APPROXIMATE SUBSONIC GAS FLOWS UNDER ASSIGNED BOUNDARY CONDITIONS

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Summary

The solution to compressible flow problems under fully assigned boundary conditions is discussed. It is shown that Schwarz's results on minimal surfaces can be immediately applied for two-dimensional flow, and several special cases and examples are given. Extensions of these results provide certain particular types of three-dimensional flow.

§ 1. *Introduction*. Chaplygin [1]) first suggested the use of a tangent to the adiabatic curve, $pv^{\gamma} =$ constant, where p and $\rho(= v^{-1})$ are respectively the pressure and density, as an approximate representation of the equation of state for the subsonic flow of a compressible fluid. Many flow problems have been solved by employing the hodograph equations, which, with this approximate equation of state, reduce to the Cauchy-Riemann differential equations. Similar incompressible flows can then be used as a basis for solution. Unfortunately, a characteristic of the hodograph method, even in the simplified case, is that it often proves difficult to satisfy given boundary conditions; thus other methods are sought for problems involving boundary values.

Now it is well-known that the equation of continuity for two-dimensional irrotational motion, using a straight line approximation to the equation of state, is related to the characteristic equation of a minimal surface in three-dimensional space (or a minimal region in four-dimensional space for a three-dimensional flow). This is discussed, for example, by Bateman [2]) and Braun [3]). Germain [4]) has discussed Enneper's representation of a minimal surface and its connection with the method given by Tsien [5]).

In this paper, the solution to flow problems under fully assigned boundary conditions is discussed. That is to say, boundaries are involved upon which, in two-dimensional flow, both components of fluid speed are given. It is shown that Schwarz's results on minimal surfaces can be immediately applied and several special cases and examples are included. Extensions of these results provide certain particular types of three-dimensional flow.

§ 2. *Relation between gas flows and minimal surfaces.* Let the approximate equation of state be written in the form

$$p = \alpha - \beta^2/\rho, \tag{1}$$

α and β being constants, and let $c^2 = c_0{}^2 + q^2$, where q is the fluid speed, c the sonic speed with value c_0 at the stagnation point, given by $c_0{}^2 = \beta^2/\rho_0{}^2$. By Bernoulli's theorem

$$\frac{\rho_0{}^2}{\rho^2} = \frac{q^2}{c_0{}^2} + 1,$$

and the equation of continuity for the velocity potential ϕ is given by

$$(c_0{}^2 + \phi_y{}^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (c_0{}^2 + \phi_x{}^2)\phi_{yy} = 0. \tag{2}$$

This differential equation is elliptic if $c_0{}^2 > 0$; thus the flow will be wholly subsonic. Setting $\phi = c_0 z$, equation (2) becomes

$$(1 + z_y{}^2)z_{xx} - 2z_xz_yz_{xy} + (1 + x_x{}^2)z_{yy} = 0, \tag{3}$$

and this is the characteristic equation of a minimal surface, where x, y, z are rectangular coordinates of a point in space.

It is interesting to note that this is also analogous to the equilibrium problem of a thin film of constant tension with equal pressures on the two sides of the film.

§ 3. *Minimal surfaces under assigned boundary conditions.* Suppose x, y, z_x, z_y can be expressed as real functions of a parameter t, then, apart from an arbitrary constant, z can be determined as a function of t from the relation

$$\mathrm{d}z = z_x\,\mathrm{d}x + z_y\,\mathrm{d}y.$$

Geometrically, assigning z_x and z_y is equivalent to assigning the direction cosines X, Y, Z of a tangent plane to a surface passing through the curve defined by $x = x(t)$, $y = y(t)$, $z = z(t)$ and so

$$(X, Y, Z) \equiv (z_x, z_y, -1)/(z_x^2 + z_y^2 + 1)^{\frac{1}{2}}.$$

The only cases in which external conditions cannot be given are those for which $dx^2 + dy^2 + dz^2 = 0$, but since we are only concerned with real values for x, y, z, the only excluded boundary is a point.

Schwarz's method (Forsyth [6])) can be applied and the coordinates of any point (x, y, z) on the surface are found to be

$$\begin{aligned} 2x &= x(\zeta) + x(\bar{\zeta}) - i\int_{\bar{\zeta}}^{\zeta} (Y\mathrm{d}z - Z\mathrm{d}y), \\ 2y &= y(\zeta) + y(\bar{\zeta}) - i\int_{\bar{\zeta}}^{\zeta} (Z\mathrm{d}x - X\mathrm{d}z), \\ 2z &= z(\zeta) + z(\bar{\zeta}) - i\int_{\bar{\zeta}}^{\zeta} (X\mathrm{d}y - Y\mathrm{d}x), \end{aligned} \tag{4}$$

where ζ and $\bar{\zeta}$ replace t on the boundary, and $\zeta = \xi + i\eta$, $\bar{\zeta} = \xi - i\eta$. Rewriting (4) as

$$x = e(\zeta) + \bar{e}(\bar{\zeta}),\ y = f(\zeta) + \bar{f}(\bar{\zeta}),\ z = g(\zeta) + \bar{g}(\bar{\zeta}),$$

it is easy to show that the curves given by $g + \bar{g} = \text{const.}$ and $g - \bar{g} = \text{const.}$ are orthogonal. Further, by considering the unit normal at any point on the surface, it follows that

$$z_x = (f'\bar{g}' - \bar{f}'g')/(\bar{e}'f' - e'\bar{f}'), \tag{5}$$

$$z_y = (g'\bar{e}' - \bar{g}'e')/(\bar{e}'f' - e'\bar{f}'), \tag{6}$$

where $e' = \mathrm{d}e/\mathrm{d}\zeta$, $\bar{e}' = \mathrm{d}\bar{e}/\mathrm{d}\bar{\zeta}$, etc.

§ 4. *Gas flow analogy*. From the preceding section we see immediately that the curves $g - \bar{g} = \text{const.}$ are streamlines, and thus for two-dimensional compressible flow under fully assigned boundary conditions we have

$$x = e(\zeta) + \bar{e}(\bar{\zeta}),\ y = f(\zeta) + \bar{f}(\bar{\zeta}), \tag{7}$$

$$\phi + i\psi = 2c_0 g(\zeta), \tag{8}$$

where ψ is the stream function, and

$$\phi_x = c_0 z_x,\ \phi_y = c_0 z_y. \tag{9}$$

Certain special cases, which simplify (7), (8) and (9), will now be considered.

(i) Boundary given by $\phi =$ constant. Here the velocity is normal to the boundary, and we define $x = x(t)$, $y = y(t)$ and the normal component of fluid speed $\phi_n = \phi_n(t)$, or equivalently $z_n = z_n(t)$, since $\phi = c_0 z$. Equations (7) and (8) become,

$$\begin{aligned} 2x &= x(\zeta) + x(\bar{\zeta}) - i\int_{\bar{\zeta}}^{\zeta} y_t(z_n^2 + 1)^{-\frac{1}{2}}\, \mathrm{d}t, \\ 2y &= y(\zeta) + y(\bar{\zeta}) + i\int_{\bar{\zeta}}^{\zeta} x_t(z_n^2 + 1)^{-\frac{1}{2}}\, \mathrm{d}t, \\ 2\phi &= \text{const.} - ic_0\int_{\bar{\zeta}}^{\zeta} z_n(z_n^2 + 1)^{-\frac{1}{2}}(x_t^2 + y_t^2)^{\frac{1}{2}}\, \mathrm{d}t. \end{aligned} \tag{10}$$

The equations are further simplified in the case of constant normal velocity, (10) becoming

$$\begin{aligned} 2x &= x(\zeta) + x(\bar{\zeta}) - iL[y(\zeta) - y(\bar{\zeta})], \\ 2y &= y(\zeta) + y(\bar{\zeta}) + iL[x(\zeta) - x(\bar{\zeta})], \\ 2\phi &= \text{const.} - iLk\int_{\bar{\zeta}}^{\zeta} \sqrt{(x_t^2 + y_t^2)}\, \mathrm{d}t, \end{aligned} \tag{11}$$

where $\phi_n = k$ and $L = c_0(k^2 + c_0^2)^{-\frac{1}{2}}$.

(ii) Boundary given by $\psi =$ constant. In this case the fluid velocity will be in a direction tangential to the curve $x = x(t)$, $y = y(t)$. Let this tangential velocity be $\phi_s = \phi_s(t) = c_0 z_s(t)$, so that on the boundary

$$z_x = z_s x_t(x_t^2 + y_t^2)^{-\frac{1}{2}}, \quad z_y = z_s y_t(x_t^2 + y_t^2)^{-\frac{1}{2}},$$

and ϕ and ψ at any point are given by

$$\begin{aligned} 2x &= x(\zeta) + x(\bar{\zeta}) - i\int_{\bar{\zeta}}^{\zeta} y_t(z_s^2 + 1)^{\frac{1}{2}}\, \mathrm{d}t, \\ 2y &= y(\zeta) + y(\bar{\zeta}) + i\int_{\bar{\zeta}}^{\zeta} x_t(z_s^2 + 1)^{\frac{1}{2}}\, \mathrm{d}t, \\ \phi + i\psi &= c_0 z(\zeta). \end{aligned} \tag{12}$$

It is hoped that, in practice, by suitable choice of variables, a close approximation to any given boundary may be achieved.

(iii) Straight boundary given by $\psi =$ constant. Geometrically, if a straight line lies in a minimal surface, then this line is an axis of symmetry of the surface. If now the x-axis is the straight boundary, by analogy the fluid motion must be symmetrical about $y = 0$. Equations (12) become

$$x = \xi,\ 2y = i\int_{\bar\zeta}^{\zeta} \sqrt{(z_s^2 + 1)}\ \mathrm{d}t, \qquad \phi + i\psi = c_0 z(\zeta). \tag{13}$$

The function $z(\zeta)$ may be interpreted as the complex potential of a certain incompressible flow in the ζ-plane, which must be symmetrical about the ξ-axis. If we let $z(\zeta) = W(\zeta)$, $z_s(\zeta) = \mathrm{d}W/\mathrm{d}\zeta$, then

$$x = \xi,$$

$$2y = i\int \sqrt{\left[\left(\frac{\mathrm{d}W}{\mathrm{d}\zeta}\right)^2 + 1\right]}\mathrm{d}\zeta - i\int \sqrt{\left[\left(\frac{\mathrm{d}\overline{W}}{\mathrm{d}\bar\zeta}\right)^2 + 1\right]}\mathrm{d}\bar\zeta,$$

$$\phi + i\psi = c_0 W(\zeta),$$

where $W(\zeta)$ is any function of ζ, real when $\eta = 0$.

§ 5. *Application of the method.* It is evident that the method is particularly suitable for various types of source flows or circulatory motions, that is, when the boundary is respectively an equipotential line or a streamline.

Example. Consider a parabolic cylinder over which the fluid velocity is constant and normal to the cylinder. The parabola must be an equipotential curve and let it be represented by $x = bt^2$, $y = 2bt$, $\phi = 0$; hence ϕ and ψ are given by

$$\begin{aligned} 2x &= b[\zeta^2 + \bar\zeta^2 - 2iL(\zeta - \bar\zeta)], \\ 2y &= b[2\zeta + 2\bar\zeta + iL(\zeta^2 - \bar\zeta^2)], \\ \phi + i\psi &= ibLk\{\zeta\sqrt{(1 + \zeta^2)} + \ln[\zeta + \sqrt{(1 + \zeta^2)}]\}. \end{aligned} \tag{14}$$

Further, with the notation of § 2,

$$e'(\zeta) = b(\zeta - iL),\ f'(\zeta) = b(1 + iL\zeta),\ c_0 g'(\zeta) = iLkb\sqrt{(1 + \zeta^2)},$$

and

$$\phi_x = iLk\left[\frac{(1+iL\zeta)\sqrt{(1+\bar{\zeta}^2)} + (1-iL\bar{\zeta})\sqrt{(1+\zeta^2)}}{(\zeta-iL)(1-iL\bar{\zeta}) - (\bar{\zeta}+iL)(1+iL\zeta)}\right], \quad (15)$$

$$\phi_y = -\,iLk\left[\frac{(\bar{\zeta}+iL)\sqrt{(1+\zeta^2)} + (\zeta-iL)\sqrt{(1+\bar{\zeta}^2)}}{(\zeta-iL)(1-iL\bar{\zeta}) - (\bar{\zeta}+iL)(1+iL\zeta)}\right]. \quad (16)$$

Equations (14), (15) and (16) provide a solution to the problem. In particular, let us consider the variation of ϕ_x along the x-axis, which is given by $\xi = 0$. Along this axis,

$$x = b\eta(2L-\eta),\ y = 0,$$
$$\phi + i\psi = iLkb\{i\eta\sqrt{(1-\eta^2)} + \ln[i\eta + \sqrt{(1-\eta^2)}]\},$$
$$\phi_x = Lk(\eta-L)^{-1}\sqrt{(1-\eta^2)},\ \phi_y = 0.$$

There are two stagnation points at $\eta = \pm 1$ and one singularity at $\eta = L$, which lies inside the parabola. The stagnation points lie one inside and one outside the curve provided $2L > 1$; that is to say, provided $k^2 < 3c_0^2$, and this condition is satisfied for subsonic flow. It is to be noted that the flow does not include the whole xy-plane, since, along the x-axis, ϕ_x is complex for $\eta^2 > 1$.

§ 6. *Three-dimensional gas flows.* The equation of continuity for irrotational motion in three-dimensions represents the characteristic equation of a minimal region in four-dimensional space, the equation being [7])

$$(1 + W_y^2 + W_z^2)W_{xx} + (1 + W_z^2 + W_x^2)W_{yy} + \\ + (1 + W_x^2 + W_y^2)W_{zz} - 2W_yW_zW_{yz} - \\ - 2W_zW_xW_{zx} - 2W_xW_yW_{xy} = 0, \quad (17)$$

where $\phi = c_0W$. In four dimensions, it is possible to have either minimal two-dimensional 'spreads' or minimal regions. The former lead to surfaces in three dimensions and the latter obviously include the whole three-dimensional space. Surface solutions have been given by Kommerell [8]) in the form, $z + iW = F(x+iy)$, where F is an arbitrary function of $x + iy$. Interpreted as a gas flow, this yields

$$\phi = -\tfrac{1}{2}c_0 i[F(x+iy) - F(x-iy)],$$
$$z = \tfrac{1}{2}[F(x+iy) + F(x-iy)],$$
$$\psi = z.$$

The projection of the streamlines on the plane $z = 0$ gives the flow pattern of a plane incompressible flow.

An extension of Schwarz's results has been derived by Eisenhart [9]) and x, y, z, ϕ can be determined for assigned boundary conditions on a given curve. However, in fluid flow, the minimal region is of more practical interest, but it appears that only particular solutions can be obtained in this case. Two simple cases are mentioned briefly below.

(i) Simple source. It is easily shown that equation (17) has a solution of the form

$$x^2 + y^2 + z^2 = E(\phi) \tag{18}$$

where $E(\phi)$ is the Weierstrass elliptic function [7]). The equipotential surfaces are concentric spheres and the streamlines are radial, and thus this solution can be interpreted as a simple source in three dimensions.

(ii) One velocity component constant. Let us suppose that the velocity potential may be written in the form $c_0[Uz + w(x, y)]$ where U is a constant, so that (17) becomes

$$(1 + U^2 + w_y{}^2)w_{xx} + (1 + U^2 + w_x{}^2)w_{yy} - 2w_x w_y w_{xy} = 0. \tag{19}$$

Let $w = \sqrt{(1 + U^2)}\chi$, then (19) yields

$$(1 + \chi_y{}^2)\chi_{xx} - 2\chi_x\chi_y\chi_{xy} + (1 + \chi_x{}^2)\chi_{yy} = 0, \tag{20}$$

and thus χ is a solution of the two-dimensional problem. To any two-dimensional solution there corresponds a three-dimensional solution obtained by adding a constant velocity normal to the xy-plane. This is similar to the result obtained by Poritsky [10]) for the more general case of compressible flow. However, (20) can be solved for many more problems than the exact equation used by Poritsky.

§ 6. *Conclusion.* In the preceding work no account has been taken of the value to be adopted for the slope of the tangent to the (p, v) curve. Unlike the method of von Kármán and Tsien [5]) there is no simple condition at infinity for which a corresponding tangent can be obtained. The value of β^2 is important since it affects the streamline pattern and the velocity distribution. However, the choice of straight line approximation which corresponds most closely to the corresponding adiabatic flow problem with $\gamma = 1.4$ seems difficult to ascertain. Jacob[11]) suggests one method and another is discussed by Power and Smith[12]). The latter consists of simply taking a mean straight line to the curve pv^γ = const. over a suitable range of values. Although no entirely satisfactory way of assessing the error seems at present available, it appears that this method yields good results at least for certain types of boundary.

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