HEAT EFFECTS IN CAPILLARY FLOW I

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Summary

The temperature distribution in a capillary due to the energy dissipation of viscous flow is calculated while the heat conductivity of the fluid and the transport of heat by convection are taken into account.

It is well known [1] [2] that the viscosity of pure liquids, as measured by flow experiments in a capillary, may show a dependence on the rates of shear occurring in the liquid. A simple explanation of this effect proposed e.g. by Hersey [1] is that high rates of shear cause a high energy dissipation and therefore a temperature rise in the capillary.

In order to judge whether this explanation is justified it is useful to have a detailed knowledge of the temperature distribution in a capillary. The local variations in viscosity due to this temperature rise may then be estimated and their influence on the flow pattern may be ascertained. Heat effects in capillary flow have been treated by several authors. However, Hersey [1] does not take into account the effect of heat transport by convection. Philippoff [2], who has given a more extensive treatment of the problem, bases his calculations on a differential equation which appears to be incorrect. Therefore it seemed worthwhile to give a new treatment of the problem.

According to Poiseuille's law the velocity of flow v in a capillary is given by

$$v = \frac{R^2 - r^2}{4\eta} \frac{dp}{dz} \tag{1}$$

where R is the radius of the capillary,

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r and z are cylindrical coordinates, the z-axis coinciding with the axis of the capillary,

η is the viscosity,

p is the pressure, which is assumed to vary linearly with z.

The heat of friction generated for such a type of flow per unit of volume and time is

$$Q = \frac{r^2}{4\eta}\left(\frac{dp}{dz}\right)^2. \tag{2}$$

This equation is easily derived by calculating the work done on an element of volume by the normal and shearing stresses. For a general treatment cf. L a m b, Hydrodynamics, section 329.

Now for the stationary state the temperature distribution in the capillary is found by considering the heat balance for an element of volume:

$$-\lambda\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right\} + vc\,\frac{\partial T}{\partial z} = \frac{r^2}{4\eta}\left(\frac{dp}{dz}\right)^2 \tag{3}$$

where λ is the heat conductivity and c is the specific heat per unit of volume.

The first term of (3) is related to the heat transport by conduction, the second term to that by convection.

A complete treatment of (3) would involve the dependence of the viscosity η on the temperature. However, only small temperature variations will be considered. Therefore the dependence of η on T will only have a small effect on the temperature distribution and may be neglected in a first approximation. It might be introduced as a correction after the solution for constant η has been obtained. A further simplification is introduced by neglecting the heat conduction in the axial direction ($\lambda\partial^2 T/\partial z^2$) which is very small compared to the convection. With these simplifications a solution of (3) is given for two cases:

a. the walls of the capillary are kept at constant temperature: $T = 0$ for $r = R$.

b. the walls of the capillary have zero heat conductivity: $\partial T/\partial r = 0$ for $r = R$.

In both cases it is assumed that the fluid is introduced into the capillary at zero temperature: $T = 0$ for $z = 0$.

The solution of (3) may be expressed in the dimensionless quantities

$$\varrho = r/R,$$
$$\zeta = \frac{4\lambda\eta z}{cR^4\,(dp/dz)},$$
$$\tau = \frac{16\eta\lambda}{R^4\,(dp/dz)^2}\,T.$$

It may be written in the form
for case a:

$$\tau = \sum_i \varphi_i\,(\varrho)\,e^{x_i\zeta} + \tfrac{1}{4}\,(1 - \varrho^4). \tag{4}$$

for case b:

$$\tau = \sum_i \varphi_i\,(\varrho)\,e^{x_i\zeta} + \varrho^2 - \frac{\varrho^4}{2} + 4\zeta. \tag{5}$$

The last terms in (4) and (5) are a solution of (3) satisfying the boundary condition at $r = R$; the series is a solution of the homogeneous equation to be chosen in such a way that (4) and (5) satisfy the boundary condition for $z = 0$ as well. The x_i are pure numbers to be determined from the boundary condition at $r = R$. The functions $\varphi_i\,(\varrho)$ are solutions of the following differential equation, found by substitution of (4) or (5) in (3):

$$\frac{1}{\varrho}\,\frac{d}{d\varrho}\left(\varrho\,\frac{d\varphi_i}{d\varrho}\right) - x_i\,(1 - \varrho^2)\,\varphi_i = 0.$$

This is a cylindrical analogon of Weber's equation (cf. Whittaker and Watson, Modern Analysis, page 347). It may be transformed into an equation of the confluent hypergeometric type. Its solutions form a complete orthogonal set with real eigenvalues. Taking into account their regularity at $\varrho = 0$, they may be found by expanding φ in a power series

$$\varphi_i\,(\varrho) = \sum_{k=0}^{\infty} b_{i,k}\,\varrho^{2k}. \tag{6}$$

Substitution in the differential equation yields the recursion formula for the coefficients

$$b_{i,k} = \frac{x_i}{4k^2}\,(b_{i,k-1} - b_{i,k-2}). \tag{7}$$

By means of (7) all $b_{i,k}$ may be found as functions of x_i and $b_{i,0}$:

$$b_{i,1} = \frac{x_i}{4}\,b_{i,0}\,,$$

$$b_{i,2} = \left(\frac{x_i^2}{64} - \frac{x_i}{16}\right) b_{i,0},$$

etc.

The $b_{i,0}$ will be determined from the boundary condition at $\zeta = 0$. The values of x_i are found by introduction of the boundary condition for $\varrho = 1$. It amounts to

for case a: $$\sum_{k=0}^{\infty} b_{i,k} = 0, \tag{8a}$$

for case b: $$\sum_{k=0}^{\infty} k b_{i,k} = 0. \tag{8b}$$

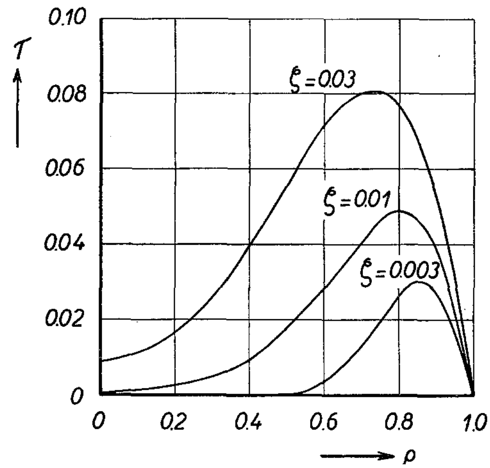

Fig. 1. Temperature distribution in a capillary. Walls at zero temperature.

The lower eigenvalues x_i were found by substitution of 7 in (8a) and (8b). The values of x_i were then determined by trial and error. They are equal to

for case a: $x_i = -7.314$; -44.61; -113.92; -213.9; . . .
for case b: $x_i = 0$; -25.68; -83.86; -174.55; . . .

The values of $b_{i,0}$ remain to be determined from the condition $\tau = 0$ for $\zeta = 0$.

As the functions $\varphi_i(\varrho)$ form a complete orthogonal set, it follows from (4) and (5), substituting $\zeta = 0$,

for case a: $$\int_0^1 \varphi_i^2(\varrho)\, \varrho d\varrho = -\int_0^1 \tfrac{1}{4}(1 - \varrho^4)\, \varphi_i(\varrho)\, \varrho d\varrho,$$

for case b: $$\int_0^1 \varphi_i^2(\varrho)\,\varrho d\,\varrho = -\int_0^1 \left(\varrho^2 - \frac{\varrho^4}{2}\right) \varphi_i(\varrho)\,\varrho d\,\varrho.$$

From these equations $b_{i,0}$ can be determined by integration. The $b_{i,k}$ than follow from (7).

In fig. 1 and 2, τ is given as a function of ϱ for various values of ζ. From these graphs the temperature distribution in a capillary can be determined in any special case. As was to be expected, the temperature is highest near the walls of the capillary where the rate of shear is highest. This maximum is reduced by the heat conductivity of the liquid.

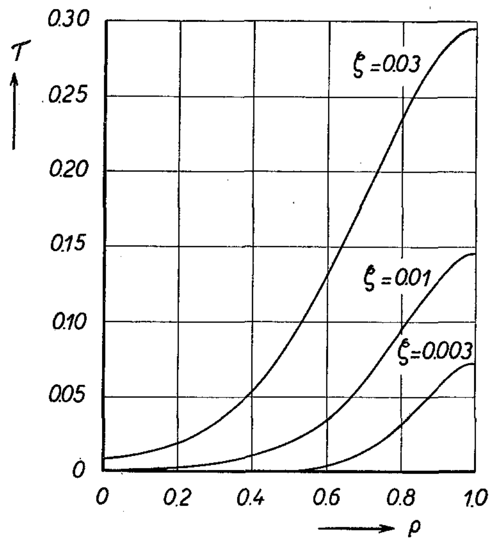

Fig. 2. Temperature distribution in capillary. Insulating walls.

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