ON THE PROPAGATION OF ENERGY IN LINEAR CONSERVATIVE WAVES

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Summary

This paper is concerned with the question when and why the rate of energy propagation in a system of waves equals the group velocity. It is shown by the method of stationary phase that this equality holds, for travelling waves without dissipation, whenever this method applies. The reason why this result can be obtained by this kinematical method is investigated by a discussion of simple harmonic waves. It is shown that the choice of an expression for the energy density to be used in connection with a given wave equation is restricted by the conservation of energy in such a way that the average rate of work done divided by the average energy density always equals the group velocity. Finally some examples of wave motion are discussed to illustrate the derived formulae.

§ 1. *Introduction.* — It is generally known that the energy of a system of progressive linear conservative waves is, as a rule, propagated with the group velocity of these waves. A consideration of a wave packet, which moves with the group velocity without appreciable deformation at least for a short time, will be sufficient evidence that this statement must be fairly exact. The total energy of the waves is confined to the region occupied by the wave packet and must therefore move with the same average velocity.

On the other hand the literature on wave motion is singularly reticent about the deeper reasons for the identity of the energy and group velocities. We have not been able to find a general proof of this property of wave packets. It is even not made clear at all what both velocities can have to do with each other. The group velocity is a kinematical property of the wave equation alone, quite independent of its physical interpretation. The energy velocity, however, which is a dynamic quantity, can be determined only when this interpretation is known.

In this paper we will first prove this kinematical property of waves with the aid of the method of stationary phase. We will find that everywhere in the wave system the energy is propagated with the group velocity corresponding to the local wave number. This result is valid to the same approximation as the application of the stationary phase method. In the second place we will show that conservation of energy restricts the possible expressions for the energy density admitted by a given wave equation in such a way, that the energy velocity in a harmonic wave always equals the group velocity. Finally, we will apply our methods to some examples of wave motion.

§ 2. *Linear conservative waves and the method of stationary phase.* We consider a homogeneous one-dimensional medium which admits a state of stable equilibrium. The waves with which we are concerned consist of travelling disturbances of this equilibrium. The deviation from equilibrium is measured by some quantity $z(x, t)$. When the principle of superposition is valid, we speak of *linear* waves. In this case it is sufficient to discuss only *harmonic waves*:

$$z = \exp j(kx - \omega t). \tag{1}$$

More extended information about the wave motion is then obtained by Fourier methods. In order to apply these we need the relation between frequency and wave number of a harmonic wave:

$$W(k, \omega) = 0. \tag{2}$$

This *characteristic equation* is sufficient to determine the wave motion. When W is a polynomial in k and ω it can be replaced by a differential equation, the *wave equation*, which can be written as

$$W\left(\frac{1}{j}\frac{\partial}{\partial x}, -\frac{1}{j}\frac{\partial}{\partial t}\right) z = 0. \tag{3}$$

Of course, when (3) is given, (2) can be obtained by substituting (1) into (3).

The characteristic equation (2) can be solved with respect to ω:

$$\omega = H(k). \tag{4}$$

$H(k)$ is known as the *Hamilton function.* $c(k) = H(k)/k$ is the *phase velocity.*

We will restrict ourselves now to those cases in which ω is real whenever k is real. The corresponding harmonic waves then have constant amplitudes and there is no dissipation of energy. This kind of waves is called *conservative*. Furthermore we will at first consider equations having only one Hamilton function. As the differential equations describing reversible phenomena in a homogeneous medium are invariant with respect to inversion of both t and x, the characteristic equation can then be written in the form

$$\omega^2 = \frac{\sum\limits_{n=0} a_n k^{2n}}{1 + \sum\limits_{m=1} b_m k^{2m}}, \tag{5}$$

where a_n, b_m are real constants and the a_n are not all zero. The right hand side must be a positive function of k^2 when k is real.

By the principle of superposition a solution of the wave problem is now

$$z(x, t) = \int_{-\infty}^{\infty} F(k) \exp j \{kx - H(k)t\} \, \mathrm{d}k + \int_{-\infty}^{\infty} F'(k) \exp j \{kx + H(k)t\} \, \mathrm{d}k. \tag{6}$$

Equation (6) describes two systems of waves going out from the disturbed region to the right and to the left. We can treat these systems separately. We write the integrals in the form

$$z(x, t) = \int_{-\infty}^{\infty} F(k) \exp j\{kx - H(k)t\} \, \mathrm{d}k = \int_{-\infty}^{\infty} G(k) \exp j\theta \, (k, x, t) \, \mathrm{d}k, \tag{7}$$

where $F(k) = G(k) \exp j\varphi(k)$, $(G, \varphi,$ both real for real $k)$ and $\theta = \varphi(k) + kx - H(k)t$. The function $G(k) = | F(k) |$ represents the *spectrum* of the wave system, $\varphi(k) = \arg F(k)$ the *eikonal*. Integrals of the type (7) can be evaluated approximately under certain conditions by the method of *stationary phase* (these conditions for G and θ and the conditions they involve for φ and F will not be discussed here). We will treat this method only very briefly and refer for a more thorough discussion to E c k a r t [1]).

When the exponential factor in (7) fluctuates very rapidly, the principal contributions to the integral arise from the regions where the phase θ is stationary. These are given by

$$\frac{\partial \theta}{\partial k} = x - x_0(k) - v(k)t = 0, \tag{8}$$

where $v(k) = \mathrm{d}H/\mathrm{d}k$ and $x_0 = -\partial\varphi/\partial k$. The first quantity is known as the *group velocity*. The solution of (8) with respect to k is denoted by $\varkappa\ (x, t)$. We now replace $G(k)$ by $G(\varkappa)$ in the integral (7) and expand the phase θ in powers of $(k - \varkappa)$. The result is

$$z(x, t) = G(\varkappa) \int_{-\infty}^{\infty} \exp j \left[\theta(\varkappa) + \tfrac{1}{2} \left(\frac{\partial^2 \theta}{\partial k^2}\right)_\varkappa (k - \varkappa)^2 \right] \mathrm{d}k.$$

When we neglect higher terms, the integration can be carried out at once and yields

$$z(x,t) = G(\varkappa) \sqrt{\frac{2\pi j}{\left(\dfrac{\partial^2\theta}{\partial k^2}\right)_\varkappa}} \ \exp j\theta(\varkappa;\, x, t).$$

In accordance with E c k a r t we write

$$S(x, t) = \theta\, [\varkappa(x, t);\, x, t],$$

$$R(x, t) = -\left(\frac{\partial^2 \theta(k;\, x,t)}{\partial k^2}\right)_\varkappa.$$

The approximate solution is then

$$z(x, t) = G(\varkappa) \sqrt{\frac{2\pi}{jR}} \exp jS. \tag{9}$$

From (8) it is seen that a certain constant value of $\varkappa$, the wave number of stationary phase, is propagated with a constant velocity $v(\varkappa)$. In a given point we find at each time from (8) a value for $\varkappa$. This $\varkappa$ is approximately the wave number of the local disturbance at that time and is therefore called the *local wave number*. The truth of this statement is seen when we differentiate

$$S(x, t) = \varphi[\varkappa(x, t)] + x \cdot \varkappa(x, t) - t \cdot H[\varkappa(x, t)]$$

with respect to x and t respectively. Using (8) we obtain:

$$\frac{\partial S}{\partial x} = \varkappa(x, t), \quad \frac{\partial S}{\partial t} = -H[\varkappa(x, t)].$$

In the neighbourhood of a point with the local wave number $\varkappa$ therefore our solution behaves as

$$z(x, t) = G(\varkappa) \cdot \sqrt{\frac{2\pi}{R}} \exp j \left[\varkappa \{x - c(\varkappa)t\} - \frac{\pi}{4} \right].$$

The amplitude of the local wave is determined by $G(\varkappa)$, the density of the initial wave spectrum at $\varkappa$, and by the factor $R = -\partial^2\theta/\partial k^2$. It should be noted that $R^{-\frac{1}{2}}$ is a measure for the width of the spectral region in which the phase is approximately constant. From the definition of R we have

$$R = -\left(\frac{\partial^2\theta}{\partial k^2}\right)_\varkappa = \left(\frac{\partial x_0}{\partial k}\right)_\varkappa + t\left(\frac{\partial v}{\partial k}\right)_\varkappa. \tag{10}$$

Differentiating (8) with respect to k and using (10) we derive the alternative form

$$R = \left(\frac{\partial x}{\partial \varkappa}\right)_t. \tag{10a}$$

R is called the *resolution* of the waves. From (10) it is seen that it is a function of $\varkappa$ and t only. From (10a) we see that when R is large (the condition turns out to be: $R\varkappa^2 \gg 1$) the difference of local wave number between succesive wave crests is small. The method of stationary phase is a good approximation then.

§ 3. *Energy propagation.* The energy density in a linear harmonic wave, averaged over a period or a wave length, is proportional to the square of the amplitude. The factor of proportionality will in general depend on the wave number. When a system of waves is so far resolved that the motion in a region containing a few wave crests differs little from a harmonic wave, this still will be nearly exact. As under these circumstances the method of stationary phase will be valid, we see from (9) that the average density of energy in this region will be approximately

$$\bar{E}(x, t) = \frac{A(\varkappa)}{R} = \frac{A(\varkappa)}{\dfrac{\partial x_0}{\partial \varkappa} + t\dfrac{\partial v}{\partial \varkappa}} = \tilde{E}(\varkappa, t). \tag{11}$$

For our purpose it is not necessary to specify the function $A(\varkappa)$, which contains $G^2(\varkappa)$ as a factor.

By differentiating (11) we obtain

$$\frac{\partial \bar{E}}{\partial t} = \frac{\partial \tilde{E}}{\partial \varkappa} \cdot \frac{\partial \varkappa}{\partial t} + \frac{\partial \tilde{E}}{\partial t} = \frac{\partial \tilde{E}}{\partial \varkappa} \cdot \frac{\partial \varkappa}{\partial t} - \frac{\tilde{E}}{R} \frac{\partial v}{\partial \varkappa}, \tag{12}$$

$$\frac{\partial}{\partial x}(v\bar{E})) = v \frac{\partial \bar{E}}{\partial x} + \bar{E} \frac{\partial v}{\partial x} = v \frac{\partial \tilde{E}}{\partial \varkappa} \cdot \frac{\partial \varkappa}{\partial x} + \tilde{E} \frac{\partial v}{\partial \varkappa} \cdot \frac{\partial \varkappa}{\partial x}. \tag{12a}$$

Now from (8) we have

$$v = \left(\frac{\partial x}{\partial t}\right)_{\varkappa} = -\frac{\partial \varkappa / \partial t}{\partial \varkappa / \partial x};$$

therefore, using (10a), we obtain, by adding (12) and (12a)

$$\frac{\partial \bar{E}}{\partial t} + \frac{\partial}{\partial x}(v\bar{E}) = 0. \tag{13}$$

An equation of this type has been derived in another way by R o s s b y [2]) for some special cases.

Equation (13) has the form of a continuity equation. As we know that the wave system is conservative (as could have been guessed beforehand, our argument is after all not purely kinematic), $v\bar{E} = \bar{T}$ must be the average density of energy flow, that is the sum of the rate of work done at a unit cross section and the convective flow of wave energy through this cross section. The quotient $\bar{T}/\bar{E} = v$ is the velocity of energy propagation, referred to in the introduction.

It is seen from this deduction that the group velocity is approximately equal to the quotient of the average energy flow and the average energy density, not necessarily to that of flow and density themselves, or the average of this quotient. This is exactly what could have been expected from considering a wave packet. The energy of this packet is proportional to the average energy density $\bar{E}$, whereas we only know that the velocity of propagation of this energy on the average must be equal to the velocity of the centre of the wave packet, i.e. to $v(\varkappa)$. The fluctuations of the energy towards or from the crests and troughs of the waves cannot be investigated unless the physical nature of the wave amplitude z, which would enable us to express E in z, is known.

Next we admit the existence of more than one Hamilton function, which means that the wave equation is of higher order than the second in $\partial/\partial t$ *). To each k then correspond several frequencies. Properties of the wave field, e.g. the energy density, as a rule cannot be expressed in terms of k alone. Our general solution (6) is now replaced by a sum of similar terms:

$$z(x,t) = \Sigma \int_{-\infty}^{\infty} [F_n(k) \exp j \{kx - H_n(k)t\} + F'_n(k) \exp j \{kx + H(k)t\}] \mathrm{d}k.$$

We assume that each term of this expression can be treated by the method of stationary phase. For a certain x, t we will find several values of $\varkappa$, each of them representing an approximately harmonic wave in the region around x. The contributions of these waves to z will simply add, the energy densities E of course will not. But, when $\partial x/\partial \varkappa$ is large enough, the cross terms in E will cancel on the average and $\bar{E}$ will be approximately additive. Treating the average energy density of each partial wave as above, we obtain a number of equations of the type (13):

$$\frac{\partial \bar{E}_n}{\partial t} + \frac{\partial}{\partial x}(v_n \bar{E}_n) = 0. \tag{13a}$$

Adding them, we find that the energy velocity now will be equal to the average group velocity, weighted with respect to $\bar{E}_n$. Therefore it depends on the distribution of energy among the various branches F_n of the wave spectrum.

Finally we remark that the results of this section are valid for any quantity, depending only on the local wave number and proportional to the square of the local amplitude, provided it is conserved during the motion. As an example we mention the charge density in wave mechanics.

§ 4. *Energy propagation in harmonic waves.* We will now investigate to what extend the conservation of energy restricts the possible expressions of E and T in terms of z and its derivatives for linear conservative waves when the wave equation or characteristic equation is given. We shall see that this restriction is strong enough to ensure that $\bar{T}/\bar{E} = v$ for a harmonic wave, and so obtain a more dynamical interpretation of the result of the preceding section.

*) The following treatment also applies to the case that different wave groups belonging to the same Hamiltonian meet each other. This can occur when (5) has more than one solution or when the two wave systems in (6) overlap.

We start with the characteristic equation (5). The corresponding wave equation, possibly of infinite order, is then

$$L(z) = \ddot{z} + \sum_{n=0} (-1)^n a_n \, z^{(2n)} + \sum_{m=1} (-1)^m \, b_m \, \ddot{z}^{(2m)} = 0. \tag{14}$$

Partial differentiation with respect to t and x is denoted by $\dot{z}$ and z' respectively. The superscript $(2n)$ denotes the number of dashes. We might assume that the right hand side of (5) cannot be simplified. If this were possible, e.g. by dividing through a factor $(1 + ck^2 + dk^4 + \ldots.)$, this clearly would mean that (14) could have been obtained by applying the differential operator

$$1 - c\frac{\partial^2}{\partial x^2} + d\frac{\partial^4}{\partial x^4} \cdots.$$

to an equation of lower order. One then can divide (5) by this factor and consider this last equation as the wave equation. Although in practice this simplification will be made whenever possible, this is not necessary for the following reasoning.

E and T are quadratic expressions in z and its derivatives. For the reasons of symmetry mentioned above, E will be even, T odd with respect both to the dots and the dashes. E must be positive, except when $z \equiv 0$, as this state is supposed to be stable. In this state T can be taken zero too, which means that E and T refer to the wave energy proper.

We will now suppose that (14) admits solutions which vanish sufficiently strongly outside of an arbitrarily extended region for some finite time, e.g. some superposition of wave packets. The total energy contained in this region then must be constant on account of the conservative character of the waves. Hence the condition.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int E \mathrm{d}x = 0 \tag{15}$$

must be a consequence of (14) whenever the wave system is confined to a certain region. This boundary condition enables us to discard the integrated terms after an integration by parts of (15). In this way we can shift over the dashes from one factor to the other at liberty in each term of the homogeneous quadratic integrand E.

We now consider a certain energy density $E(z)$. If it is possible, after performing the differentiation under the integral, to recast (15) by integration by parts into the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int E(z)\,\mathrm{d}x = \int \frac{\partial F(z)}{\partial t}\cdot L(z)\mathrm{d}x, \tag{16}$$

(where $F(z)$ will be a linear expression in z and its derivatives, even in the dashes and the dots), the energy will be constant for the solution of $L(z) = 0$.

We now distinguish three cases:

a. $F(z) =$ const. z. It will turn out that this case already yields all essential information. We have to look for homogeneous quadratic expressions $E(z)$ which permit the transformation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int E(z)\mathrm{d}x = \int \frac{\partial E(z)}{\partial t}\mathrm{d}x = \text{const.}\int \dot{z}L(z)\mathrm{d}x.$$

It is easily verified that a special solution of this problem is furnished by

$$E_0(z) = \frac{B}{2}\,[\dot{z}^2 + \sum_{n=0} a_n\{z^{(n)}\}^2 + \sum_{m=1} b_m\,\{\dot{z}^{(m)}\}^2]. \tag{17}$$

The general solution is then

$$E(z) = E_0(z) + \frac{\partial J}{\partial x},$$

where J is any homogeneous quadratic expression in z, odd in the dahes, even in the dots. Its contribution to the total energy vanishes.

We now must derive the corresponding density of energy flow $T(z)$. On account of the conservation of energy T has to satisfy the equation

$$\frac{\partial E(z)}{\partial t} + \frac{\partial T(z)}{\partial x} = 0, \tag{18}$$

when z is a solution of the wave equation. We need that solution of (18) which vanishes when $z \equiv 0$. This solution is

$$T = B\,[\sum_{n=0} a_n T_n + \sum_{m=1} b_m U_m] - \frac{\partial J}{\partial t}, \tag{19}$$

where

$$T_n = -[z^{(n)}\,\dot{z}^{(n-1)} - z^{(n+1)}\,\dot{z}^{(n-2)}\,\ldots\ldots + (-1)^{n-1}\,z^{(2n-1)}\,z],$$

$$U_m = -[\ddot{z}^{(m)}\,\dot{z}^{(m-1)} - \ddot{z}^{(m+1)}\,\dot{z}^{(m-2)}\,\ldots\ldots\,(-1)^{m-1}\,\ddot{z}^{(2m-1)}\,\dot{z}].$$

To verify this we compute

$$\frac{\partial E}{\partial t} = B[\dot{z}\,\ddot{z} + \Sigma\, a_n\, z^{(n)}\dot{z}^{(n)} + \Sigma\, b_m\, \dot{z}^{(m)}\, \ddot{z}^{(m)}] + \frac{\partial^2 J}{\partial t \partial x} =$$

$$= B\,[\Sigma\, a_n\, (z^{(n)}\, \dot{z}^{(n)} - (-1)^n\, z^{(2n)}\, \dot{z}) + b_m(\dot{z}^{(m)}\, \ddot{z}^{(m)} -$$

$$-(-1)^m\, \dot{z}\, \ddot{z}^{(2m)})] + \frac{\partial^2 J}{\partial t \partial x},$$

using the wave equation to eliminate $\ddot{z}$. But

$$\frac{\partial T_n}{\partial x} = -\,[z^{(n)}\, \dot{z}^{(n)} + (-1)^{(n-1)}\, z^{(2n)}\, \dot{z}],$$

$$\frac{\partial U_m}{\partial x} = -\,[\ddot{z}^{(m)}\, \dot{z}^{(m)} + (-1)^{m-1}\, \ddot{z}^{(2m)}\, \dot{z}].$$

Therefore (19) satisfies (18) when z is a solution of the wave equation.

We proceed by calculating the energy velocity $\bar{T}/\bar{E}$ for a harmonic wave from (16) and (19). The most convenient way to do this is to use the exponential form (1) for this wave and to apply the formula

$$\overline{Re(a)\; Re(b)} = \tfrac{1}{2} Re\; ab^*, \tag{20}$$

valid when a and b have the same period. We find in this way, observing that the averages of $\partial J/\partial x$ and $\partial J/\partial t$ over a wavelength or period must vanish,

$$E = \frac{B}{2}\left[\frac{\omega^2}{2} + \Sigma\, \frac{a_n}{2}\, k^{2n} + \omega^2\, \Sigma\, \frac{b_m}{2}\, k_{2m}\right] =$$

$$= \frac{B}{2}\, \Sigma\, a_n\, k^{2n},$$

by using the characteristic equation to eliminate ω^2. Furthermore, by repeated application of (20),

$$T_n = \tfrac{1}{2} n\, \omega\, k^{2n-1},$$

$$U_m = -\,\tfrac{1}{2} m\, \omega^3\, k^{2m-1};$$

therefore:

$$T = \frac{B}{2}\, \omega\, \sum_n\, (a_n - b_n\, \omega^2) n\, k^{2n-1}.$$

In this way we obtain

$$\frac{\bar{T}}{\bar{E}} = \frac{\omega \sum_n (a_n - b_n \omega^2)\, n\, k^{2n-1}}{\sum_n a_n k^{2n}} = \frac{\sum_n (a_n - b_n \omega^2)\, n\, k^{2n-1}}{\omega(1 + \sum_n b_n k^{2n})} \tag{21}$$

by using (5).

On the other hand we find on differentiating (5) with respect to k

$$2\omega \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\Sigma\, 2n\, a_n\, k^{2n-1}}{1 + \Sigma\, b_n\, k^{2n}} - \frac{\Sigma\, a_n k^{2n} \cdot \Sigma\, 2n\, b_n\, k^{2n-1}}{(1 + \Sigma\, b_n\, k^{2n})^2} =$$

$$= \frac{\Sigma\, 2n(a_n - b_n\omega^2)k^{2n-1}}{1 + \Sigma b_n k^{2n}}. \tag{22}$$

Comparing this with (21) we see that

$$\frac{\bar{T}}{\bar{E}} = \frac{d\omega}{\mathrm{d}k} = v(k), \tag{23}$$

which is the required result.

b. $F(z)$ is a linear expression in z and its even derivatives with respect to x (it must be even because L and E are so). In this case we can proceed with the process of partial integration until we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int E\mathrm{d}x = \int \frac{\partial}{\partial t} F(z) \cdot L(z)\mathrm{d}x = \int \dot{z} \cdot FL(z)\mathrm{d}x. \tag{24}$$

$FL(z)$ signifies the result of applying the operator F to $L(z)$. Since the coefficients in L and F are constants on account of the homogeneous character of the medium, L and F are commutable.

The solutions of (24) can be constructed from the coefficients of the equation $LF(z) = 0$ in exactly the same way as $E(z)$ was obtained from $L(z) = 0$. The energy velocity in a harmonic wave therefore now is found to be the group velocity associated with the former equation. But both equations have the same group velocity, since the effect of the differential operator F on the characteristic equation cancels in (5).

c. $F(z)$ could contain also even derivatives with respect to the time. We will not enter upon details but only mention that, when L is restricted to the second order with respect to t, no suitable expressions for the energy density are found in this way.

Finally we consider briefly the case of a multiple Hamilton function. We restrict ourselves to two branches and suppose that both of them correspond to conservative waves. The characteristic equation then factorises into two equations of the type (5)

$$W(k, \omega) \equiv W_1\,(k, \omega) W_2\,(k, \omega) = 0.$$

The wave equation is of the fourth order in t and can be written

$$L(z) = L_1 L_2(z) = 0. \tag{25}$$

A possible energy density now is

$$E = E_1(L_2 z) + E_2(L_1 z),$$

where $E_1(z)$, $E_2(z)$ are arbitrarily chosen energy densities corresponding to the wave equations $L_1(z) = 0$, $L_2(z) = 0$. In fact, using the foregoing results, we obtain for a wave system confined to a certain region.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{E}\mathrm{d}x = \int \{L_2(\dot{z}) \cdot L_1 L_2(z) + L_1(\dot{z}) \cdot L_2 L_1(z)\}\,\mathrm{d}x = \int (L_1 + L_2)\,L(z)\mathrm{d}x,$$

which vanishes on account of (24).

The corresponding flow density then is

$$T = T_1(L_2 z) + T_2(L_1 z)$$

and the energy velocity for a harmonic wave will be

$$\frac{\bar{T}}{\bar{E}} = \frac{\bar{T}_1 + \bar{T}_2}{\bar{E}_1 + \bar{E}_2} = \frac{v_1 \bar{E}_1 + v_2 \bar{E}_2}{\bar{E}_1 + \bar{E}_2},$$

which is again the weighted average of the group velocities.

The agreement of the results of this section and the preceding one is therefore complete, in accordance with the fact that the stationary phase method is a good approximation when the local wave spectrum is narrow. The wave system in the considered region then closely resembles a progressive harmonic wave.

§ 5. *Examples.* A very simple application of the formulae of § 4 can be made to flexural waves of a thin elastic rod. (The methods of the foregoing section in fact were found by a straight-forward generalisation of the following considerations).

Denoting the lateral displacement by z we will have for the bending moment

$$M = Bz'',$$

where B is the rigidity. Since for a stationary load q

$$M'' = -q,$$

the differential equation for small displacements will be

$$\ddot{z} = -\frac{B}{\varrho} z^{(4)} = -\lambda^2 z^{(4)}.$$

The characteristic equation now will be

$$\omega^2 = \lambda^2 k^4,$$

which yields for the phase and group velocities respectively

$$c = \frac{\omega}{k} = \lambda k.$$

$$v = \frac{\mathrm{d}\omega}{\mathrm{d}k} = 2\lambda k.$$

The energy density is the sum of the kinetic energy and the elastic deformation energy per unit length:

$$E = \tfrac{1}{2}\varrho\dot{z}^2 + \tfrac{1}{2}Bz''^2 = \tfrac{1}{2}\varrho[\dot{z}^2 + \lambda^2 z''^2].$$

From (19) we obtain therefore, as $a_2 = \lambda^2$, all other a_n, b_n being zero

$$T = -\varrho\lambda^2[\dot{z}'z'' - \dot{z}z'''] = M'\dot{z} - M\dot{z}'.$$

This is indeed the correct result for the rate of work done at a cross section (M' is the resultant force at this cross section, $\dot{z}\mathrm{d}t$ the displacement; M the bending moment, $-\dot{z}'\mathrm{d}t$ the increment of the deflection angle in the direction of positive M; cf. any treatise on the theory of elasticity).

If we now take for z a travelling wave

$$z = A \sin(kx - \lambda k^2 t) = A \sin\theta,$$

we get

$$E = \tfrac{1}{2}\varrho A^2[\lambda^2 k^4 \cos^2\theta + \lambda^2 k^4 \sin^2\theta] = \tfrac{1}{2}\varrho A^2\lambda^2 k^4,$$

$$T = -\varrho\lambda^2 A^2[-\lambda k^5 \sin^2\theta - \lambda k^5 \cos^2\theta] = \varrho A^2\lambda^3 k^5.$$

Therefore $T/E = 2\lambda k = v$.

In this case T and E are constant and $T/E = v$. In general this will not be the case. For instance, for the dispersionless waves on a string

$$\ddot{z} = c_0^2 z''.$$

T and E will vary like $\cos^2\theta$. In this case we still have $T/E = v$. An example where this is not true either, is furnished by the wave-guide equation

$$\ddot{z} = c_0^2 z'' - \omega_0^2 z.$$

The energy then fluctuates in such a way that it is propagated at the rate v only on the average.

A more complicated affair is the classical theory of surface waves on a non-viscous liquid under influence of gravity. The difficulty in applying the considerations of § 4 to these waves is that, although the propagation of these waves is in the horizontal direction only, "vertical" derivatives of the velocity potential do occur in the usual, most simple, treatment. We will therefore have to show at first how this vertical coordinate can be eliminated.

We consider an infinitely extended canal of uniform rectangular cross section, depth h. The axis of the canal is chosen as the x-direction, the vertical as the y-direction. The origin is located in the surface of the undisturbed liquid.

The motion is then described in terms of the velocity potential $\varphi(x, y, t)$, satisfying the equation

$$\varphi_{xx} + \varphi_{yy} = 0. \tag{26}$$

The velocities are $u = \varphi_x$, $v = \varphi_y$. At the bottom we have the boundary condition

$$(\varphi_y)_{-h} = 0, \tag{27}$$

since the flow must be horizontal there. The condition at the surface can be found from the Bernouilli equation

$$p + \tfrac{1}{2}\varrho(\varphi_x^2 + \varphi_y^2) + \varrho g y + \varrho\dot{\varphi} = 0. \tag{28}$$

When we apply this to the surface and discard the non-linear terms, we get

$$g\eta + (\dot{\varphi})_s = 0, \tag{29}$$

where η is the elevation of the surface above the plane $y = 0$. Now in a linear approximation we can put $\dot{\eta} = v_s$ and take the boundary condition at the undisturbed surface instead of at the real surface. (Details are given in any textbook on hydrodynamics, e.g. in L a m b [3]), or C o u l s o n [4])). In this way the boundary condition finally reduces to

$$g(\varphi_y)_0 + (\ddot{\varphi})_0 = 0. \tag{29a}$$

Instead of discussing the usual solution of (26) with the boundary conditions (28) and (29a), we try to describe the wave motion in terms of a variable $z(x, t)$, for which we take the surface value of the potential, φ_0, and to find the characteristic equation.

We start by expanding φ in a power series in y:

$$\varphi = \varphi_0 + (\varphi_y)_0 y + \tfrac{1}{2}(\varphi_{yy})_0 y^2 + \dots . \qquad (30)$$

(It can be verified that this series is convergent). The coefficients of the even terms can be expressed in z at once by means of (26), which yields

$$(\varphi_{yy})_0 = -z_{xx} = -z'',$$
$$(\varphi_{yyyy})_0 = z^{(4)}, \text{ etc.}$$

In the same way the coefficients of the odd terms become:

$$(\varphi_y)_0, -(\varphi_y)_0'' \dots \text{ etc.}$$

The only problem is now to express $(\varphi_y)_0$ in terms of z. This can be done step by step when we substitute (30) into the condition (27) and then differentiate with respect to x. When we want $(\varphi_y)_0$ e.g. up till terms of the sixth order, we write (27) in the form

$$0 = (\varphi_y)_0 + z'' \cdot h - \tfrac{1}{2}(\varphi_y)_0'' h^2 - \tfrac{1}{6} z^{(4)} h^3 + \tfrac{1}{24}(\varphi_y)_0^{(4)} h^4 + \tfrac{1}{120} z^{(6)} h^5, \dots . \qquad (31)$$

We then differentiate the expression for $(\varphi_y)_0$ given by (31) two and four times with respect to x, obtaining

$$(\varphi_y)_0'' = -z^{(4)} h + \tfrac{1}{2}(\varphi_y)_0^{(4)} h^2 + \tfrac{1}{6} z^{(6)} h^3 \dots \qquad (31a)$$

and

$$(\varphi_y)_0^{(4)} = -z^{(6)} h \dots \qquad (31b)$$

Now we substitute (31*b*) in (31*a*), which yields

$$(\varphi_y)_0'' = -z^{(4)} h - \tfrac{1}{3} z^{(6)} h^3 \dots , \qquad (31c)$$

and finally we use (31*b*) and (31*c*) to obtain from (31)

$$(\varphi_y)_0 = -z'' h - \tfrac{1}{3} z^{(4)} h^3 - \tfrac{2}{15} z^{(6)} h^5 . \qquad (32)$$

Once the existence of an expansion of this type is established, we can find the general term in the following way: For a harmonic wave at a certain instant we will have

$$z = \exp jkx, \ (\varphi_y)_0 = a \exp jkz,$$
$$z^{(2n)} = (-1)^n k^{2n} z, \ (\varphi_y)_0^{(2n)} = (-1)^n k^{2n} (\varphi_y)_0$$

Substituting into (31) we obtain therefore

$$0 = a \cosh kh - k \sinh kh.$$

By comparing the coefficients of h we obtain the expression (32) in the form

$$(\varphi_y)_0 = \Sigma c_n \cdot (-1)^n z^{2n}, \tag{32a}$$

where the coefficients c_n are defined by

$$\Sigma c_n k^{2n} = k \operatorname{tgh} kh. \tag{33}$$

The wave equation for z is supplied by (29a), vhich now becomes on account of (32a)

$$\ddot{z} + g \Sigma c_n (-1)^n z^{(2n)} = 0.$$

Observing (33) we obtain the characteristic equation in the well-known form

$$\omega^2 = gk \operatorname{tgh} kh.$$

The energy of the wave system in a section $x_1 - x_2$ of the canal is, per unit width,

$$\bar{E} = \tfrac{1}{2}\varrho g \int_{x_1}^{x_2} \eta^2 dx + \tfrac{1}{2}\varrho \int_{x_1}^{x_2} \int_{-h}^{0} (\varphi_x^2 + \varphi_y^2) \mathrm{d}x\mathrm{d}y.$$

Integrating the second term by parts by means of Greens' theorem and observing (27) and (29) we get

$$E = \tfrac{1}{2} \frac{\varrho}{g} \int_{x_1}^{x_2} \dot{z}^2 \, \mathrm{d}x + \tfrac{1}{2} \varrho \int_{x_1}^{x_2} z \cdot (\varphi_y)_0 \, \mathrm{d}x + \left| \int_{-h}^{0} \varphi \varphi_x \, \mathrm{d}x \right|_{x_1}^{x_2}.$$

When z vanishes outside of the region considered the last term will vanish. Substituting (32a) in the second term and integrating by parts we obtain

$$\bar{E} = \tfrac{1}{2} \frac{\varrho}{g} \int \mathrm{d}x \, [\dot{z}^2 + g \Sigma c_n \{z^{(n)}\}^2].$$

The surface waves therefore really do belong to the class of wave motions discussed in this paper.

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REFERENCES

1) Eckart, C., Rev. mod. Phys. **20** (1948) 399.
2) Rossby, C. G., J. Meteorol. **2** (1945) 187.
3) Lamb, H., Hydrodynamics 6th ed, Cambridge, 1932.
4) Coulson, C. A., Waves 5th ed., Edinburgh, 1948.