a simple theoretical proposition which the Yes-no test implies. Birkhoff and von Neumann seemed to be aware of the problem: they spoke of an equivalence class of experimental propositions. But none of the recent literature mentions it. Perhaps, the authors thought it was too trivial a matter to point out. Trivial though it may seem to be, however, big and important consequences follow from it.

In the general case, then, there is at least one other simple empirical proposition b, of type a but incompatible with a, which also implies ϕ. In other words, $a \supset \phi$, $b \supset \phi$ but a is not compatible with b. Either a or b gives full empirical warrant to the proposition ϕ, but the truth of ϕ does not justify a nor does it justify b, although it does justify

$$a \vee b \vee \cdots$$

where the disjunction includes all possible simple empirical propositions of type a which imply ϕ.

Now the empirical propositions which constitute the space of quantum logic are not propositions of type a, simple empirical propositions (since they are not in biconditional relation to ϕ) but they are propositions of the type

$$a \vee b \vee \cdots =_{\text{def}} a^*(\phi) \tag{3}$$

I shall call $a^*(\phi)$ the 'empirical interpretation of the simple theoretical proposition ϕ'. A biconditional relation exists between ϕ and the chain of disjunctions $a^*(\phi)$, that is,

$$a^*(\phi) = (a \vee b \vee \cdots) \quad \text{if and only if} \quad \phi \tag{4}$$

The $a^*(\phi)$'s then constitute a space of propositions isomorphic with the space of ϕ under the logical functors 'and', 'or', 'implies' and 'not' *as these function within quantum logic.*

We have distinguished then three types of propositions in quantum mechanics: simple empirical propositions (or propositions of type a), simple theoretical propositions (propositions of type ϕ) and the empirical interpretations of simple theoretical propositions (or propositions of type $a^*(\phi)$).

A simple empirical proposition a implies the fulfillment of the following conditions, both necessary and sufficient in their respective orders: (1) in the linguistic order: that there be an event-language L_a with the names of objects (or other referring expressions), descriptive predicates corresponding to the observables of the commuting set X and Y, and the logical grammar of classical logic. Note that L_a does not contain the descriptive predicates for observables U or V which are not compatible with X and Y. (ii) In the physical order: some standard measuring conditions for measuring the ranges Δ_x and Δ_y of the observables X and Y in question. Let me call this the 'physical context of the system'. (iii) In the mathematical order: that the subspace $S \subseteq H$ be coordinatized by the eigenvectors of the commuting self-adjoint operators which represent the physical observables X, Y mentioned in (ii), and which the descriptive predicates mentioned in (i) stand for.

Returning to the language L_a. Let us identify it with the set of those sentences which can be correctly formulated, asserted or denied in the standard measuring conditions described in (ii) above. L_a is a *picturing language*. Out of its resources a variety of linguistic pictures can be put together. One of these is a; another is $\sim a$. But b, which is a proposition incompatible with a, and which implies its own language L_b, is not one of these pictures. That is, $a \in L_a$, but $b \notin L_a$. Similarly, $b \in L_b$, but $a \notin L_b$. The set of linguistic pictures that can be formulated in L_a is the mapping of the manifold of possible physical events that could occur within the limitations of the physical milieu described in (ii) above.

The standard measuring conditions of a physical context are themselves invariant relative to the manifold of possible events that could occur within that physical context. The constancy of the standard measuring conditions is correlated with the domain of applicability of language L_a, L_a being the linguistic invariant of the manifold of possible pictures. There is nothing special about the logic intrinsic to L_a or to L_b separately; it is or could be the classical propositional logic of *PM*, a Boolean algebra of statements under the operations of 'and', 'or', 'implies' and 'not'. The logic of simple empirical propositions is or could be classical.

The second type of proposition I mentioned is a simple theoretical proposition or a proposition of type ϕ. The space of these propositions constitute an orthocomplemented non-distributive lattice of the kind called a quantum logic. This was proved by Birkhoff and von Neumann in 1936 for quantum mechanics in what we now call its received form.

Are the ϕ's, however, the basic propositions about quantum mechanics to which the quantum logicians refer when they make their claim that quantum mechanical propositions obey a special non-classical logic? It would be easy to quote passages from the writings of Jauch, Finkelstein, Putnam and others to show that whatever else they had in mind this at least was part, and possibly even the whole of what quantum logic was all about. Jauch, for example, writes: "Elementary propositions of quantum mechanics are represented by projection operators, or equivalently by closed linear subspaces."[11] But if this is all the quantum logicians intend to say, then I have no criticism to make. All I would say is that their claim is neither novel, exciting, significant nor controversial.

In fact, however, additional claims are made, that these simple theoretical propositions are also *empirical propositions*, in the sense that they represent the answer to Yes-no experimental questions. Jauch writes: "We shall refer to Yes-no experiments simply as *propositions* of the physical system."[12] But Yes-no experiments are not isomorphic to simple theoretical propositions, as I have shown. A simple theoretical proposition is correlated with a disjunction of simple empirical propositions expressing the outcome of competing and generally mutually incompatible Yes-no tests. In this fact lies the empirical relevance of ϕ. But is ϕ itself a statement of empirical fact? To state an empirical fact is generally to *picture* a state of affairs in a picturing language. A picturing language generally has no more than the following resources: names of objects (or other referring expressions), empirical predicates and a logical vocabulary. The empirical picturing predicates of quantum mechanics are the observables. To say of a quantum system that its state vector lies in a certain subspace of a Hilbert space is then not to picture the quantum system since no observable predicate has been used.

Perhaps, however, it is the implicit but esoteric claim of the quantum logicians that ϕ is a *picturing*.[13] That would indeed be a novel, exciting, significant and controversial claim. But if that is the significant claim they want to make, they seem not to have noticed its unusual character. For if ϕ were a true picturing, then each subspace of H would have to correspond to a unique empirical predicate. There is at present no warrant for this. The traditional interpretation of quantum mechanics permits and indeed implies two different kinds of *picturings*, (i) of the sentences within a particular language L_a which picture the manifold of events in a physical context and (ii) of the set of competing languages $L_a, L_b, \ldots$ which themselves picture the manifold of possible physical contexts. (Note: A physical context constitutes the conditions for a definite manifold of events; it is then invariant relative to the events of the manifold. In other words, two different events of the same manifold share the same physical context; two events from different manifolds do not share the same physical context.)

I said above that the empirical interpretation of ϕ, authorized by ϕ, is:

$$a^*(\phi) = (a \vee b \vee \cdots)$$

We are now in a position to amplify and correct this assertion. Since $a \in L_a$, but $a \notin L_b$ and $b \in L_b$, but $b \notin L_a$, the members of the disjunction $a^*(\phi)$ are sentences belonging to different languages. Let us group together in $a^*(\phi)$ the sentences that belong to each language, repeating a sentence, if necessary, so that each language group is complete. We write,

$$a^*(\phi) = (a_1 \vee a_2 \vee \cdots) \vee (b_1 \vee b_2 \vee \cdots) \vee (c_1 \vee c_2 \vee \cdots) \vee \cdots \tag{5}$$

where $a_1, a_2 \ldots$ all belong to L_a, $b_1, b_2 \ldots$ all belong to L_b and so on. What is entailed then by ϕ, is not a disjunction of sentences within a common language, but a *disjunction of languages*. Thus, the correct revised form of $a^*(\phi)$ is:

$$a(\phi) = (\text{`}L_a\text{'} \vee \text{`}L_b\text{'} \vee \text{`}L_c\text{'} \vee \cdots) \tag{6}$$

where 'L_a', 'L_b', etc. represent statements within a higher-order meta-context-language $m(L)$; 'L_a', for instance standing for the statement 'This physical context is an 'L_a'-type context' or some equivalent formulation. $a(\phi)$ is then a statement in $m(L)$; not in any one of the event languages L_a, L_b, etc.

Let us look at the lattice structure of $m(L)$. A partial ordering relation '$\rightarrow$' (read 'implies') can be introduced into a set of propositions $a(\phi)$: let

$$a(\phi_1) \rightarrow a(\phi_2) \quad \text{if and only if} \quad \phi_1 \rightarrow \phi_2$$

then, since $a(\phi)$ is isomorphic with ϕ, the logic of the $a(\phi)$'s, i.e., of $m(L)$ is isomorphic with the logic of the ϕ's; that is, is isomorphic with the non-distributive lattice of the subspaces of a Hilbert space. We can define greatest lower bounds (g.l.b.) of two sentences, $a(\phi_1)\otimes a(\phi_2)$ (read '$a(\phi_1)$ *and* $a(\phi_2)$'), least upper bounds (l.u.b.) of two sentences, $a(\phi_1)\oplus a(\phi_2)$ (read '$a(\phi_1)$ *or* $a(\phi_2)$') and complements, $a'(\phi_1)$ (read 'not-ϕ') in the usual way:

$$
\begin{array}{lll}
a(\phi_3)=a(\phi_1)\otimes a(\phi_2) & \text{if and only if} & \phi_3=\phi_1\otimes\phi_2 \\
a(\phi_4)=a(\phi_1)\oplus a(\phi_2) & \text{if and only if} & \phi_4=\phi_1\oplus\phi_2 \\
a(\phi_5)=a'(\phi_1) & \text{if and only if} & \phi_5=\phi_1 \\
a(\phi_6)=\emptyset & \text{if and only if} & \phi_6=\emptyset \\
a(\phi_7)=1 & \text{if and only if} & \phi_7=H
\end{array}
$$

where '$\otimes$', '$\oplus$' and complementation on the right hand side refer to the logical operations of conjunction, disjunction and negation in the domain of simple theoretical propositions. The functors '$\otimes$', '$\oplus$' and '$\rightarrow$' on the left hand side can then be interpreted as the logical functors 'and', 'or' and 'implies' of a quantum logic in the meta-context-language $m(L)$, with complementation as negation. The logic of the $a(\phi)$'s on the left hand side is then an orthocomplemented non-distributive lattice isomorphic with the lattice of subspace of a Hilbert space.

Let me point out some of the attributes of the meta-context-language $m(L)$. In the first place, $m(L)$ is not an event-language; its objects are the manifold of possible event-language and the manifold of conditions for the correct use of each. Its subject matter is the set of physical contexts in which it is relevant to use one linguistic or conceptual framework rather than another. I have called it a meta-context-language, because its subject matter is the semantical applicability of event-languages. As I have already pointed out, $m(L)$ is a *picturing language*; it pictures the manifold of physical contexts; each of the physical contexts constitutes a domain of possibility for some manifold of events.

It is on the level of $m(L)$ that I believe the real significance of quantum logic lies. For while the logic of the meta-context-language of quantum mechanics is a non-distributive lattice, the analogous context-language of classical physics is a Boolean algebra isomorphic with the logic of subsets of the Euclidean phase-space of classical physics.[14]

The situating of the characteristic logics of classical and quantum physics in a meta-context-language throws light on the root cause of the difference in form between these two sciences. The quantum logic part of quantum physics expresses the context-dependent character of simple empirical propositions in quantum physics. The equation:

$$a(\phi_1) \oplus a(\phi_2) = a(\phi_1 \oplus \phi_2) \tag{7}$$

can be interpreted to mean that if one were to enlarge the physical context of a quantum system satisfying $a(\phi_1)$ so as to include conditions which satisfy $a(\phi_2)$, then one would obtain a new kind of physical context $a(\phi_1 \oplus \phi_2)$ in which the new manifold of possible events is different from the set theoretic union of the separate manifolds of events which corresponded to the former physical contexts in isolation from each other. That it is the context dependent character of simple empirical statements in quantum mechanics which generates a non-classical logic at the level of the meta-context-language $m(L)$ can be shown by examples constructed outside of the realm of physics.

For example, consider the philosophical languages used by two philosophers – let us call them, Joseph and Abner – each possessing a distinct and different philosophical perspective, but each capable of entering into dialogue with the other without yielding anything of which he considers essential to his perspective. Let L_A be the set of philosophical statements which Abner uses in a context that does not involve dialogue with Joseph; L_B be the set of philosophical statements used by Joseph in a context that does not involve dialogue with Abner, and L_{AB} the common language of dialogue. In order to get a non-distributive lattice all we need to suppose is (i) that L_{AB} contains sentences not in L_A or in L_B and, (ii) since within L_{AB} both Joseph and Abner can preserve their different philosophical prespectives, L_A and L_B are subsets of L_{AB}. We can then define complements of L_A and L_B in the following set theoretic way:

$$L'_A = (L_{AB} - L_A) \cup L_B$$
$$L'_B = (L_{AB} - L_B) \cup L_A$$

And we can define implication ('$\rightarrow$') in such a way that $L_X \rightarrow L_Y$, if and only if every sentence of L_X is also a sentence of L_Y. Then, the following objects constitute a lattice ($L_0, L_A, L_B, L'_A, L'_B, L_{AB} = 1$), where L_0 is the set theoretic intersection of L_A and L_B. The operations of '$\otimes$', '$\oplus$' and '$\rightarrow$' are schematized in the diagram below. We can interpret the objects in the figure below as sentences in a meta-context-language of conditions $m(L)$: for $L_A \oplus L_B$, read 'This is an 'L_A'-type situation and/or an 'L_B'-type situation'; for $L'_B \otimes L_A$, read 'This is an 'L_A'-type situation *and* it is *not* an 'L_B'-type situation'; and so on. Then we have the result that $m(L)$ is isomorphic to the orthocomplemented lattice represented in Figure 2.

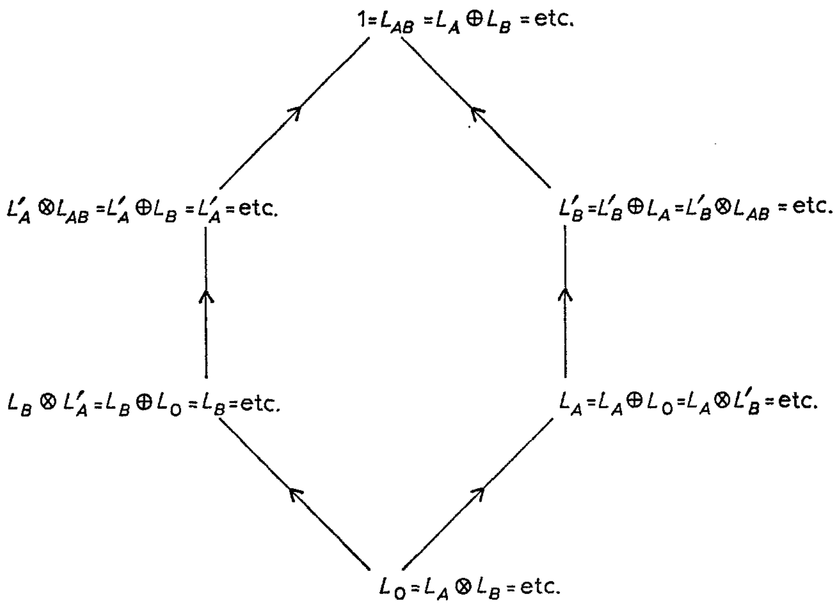

Fig. 2. Lattice of the meta-context language $m(L)$.

The arrows represent the partial ordering '$\rightarrow$' (or 'implies') among the elements. The l.u.b.'s (representing disjunctions in $m(L)$ and the g.l.b.'s (representing conjunctions in $m(L)$) can be read off from the figure. The lattice is non-distributive as can be seen from the relations.

$$L_A \oplus (L_B \otimes L'_B) = L_A$$
$$(L_A \oplus L_B) \otimes (L_A \oplus L'_B) = L'_B$$

but,

$$L_A \neq L'_B$$

Hence the logic intrinsic to the meta-context-language $m(L)$, which has its subject matter L_A, L_B, L_{AB} and the conditions for their correct use is an orthocomplemented non-distributive lattice like quantum logic.

The point I am making is that one does not have to look to physics to find quantum logic. One finds it in the meta-context-language of context-dependent statements. You can construct your own examples: the language of work and the language of leisure, the language of esthetic beauty and the language of functional utility. Nor is it necessary to appeal to the actions or emotions of human subjects. In chemistry, there is the language of heterogeneous gasses. In quantum physics, there is the language of single slit experiments and the language of double slit experiments.

Since I started this paper with the example of the two slit experiment, I might point out how to read the above diagram as descriptive of the meta-context-language of the two slit experiment. Let L_A be the single slit language relevant to the physical situation of slit 1 open, and let L_B be the single slit language relevant to the physical situation of slit 2 open, and let L_{AB} be the double slit language relevant to the situation of both slits open. Then, the solution to the problem we started with, namely, how to void the argument

$$\begin{array}{l} (a_1 \vee a_2) \supset b \\ \sim b \\ \hline \sim (a_1 \vee a_2) \end{array}$$

is the following. Either a_1 belongs to L_A and a_2 belongs to L_B, then the major is not well-formed, since we have no rules for disjoining sentences belonging to different languages. Or a_1, a_2 and b are to be interpreted as sentences in L_{AB}. The major of the argument is then false both empirically and theoretically, not, however, logically by virtue of a special logic.

I hope it is clear from these examples that the general character of non-distributivity in the meta-context-language is a direct consequence of the fact that although there are disjoint contexts, say, of Joseph philosophizing to the exclusion of dialogue with Abner and vice versa, the joint interaction context of Joseph and Abner in dialogue has broader possibilities than the sum total of those offered by the disjoint non-interacting contexts. We find just such context dependence in quantum mechanics.

VI. GENERAL QUANTUM MECHANICAL LANGUAGE AND GENERALIZED COMPLEMANTARITY

Remembering that a physcial context is those physical conditions which specify, antecedently to the outcome of any experimental observation, the manifold of possible events to which any experimental outcome will belong, and which, consequently, remain invariant throughout the course of any experimental test, one can ask how the separation and union of contexts is represented in quantum mechanics. Consider the following two equations:

$$a(\phi)=(\text{`}L_a\text{'} \vee \text{`}L_b\text{'} \vee \cdots) \tag{6}$$

and

$$a(\phi_1)\oplus a(\phi_2)=a(\phi_1\oplus\phi_2) \tag{7}$$

We can ask; (i) how can one physical context – somehow associated with ϕ – be consistent with a set of mutually exclusive physical contexts like those associated with L_a, L_b, etc.? and (ii) Does it not seem from the form of (7) that ϕ (and not 'L_a' or 'L_b' etc.) *pictures* the physical context of the system?

The answer to the former is that the physical context associated with ϕ is that of the preparation of state, while those associated with 'L_a', 'L_b' etc. are those associated with a subsequent measuring process.

Turning to the latter question: to say that the system is prepared in state ϕ is to say something about the physical world, actually about the preparation of state. *But is it a picturing*? We have already raised this question and the answer given was, No! However, at that time, all we had at our disposal was quantum event-language, i.e., the language of Yes-no tests. The suggestion is now made that possibly 'ϕ' is a picturing in $m(L)$, the meta-context-language that deals with the kind of language appropriate to a physical context rather than with specific quantum events. But if the 'L_a'-type context and the 'L_b'-type context are *absolutely* mutually exclusive, then there is no term in the linguistic resources of $m(L)$ with which to designate a *general physical context*, uniquely correlated with 'ϕ', of which the 'L_a'-type context and the 'L_b'-type context are specific particularizations. Under these circumstances 'ϕ' can only denote a *class* of mutually exclusive contexts – not a general type of physical context.

If, however, the 'L_a'-type physical context and the 'L_b'-type physical context are not absolutely mutually exclusive, a general physical context might exist of which the 'L_a'-type and the 'L_b'-type were specifications (as it were, limiting or special cases of the general case). This would imply the existence of a general language appropriate to this context, say, $L_{abc..}$, of which L_a, L_b, L_c etc. would be subsets corresponding to special or limiting cases. The enlarged possibilities of united contexts then imply that, although $L_a \subseteq L_{abc..}$, $L_b \subseteq {}_{abc..}$ etc., it is also the case that

$$L_{abc..} \neq L_a \cup L_b \cup L_c \cup \cdots$$

The meta-context-language, $m(L)$, then, would contain, besides the names 'L_a', 'L_b', 'L_c' etc., the name '$L_{abc..}$' of the general context $L_{abc..}$. It seems plausible to the author that the mutual exclusivity of complementary physical contexts may not be absolute, and that there may exist a general physical context '$L_{abc..}$' uniquely correlated with 'ϕ'. This is suggested by the use which Bohr and Heisenberg make of the Indeterminacy Principle when they apply it to an individual quantum system in one and the same physical context. This cannot be done without the simultaneous attribution of both position and momentum predicates (though evidently not the classical ones) to a quantum system in a general physical context. If $L_{abc..}$ exists, then there is a general physical context pictured by '$L_{abc..}$', and it would indeed become plausible to say that 'ϕ' pictures – not, however, within any of the event-languages L_a, L_b, L_c etc., but within the meta-context-language $m(L)$. The disjunction $a(\phi)$ would then include an extra term,

$$a(\phi) = (`L_a' \vee `L_b' \vee `L_c' \vee \cdots \vee `L_{abc..}')$$

and since L_a, L_b, L_c etc. are all subsets of $L_{abc..}$, this reduces simply to,

$$a(\phi) = (`L_{abc..}')$$

Under these circumstances, we can drop the assumption made earlier that the observables X and Y referred to in the description of a Yes-no test be commuting observables. Even if they do not commute, they belong to the general language $L_{abc..}$ within which the physical context of the Yes-no test can now be uniquely described.

The situation we have just described suggests that the notion of complementarity be extended to the set of mutually exclusive event-languages L_a, L_b, L_c etc. on condition that there exist a general event-language $L_{abc..}$ which contains all of them, but is richer in expressive power than L_a, L_b, L_c etc. taken in contextual isolation. This supposes that the contexts A, B, C, etc. which specify the conditions for the valid use of L_a, L_b, L_c etc. are mutually exclusive but not absolutely so. It supposes that there exists a general or synthetic context $ABC..$ to which there corresponds a general language $L_{abc..}$ satisfying the following conditions:

$$L_a \subseteq L_{abc..} \qquad (8)$$
$$L_b \subseteq L_{abc..}$$
$$L_c \subseteq L_{abc..} \text{ etc.}$$

and

$$L_{abc} \neq L_a \cup L_b \cup L_c \cup \cdots \qquad (9)$$

It further supposes that there is a way of defining complements L'_a, L'_b, L'_c etc. of L_a, L_b, L_c etc. respectively such that $(L_0, L_a, L_b, L_c, \ldots, L'_a, L'_b, L'_c, \ldots, L_{abc..})$ constitutes an orthocomplemented non-distributive lattice.

This claim is illustrated and justified by the author in another paper, for the case of two event-languages L_a and L_b.[15] The complements L'_a and L'_b can be defined in the following way:

$$L'_a = (L_{ab} - L_a) \cup L_b$$
$$L'_b = (L_{ab} - L_b) \cup L_a$$

The partial ordering '→' ('implies') of the lattice is to be understood in such a way that, $L_x \rightarrow L_y$ if and only if every sentence that can be correctly used (either by affirmation or negation) in context X can also be correctly used in context Y. L_a and L_b are said to be *complementary* in L_{ab}. In this situation, the meta-context-language $m(L)$ will be a domain of non-classical logic, even though each of the event-languages L_a, L_b and L_{ab} obey or could obey classical *PM*-logic.

In the paper referred to, the author worked out some examples taken from quantum mechanics, as, for example, the paradigm case of the relation between quantum mechanical position language (L_a) and quantum mechanical momentum language (L_b) and the general quantum mechanical kinematical language (L_{ab}). The author also uses this logic of contexts (or context-logic) to show the relationship between the

Aristotelian and the Augustinian-Platonic languages in the philosophy of Aquinas and suggests the usefulness of the method of context-analysis in the history of philosophy, the history of science and in the metascience of both philosophy and science.[16]

In the case of four event-languages, L_a, L_b, L_c and L_{abc} which satisfy (8) and (9), it may be possible to define appropriate complements so that the set (L_0, L_a, L_b, L_c, L'_a, L'_b, L'_c, L_{abc}) constitutes an orthocomplemented non-distributive lattice. In this case, L_a, L_b and L_c are said to be *complementary in* L_{abc}. The structure of such a lattice is represented in Figure 3 below.

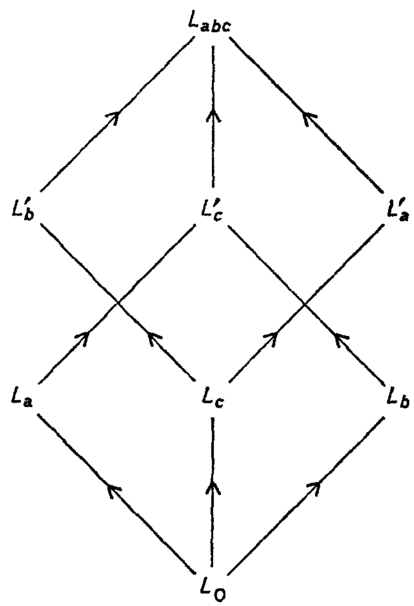

Fig. 3. Diagram of a non-Boolean lattice for three complementary languages L_a, L_b and L_c.

In the light of what has been said about the pervasive character of contextual dependence in our experience, classical physics with its context-independence begins to appear in some sense as incomplete, as lacking a contextual dimension one feels every physical statement should have, and one begins to suspect that perhaps this is due to a smoothing, smearing or averaging of many context-dependent results. That this may be so, is suggested by the fact that we can construct certain context-independent variables, called expectation-values, for ensembles of identically prepared quantum systems, which obey the same laws as their counterparts in classical physics.

The separation of two levels of picturing in quantum mechanics also throws light on the reduction of the wave packet. If it is true that whatever concerns the real order is expressible either in the picturing metalanguage $m(L)$ or in one of the competing first level picturing languages, L_a, L_b,... then nothing need be postulated to belong to the real world if it cannot be stated in $m(L)$ or in one or other of the first level languages. The conversion from the pure state to a mixture (the so-called 'reduction of the wave packet') is consequently neither a rule of nor a proposition about reality, but is part of the manipulation of certain abstract mathematical objects. This, of course, is not a novel conclusion; but it has never been reached before in this way. The dual level of picturing can also suggest how one might go about working out the (descriptive) ontology of the quantum domain of physical systems. But I shall not pursue this matter now.[17]

VII. CRITICAL REMARKS

In Section IV, I made the claim that much of the recent writing on quantum logic was fundamentally confused and had failed to appreciate the true significance of quantum logic. I ought to make explicit, if it is not already clear, what these confusions are and where their source lies.

The principal confusion is in the characterization of those basic quantum mechanical propositions which, it is claimed, obey quantum logic and not classical logic. Birkhoff and von Neumann describe these propositions as equivalence classes of categorical statements of fact, each statement supposing a definite set of commuting observables. They have the logical form of Equation (3) above, and Birkhoff and von Neumann do not pursue the analysis to the point of deriving Equation (6).

Putnam and Finkelstein, on the other hand, are dealing, not with categorical propositions but with subjunctive conditionals of the form: 'If a certain test were to be made, then the system would pass it'. They are not statements of fact but of the dispositions which a quantum system has. They have the desirable property, however, of being isomorphic with simple theoretical proposition, i.e., if p is one of those propositions, then

$$p \leftrightarrow \phi$$

The logic of these subjunctive conditionals is then a quantum logic isomorphic with the lattice of subspaces of a Hilbert space.

How satisfactory is this solution from the logical point of view? And with what warrant are these subjunctive conditionals called the 'empirical propositions of quantum mechanics'?

From the point of view of either classical or quantum logic a subjunctive or counterfactual conditional is not a clear and definite structure. It suffers from the ambiguities and latent paradoxes which are the subject matter of an extensive and inconclusive literature.[18] Moreover, if the implications of a single subjunctive conditional are not always clear, the logical product of two such sentences is perilous indeed. Consider the sentences: 'If I were drunk, I would dance a jig' and 'If I were sober, I would trim the hedge'. The former implies I am not drunk; the latter implies I am not sober. What then would the conjunction of the two sentences imply? If the basic sentences of quantum mechanics were subjunctive conditionals, the antecedents of the basic sentences would include all the mutually exclusive physical conditions into which the system could be placed. Just as in the example I have chosen and for the same reason, if a conjunction of a set of basic sentences were formed, it would not be clear what such a conjunction would imply.

Again, as statements about the physical constitution of the world, counterfactual conditionals leave open and undetermined, or if you like, to be determined by the subject or by nature (but more often by the subject), the final conditions (or physical context) without which nothing is or can be observed, and no matter of fact established. In Heisenberg's language, they refer to the *potentiae* of nature. One might perhaps be content with this situation if one subscribed to the view espoused by von Weizsäcker and endorsed by Heisenberg, that the subject matter of quantum physics is the subject-object interaction between a human observer and nature. Then, the empirical content of physics might be expected to reflect the initiatives which a scientist-observer takes spontaneously and unpredictably in his relation with the world. But it is not necessary, I believe, and it is certainly contrary to the ideal aims of natural science, that quantum physics renounce its right to make objective and *categorical* statements about the world and not merely about its *potentiae* which are true independently of observers.

Moreover, calling these subjunctive conditionals the 'empirical propositions of science' is unwarranted, for the thrust of work on quantum logic has been and is the attempt to articulate the 'logic of nature' as revealed by quantum mechanics in the sense of the 'interrelatedness of things' in nature. Such a 'logic of nature' would be pictured in empirical propositions of a *categorical* kind, not in counterfactual conditionals. It is true that certain experimental conditions are mentioned which anchor the conditional statement in the real world. These give empirical warrant to the subjunctive conditionals. But wherever there is a warranted subjunctive conditional its warrant is in some set of objective laws linking the antecedent and the consequent.[19] These are the links which express the 'logic of nature', and these are the empirical propositions of quantum mechanics. Now these links are categorical propositions. There are, I claim, two levels of such propositions: the level of quantum event-languages and the level of a meta-context-language $m(L)$ about the conditions for the use of quantum event-languages.

My criticism of Jauch's very elegant treatise *The Foundations of Quantum Mechanics* is that he too seems not to have considered the variety of propositional types employed in quantum mechanics. Sometimes he speaks as if the basic quantum mechanical propositions were simple empirical propositions, the results of Yes-no experiments, sometimes as if they were simple theoretical propositions about subspaces of a Hilbert space, and at other times as if they were counterfactual conditionals. Moreover, while it is understandable that quantum logicians would want to ground their claims in empirical fact, it happens that this empiricist urge sometimes caricatures itself, as we see, for example, in the case of Jauch and Ludwig. They describe the construction of an *infinite experimental filter* for two incompatible propositions $a \otimes b$.[20] One wonders with what right an infinite experimental filter that cannot be made or used is termed 'experimental'.

APPENDIX: SOME TERMS, DEFINITIONS, AXIOMS AND FORMULAS

I. *By classical logic* is meant a *Boolean algebra of sentences*

A. A Boolean algebra is a set B of at least two distinct elements with two binary operations $\cup$(cup) and $\cap$(cap) and one unary operation $'$ (prime) such that B is closed with respect to each of these three operations and for all a, b and c belonging to B the following axioms are satisfied:

A1	$a \cup b = b \cup a$
A2	$a \cap b = b \cap a$
A3	$a \cup (b \cup c) = (a \cup b) \cup c$
A4	$a \cap (b \cap c) = (a \cap b) \cap c$
A5	There is an element $\emptyset$ belonging to B such that $a \cup \emptyset = a$
A6	There is an element I belonging to B such that $a \cap I = a$
A7	$a \cup a' = I$
A8	$a \cap a' = \emptyset$
A9	$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$
A10	$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$

A9 and A10 are the *distributive laws.*

Def.: $a \subseteq b$ if and only if $a \cup b = b$

B. A Boolean algebra of sentences is obtained by adding the following interpretation to the formal syntax A:

a, b, c,	stand for sentences
$\emptyset$	stands for the absurd sentence (always false)
I	stands for the trivial sentence (always true)
$a \cup b$	stands for *a and/or b* (the logical sum)
$a \cap b$	stands for *a and b* (the logical product)
a'	stands for *not-a*
$a \subseteq b$	stands for *a implies b*

II. By a *quantum logic* is meant an *orthocomplemented non-distributive lattice* of sentences.

A. An orthocomplemented non-distributive lattice is obtained (rather than an algebra) if the axioms A9 and A10 are dropped in IA. Using '$\oplus$', '$\otimes$' and '$\rightarrow$' instead of cup, cap and '$\subseteq$' to distinguish the non-distributive lattice operations from the Boolean operations, we use the following terminology for a lattice:

$a \oplus b$	is called the 'least upper bound' or 'l.u.b. of *a*, *b*'
$a \otimes b$	is called the 'greatest lower bound' or 'g.l.b. of *a*, *b*'
$a \rightarrow b$	stands for *a implies b*

B. A quantum logic is obtained by adding to the axioms of II A, the following interpretation:

$a \oplus b$ stands for *a and/or b*
$a \otimes b$ stands for *a and b*
a' stands for *not-a*
$a \rightarrow b$ stands for *a implies b*

where $a, b, c, \emptyset, I$ are sentences, $\emptyset$ is the absurd sentence and I is the trivial sentence.

III. Fig. 4 representing the relations between the elements of an *ortho-complemented non-distributive lattice* of six elements. The arrows indicate implication.

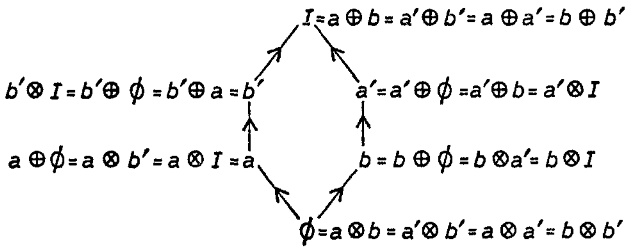

Fig. 4.

This lattice is non-distributive since:

$$a \oplus (b \otimes b') = a \oplus \phi = a$$
$$(a \oplus b) \otimes (a \oplus b') = I \otimes b' = b'$$
and $a \neq b'$

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[1] G. Birkhoff and J. von Neumann, 'The Logic of Quantum Mechanics', *Annals of Math.* **37** (1936) 823–843.

[2] See references in the bibliography and also note 4. M. S. Watanabe who shares in his own way the belief that the logic of quantum mechanics is a non-distributive lattice, has based his theory on a different and subtler analysis of the use of ordinary and scientific language than the authors I am explicitly criticizing in this paper. I do not wish to include his views among those I am attacking, although it is possible that some of the positions I take in the latter part of this paper conflict with those of Watanabe.

[3] J. M. Jauch, *The Foundations of Quantum Mechanics*, New York 1968, p. 73.

[4] For an exposition of the properties of lattices, see G. Birkhoff, *Lattice Theory*, 2nd ed. 170, Amer. Math. Soc., 1948, and for the application of lattices to quantum mechanics, see J. M. Jauch, *The Foundations of Quantum Mechanics*. In the appendix to this paper will be found a convenient summary of the differences between a Boolean algebra (like *PM*-logic) and a non-Boolean lattice (like quantum logic).

[5] The sign '$\supset$' denotes material implication; '$\vee$' denotes alternation.

[6] H. Putnam, 'Is Logic Empirical?' in *Boston Studies in the Philosophy of Science*, vol. V (ed. by R. S. Cohen and M. Wartofsky), Humanities Press, New York, and D. Reidel, Dordrecht, 1969.

[7] At the biennial meeting of the Philosophy of Science Association at Pittsburgh, October 1968, Putnam challenged his audience, among whom the author found himself on that occasion, to show that he had violated in his argument the principles of quantum logic.

[8] The example given in the first draft of this paper was incorrect as D. Finkelstein pointed out to me. This example is due to him.

[9] Putnam claims that this part of his paper has the value of an inductive argument. The author sees it as being merely of expository value.

[10] See above, Section I.
[11] J. M. Jauch, *Foundations of Quantum Mechanics*, p. 131.
[12] Jauch, *op. cit.*, p. 73.
[13] W. Sellars in chap. 3 of *Science and Metaphysics* (London 1968) develops a notion of *picturing* close to what the author would be willing to subscribe to.
[14] The author has treated the interrelationship of language, inquiring behaviour (as a form of life) intentionality, horizon and objectivity in 'Horizon, Objectivity and Reality in the Physical Sciences', in *Intern. Philos. Quart.* **7** (1967) 375–412. The analysis given there, although expressed in the language of continental philosophy, is nevertheless very applicable here.
[15] P. A. Heelan, 'Complementarity, Context-Dependence and Quantum Logic' (to be published).
[16] P. A. Heelan, 'The Role of Subjectivity in Natural Science', *Proc. Amer. Cath. Philos. Assoc.*, Washington, D.C., 1969.
[17] The author has tried to do this in his *Quantum Mechanics and Objectivity*, The Hague 1965, but using a method of intentionality-analysis rather than the analysis of picturing.
[18] For example, N. Goodman, *Fact, Fiction and Forecast*, 2nd ed. (Bobbs-Merrill, New York 1965), chap. I; R. Chisholm, 'The Contrary-to-Fact-Conditional', *Mind* **55** (1946) 289–307; S. Hampshire, 'Subjunctive Conditionals', *Analysis* **9** (1948) 9–13; D. Pears, 'Hypotheticals', *ibid.*, **10** (1950) 49–62; W. Kneale, 'Natural Law and the Contrary-to-Fact Conditional', *ibid.* **10** (1950) 121–125; J. C. D'Alessio, 'On Subjunctive Conditonals', *Journ. Philos.* **64** (1967) 306–310.
[19] R. Harré, *Introduction to the Philosophy of Science*, London 1960, pp. 17–24.
[20] J. M. Jauch, *op. cit.* p. 75.