A. ORTON

STUDENTS' UNDERSTANDING OF DIFFERENTIATION

ABSTRACT. A clinical interviewing method was used to investigate students' understanding of elementary calculus. The analysis of responses to tasks concerned with differentiation and rate of change led to detailed data concerning the degree of understanding attained and the common errors and misconceptions. Some conclusions were drawn concerning the teaching of differentiation and rate of change. This article is a companion to a previous article concerning integration and limits.

INTRODUCTION

In a previous article (Orton, 1983) the research study on which this article is based was described (Orton, 1980). The study involved individual interviews with 110 students across the age range 16–22 years. Sixty of the students were selected from the sixth forms (age 16–18) of four schools, and the other 50 students (age 18–22) were selected from two colleges where they were training to become teachers of mathematics. The complete sample included 55 males and 55 females. In the first interview students were presented with tasks largely involving limits, area and integration. The second interview involved rate of change, differentiation and applications. Inevitably, both sets of tasks tested the understanding of certain algebraic skills or processes. This article is concerned only with students' understanding of differentiation and associated background mathematics such as rate of change.

The tasks used, being based largely on mathematical situations, such as graphs, usually tested a variety of concepts and skills. In the analysis of results appropriate subdivisions of the tasks were therefore brought together and called items, each item relating to just one aspect of differentiation or rate. There were 21 such items. A selection of the tasks concerned with differentiation and rate is detailed in the Appendix to this paper. The descriptions of the main items derived from tasks referred to in this paper are listed in Table I, together with the associated mean scores for the two groups of subjects separately, school students and college students.

Responses to all items were assessed on a five-point scale (0, 1, 2, 3, 4). Study of the mean scores shows that the two groups of students experienced similar successes and failures on the items. Among the more difficult items were those concerned with understanding differentiation (Items 33 and 34) and graphical approaches to rate of change (Items 27, 28, 29). Students generally found the applications of differentiation relatively easy (Items 35 and 36), though Item 37 was rather more difficult.

Educational Studies in Mathematics **14** (1983) 235–250. 0013–1954/83/0143–0235$01.60.

TABLE I
Some differentiation and rate items and mean scores

Item	Description	Related tasks	Mean scores	
			School	College
3	Infinite geometric sequences	A3(b), A4, A5(a), A6(b)*	2.88	2.56
4	Limits of geometric sequences	A3(a), A5(b), A6(a)*	2.92	2.78
21	Substitution and increases from equations	C4(i)–(iii), C6(i)–(iii)	3.32	3.68
27	Rate of change from straight line graph	C3(v), C4(vii)	2.22	2.02
28	Rate, average rate and instantaneous rate	C4(iv), C6(iv)–(v)	0.88	1.18
29	Average rate of change from curve	D1(vi)–(viii)	2.22	1.92
32	Carrying out differentiation	C7(ii)–(iii)	3.62	3.50
33	Differentiation as a limit	C8	1.88	1.14
34	Use of δ-symbolism	D2	1.52	1.40
35	Significance of rates of change from differentiation	D3	3.43	3.62
36	Gradient of tangent to curve by differentiation	D7	3.63	3.76
37	Stationary points on a graph	D7	2.30	2.54

* Tasks A3, A4, and A6 are not included in the Appendix, as they are not discussed in the article.

ERRORS AND MISCONCEPTIONS

In attempting to classify errors made by students the scheme described by Donaldson (1963) has been used alongside a more mathematical analysis. Donaldson's three types of error, structural, arbitrary and executive, outlined in the previous article, were found to be useful as providing a broader categorization than the one based on mathematical skills and concepts. For most items there was a very large number of different responses and, although in the discussion which follows an attempt has been made to simplify and generalize, a certain amount of detail has also been retained. Donaldson's simple scheme contrasts well with the more detailed mathematical topic structure. At the end of this section Table II summarizes the type of errors found in responses to those items which are discussed in detail.

Some of the difficulties experienced by students were algebraic. One such algebraic error arose in response to Task D7 (Item 37). In carrying out this task the solution of the equation $3x^2 - 6x = 0$ was required. In all, 24 of the students were unable to solve this correctly. Twelve students lost $x = 0$ by incorrectly 'cancelling' x, or dividing throughout by x, a very familiar error to teachers, still being committed by older, intelligent, students after many years' experience of encountering such equations. A further 6 of the 24 students incorrectly factorized into $3x(x-6)=0$, yet another common school error. The errors appeared to be both structural and executive.

Another algebraic error which arose on the rate of change and differentiation items occurred in responses to Item 21 (Task C6). In the expansion of $3(a+h)^2$, 17 students lost the middle term, $6ah$, yet another very common school error. There could have been some lack of understanding of quadratic expansion involved in this, and so, as with the previous situation, there is a strong possibility that errors were structural. Other incorrect responses to C6(i)–(iii) suggested, however, that executive errors were being committed.

Errors concerning limits, rather than algebraic manipulation, were committed in responses to Task A5 (Items 3 and 4). Some students gave numerical responses to part (a) and to similar questions in other tasks. It must be admitted that, to the mathematically unsophisticated student, the question may well suggest that there was a numerical answer. In fact, it was the degree of sophistication of the response which was the clue to level of understanding. Other responses suggested concern about the practicality of carrying out the task, for example some students said, in effect, "You'd get more the finer your pencil lines."

Part (b) produced many unsatisfactory responses which appeared to suggest a widespread misconception. The idea of the rotating secant was intended to relate to the approach to differentiation and so was considered to be an important task in giving further evidence concerning level of understanding of the tangent as a limit. It seems very significant that 43 students were unable to state that the secant eventually became a tangent, despite considerable encouragement, through further questioning, to say more about what happened to the secant, until they ran out of things to say about it. There appeared to be considerable confusion in that the secant was ignored by many students; they appeared only to focus their attention on the chord PQ, despite the fact that the diagram and explanation were intended to try to ensure that this did not happen. Typical unsatisfactory responses included: "The line gets shorter"; "It becomes a point"; "The area gets smaller"; "It disappears". It appeared that, in the normal approach to differentiation, students may need considerable help in understanding the tangent as the limit of the set of secants. Some curriculum implications of this are discussed in Orton (1977). Errors made by students on Task A5 therefore appeared to be structural.

The basic calculation of gradient or rate of change from a graph depends on obtaining the difference in y-values which corresponds to a unit difference in x-values. In the case of a straight line this can be obtained very easily; any right-angled triangle with part of the line as hypotenuse provides a y-difference and an x-difference from which the ratio, y-difference/x-difference, gives the rate of change. This simple rule for obtaining the ratio is important, and the students would have met it in a study of graphs and algebra before commencing calculus. At the same time, however, it is in this aspect of the study of rate of change that it is clear that proportionality is involved. The student must accept that the ratio y/x is the same whatever the triangle, as long as the hypotenuse lies along the given line. In general, throughout the tasks, this elementary rule, y-difference/x-difference, was not found to be elementary to apply by a very large number of students. This was true whether the graph was a straight line or a curve.

One particular task which demanded the use of y-difference/x-difference was Task D1 parts (vi), (vii) and (viii) (Item 29). The most elementary application of the rule was in D1(vi), in which both the y-increase and the x-increase were 1. Thirty-nine students did not obtain the correct answer. In the two remaining parts of the item there were even more errors made by students. In part (vii) a quarter of the students omitted the negative sign; others calculated a ratio, but appeared to divide the correct y-increment by the x-coordinate of E. There were incorrect answers to part (viii) from nearly half of the students. The largest identifiable proportion of these were the students who could give no answer. A smaller group comprised those who gave an answer involving both 6 and 5, the coordinates of J being (6, 5). Overall, in Item 29, although many errors were executive, there were also misunderstandings which led to structural problems.

A major point which students must grasp in a graphical study of rate of change concerns the difference between straight lines and curves. For a curve an average rate of change can be calculated in the same way as for a straight line, but there is also the idea of rate of change at a point on the curve, and every point on the curve may lead to a different value for the rate of change. In the case of a straight line one can also obtain the rate of change at a point, but its value is the same for every point, indeed its value is the same as the average rate of change over an x-interval. It can be difficult to study rate of change at a point for this reason. The most elementary graph to begin with, the straight line, has a constant rate of change, so the distinction between average rate of change and rate of change at a point may have little meaning to some students. Both Tasks C3 and C4 included questions about the rate of change at a point on a straight line (Item 27).

In Task C3, students were informed that water was flowing into a tank at a constant rate; the rate was given as 2 units of depth per unit of time, but the meaning of this had apparently not been grasped when it came to answering questions on the rate of change at a given point (C3(v)). At $x = 2\frac{1}{2}$, 22 students responded with the y-value, 5, and not with the rate. At the general point $x = T$, a larger number of students either failed to respond or they said "$2T$." Responses to the corresponding questions in Task C4 (part (vii)), based on a graph with a given equation, were a little worse. Twenty-nine students either gave the value of y when $x = 2\frac{1}{2}$, or what they did give suggested that they had attempted to give the value of y. The same students generally responded with $3x - 1$, or similar, for the final question. These errors certainly appeared to be structural. Janvier (1978) discovered a similar phenomenon which he discussed as "attraction to high values."

Where the distinction between average rate of change over an interval and rate of change at a point becomes meaningful, in non-linear relationships, was tested in Task C6(iv)–(v) (Item 28). The extent to which students found C6(iv) difficult is reflected in the fact that 30 students were unable to attempt the question and a further 44 students attempted but failed to complete the solution. The high incidence of poor attempts tallies with the results of the corresponding part of Task C4. Again, the errors were structural.

The extension of the idea of ratio of y-difference/x-difference to rate of change at a point appeared in Task C6(v), in which question a substitution of $a = 2\frac{1}{2}$ and a limit as $h \to 0$ were both involved. Ninety-six students could not answer this question correctly, a very large number of them being unable to give any answer at all. A response was considered to be correct if the student said, in effect, "put $a = 2\frac{1}{2}$ and $h = 0$ in part (iv)," so part (iv) did not need to be correct. The main point is that C6(v) was not answered well even by students whose previous response was either correct or was of a form which helped the student. This was not a case of mistakes being made, but rather a case of students having little conception of what to do, so the basic error was structural.

Both limit and rate of change were involved in the explanation of differentiation in Task C8 (Item 33). A very common error was to state that the formula measured the rate of change "from P to Q," that is, over the whole section of curve between P and Q. Such a response may have been something of a guess in an unfamiliar situation, but it does indicate how many students ignored the limit in the given definition when answering part (i). However, a few of these students did reveal some understanding of limit in their response to part (ii). Overall, the errors made were again structural.

Elementary calculus makes use of a number of symbols, all of which are standard and should eventually be understood by students. The symbols δx, δy, $\delta y/\delta x$, $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}y/\mathrm{d}x$ were used in the tasks, and in Task D2 (Item 34) students were asked to explain what they understood by them. In their explanations students revealed many misunderstandings and the overall level of understanding was very poor. Errors were fundamentally structural.

The majority of students were able to explain δx and δy satisfactorily and this also suggested that most of the students had met the symbols before. It is important to make this point because from there on in Task D2 many students had difficulty in making sense of the other symbols. For example, $\delta y/\delta x$ is a straightforward ratio or quotient of the two small increments, but seventy-one students could not answer correctly. The largest number of these students gave an answer involving rate which was unacceptable, for example "rate of change of y/rate of change of x," "rate of change at a point," "a small increase in the rate of change."

The symbols which caused the greatest problems to students were $\mathrm{d}x$ and $\mathrm{d}y$. This was expected in the sense that the symbols are not really meaningful except when used together as $\mathrm{d}y/\mathrm{d}x$ or when used in an integration, for example, $\int_a^b f(x)\,\mathrm{d}x$. Three main types of incorrect response were apparent in addition to nil responses. Twenty-nine students explained $\mathrm{d}x$ as "the differential of x," or "the rate of change of x." A further twenty-five students explained $\mathrm{d}x$ as "the limit of δx as $\delta x \to 0$." Another twenty students thought that $\mathrm{d}x$ was an "amount of x" or "x-increment," in other words, was more or less the same as δx.

Part (vi) of Task D2 was the next easiest, as far as the students were concerned, after parts (i) and (ii), though forty-seven students did not give acceptable responses. The only large group within this forty-seven consisted of the seventeen students who answered in terms of a ratio or, more rarely, a gradient. Examples of such responses were "the difference in y/the difference in x," "the average change of y per x," and "the gradient, y-step/x-step."

Amongst the seventy-eight responses to part (vii) which showed no progress there were five easily identifiable categories. Firstly, twenty-one students could not give an answer at all. Secondly, eleven students said that $\mathrm{d}y/\mathrm{d}x$ and $\delta y/\delta x$ were the same as each other. Thirdly, eleven students said something like "$\delta y/\delta x$ gets smaller until it is so small it is called $\mathrm{d}y/\mathrm{d}x$." Fourthly, six students said that $\mathrm{d}y/\mathrm{d}x$ and $\delta y/\delta x$ were approximately equal. Finally, ten students answered in terms of limits, but actual statements made were very varied and included "as $\delta y/\delta x \to 0$ the gradient $\to \mathrm{d}y/\mathrm{d}x$," and "as δy gets smaller it tends to $\mathrm{d}y$, and similarly $\delta x \to \mathrm{d}x$."

The routine aspect of differentiation was well understood. Only four students failed to differentiate $y = x^2 - 4x + 1$ in Task D3 (Item 35), and a different six students could not differentiate $y = x^3 - 3x^2 + 4$ in Task D7 (Item 36). The only common error made in differentiating was in handling $y = 2/x^2$ in Task C7 (Item 32). In the differentiation of $2/x^2$, twenty students gave $-4/x$ as their answer instead of $-4/x^3$. The errors which were made appeared to be executive.

Failures of interpretation of negative and zero rates of change occurred in responses to Item 35 (Task D3). Twelve students could not respond at all when asked to interpret $dy/dx = -2$, and a further ten students could only say "decreasing," or "decreasing gradient," or similar, rather than "decreasing function." Twelve students were unable to interpret $dy/dx = 0$, yet in Task D7, six of these students were able to put $dy/dx = 0$ and obtain values of x for stationary points, so they must have experienced some confusion in one context but not in the other. A few students made careless numerical errors in substituting $x = 2$ into $dy/dx = 2x - 4$. Overall, the many errors of interpretation were structural, but executive errors were also committed.

Two elementary applications of differentiation encountered early in a school study of calculus were tested in Task D7 (Items 36 and 37). Item 36 concerned obtaining the gradient of the tangent to $y = x^3 - 3x^2 + 4$ at $x = 3$ and was answered quite well by most students. Numerical errors were rare, but six such executive errors did occur in substituting $x = 3$ into $dy/dx = 3x^2 - 6x$. Item 37 concerned obtaining the coordinates of the stationary points of $y = x^3 - 3x^2 + 4$ and identifying their nature. The most striking errors made by students have already been discussed and concerned elementary algebra in the solution of $3x^2 - 6x = 0$. Other errors were made

TABLE II

Classification of errors

Item	Description	Errors		
		Structural	Executive	Arbitrary
3	Infinite geometric sequences	√		
4	Limits of geometric sequences	√		
21	Substitution and increases from equations	√	√	
27	Rate of change from straight line graph	√		
28	Rate, average rate and instantaneous rate	√		
29	Average rate of change from curve	√	√	
32	(35, 36) Carrying out differentiation		√	
33	Differentiation as a limit	√		
34	Use of δ-symbolism	√		
35	Significance of rates of change from differentiation	√	√	
36	Gradient of tangent to curve by differentiation	√	√	
37	Stationary points on a graph	√		√
37	Algebraic	√	√	

by only one or two students and appeared to be either structural or arbitrary. A very significant feature of responses to this item was the widespread use of d^2y/dx^2 to determine the nature of the stationary points.

DISCUSSION AND CONCLUSIONS

There are a number of implications for the school curriculum and teaching methods suggested by the results of the study. It is known that some students are introduced to differentiation as a rule to be applied without much attempt to reveal the reasons for and justifications of the procedure. Many regard this as bad educational practice, and, in fact, it should not be necessary. However, the question of how much of the approach to calculus should be attempted before the sixth form still remains, and this will be referred to throughout the discussion which follows.

The electronic calculator should be of great benefit in the approach to differentiation. Consider the graph of $y = x^2$. The gradient or rate of change or derivative at a point, P, on the curve may be studied through the gradients of secants PQ, where Q takes a variety of positions on the curve. This is not a new idea. The difference is now that the electronic calculator enables pupils to calculate differences and ratios more quickly and more accurately. Recent papers, such as Neill (1978), have already drawn attention to this important development. The outcome of the procedure ought to be the conviction that the gradient (rate of change) at, say, $x = 1$ is 2. Further investigations, at $x = 2$, 3, ..., leads eventually to the acceptance of a pattern and ultimately the formula $g = 2x$. Of course, the support of background studies of rate of change and tangents is vital. Care must also be taken that students do understand the tangent as the limit of the secants; teaching must aim to avoid confusion about disappearing chords (Orton, 1977). The same procedure can then be adopted for other curves, such as $y = x^3$, $y = x^3 + x^2$, and so on. By such means students may discover the gradient formulae for a variety of polynomial functions and thus discover the general principle involved. No algebraic proof has so far been necessary and this may be an adequate first introduction to differentiation, though the way is then clear for more able and mature students to consider a more general proof.

It has been suggested already that one of the problems of learning about rate of change is that the ideas are basically concerned with ratio and proportion. Pupils' difficulties in this mathematical topic are well documented, see, for example, the chapter on ratio and proportion in Hart (1981). We cannot, therefore, necessarily expect to be able to find new ways to make the numerical aspect of rate of change a more accessible idea to pupils. It is possible that a more lively approach to the teaching of ratio might help, and many ideas for class discussion of ratio have emerged over recent years from the test questions devised. We must certainly take every opportunity to lay foundations of ideas of rate of change throughout a pupil's school life and, as with limits, not leave the study of this important idea until it is required in order to make sense of differentiation.

Graphical work is also of great importance in developing concepts of rate of change. However, pupils' graphical understanding may be limited, as revealed, for example, by Kerslake in Hart (1981). Nevertheless we should take the attitude that a developing understanding of rate of change should go hand in hand with a developing understanding of ratio and of graphical representation. Real-life situations need to be used as the data for graphs before more algebraic approaches are used. Non-linear as well as linear graphs should also be introduced. The relationship between the tangent at a point on a curve and rate of change must be investigated, and this needs to follow studies of average rate, gradient of a line and secants to the curve. We also need to introduce the idea of points on the curve where the function is increasing, where it is decreasing, and where it is increasing or decreasing most rapidly. At the same time it is important to link such considerations with the nature of the numerical measure of the gradient of the curve, whether it is positive or negative, whether it is numerically large or small. An elementary grasp of what is meant by stationary points, turning points, minimum points, maximum points and points of inflexion, and the nature of the gradient or rate of change at such points, may all be obtained by pupils at this stage. A second, analytic, look at such points, using elementary calculus, should be much more meaningful if it has all been studied previously from a purely graphical point of view.

Few of the suggestions so far included, concerning rate of change, are new, let alone revolutionary. However, the writer believes that no opportunity should be lost by teachers to develop these ideas, and that it is wrong to attempt to introduce calculus without a long and persistent study of graphs and rate of change. The same applies to ideas of limit. Moves towards a common syllabus at 16 + in Britain may involve removing the small element of calculus. This could have both advantages and disadvantages. There are certainly arguments for and against including some calculus before the age of 16. The main argument for is that as many pupils as possible should see something of the power of this very important branch of mathematics. The main argument against is that it is too difficult for all but a few pupils before the age of 16. These arguments are elaborated in Shuard and Neill (1977). The main danger in removing calculus, as the writer sees it, is that teachers, freed from the worry of eventually having to get to calculus, will do even less work with their pupils on rate of change and limit. It is to be hoped that they will do more, not less. Perhaps the real challenge lies for those who set external examination questions, in testing rates of change, limits and properties of graphs of functions without calculus. It is worth pointing out that many of the college students interviewed in the study appeared to have ceased to think about rate of change. Their studies of calculus, or analysis, had moved on, and they had lost touch with the earlier considerations, for example through graphical work. This leads on to my next consideration.

An obvious educational implication of the study is that the foundations of calculus need to be returned to and developed anew at various times throughout the students' mathematical education. A first approach to differentiation may be very informal and may be based largely on numerical and graphical explorations assisted by an electronic calculator. This may be all that is appropriate before the age of 16; the rule for differentiation is not then given as a rule to be learned, it is discovered as a summary of investigatory work carried out by students. Then, in later years, those who continue to study mathematics can be taken back to look in a different, more abstract, way at the results which they have been using. Revision, extension and redevelopment of the approach to calculus is important at each stage of education.

It has already been mentioned that the majority of students chose to investigate the nature of the stationary points of $y = x^3 - 3x^2 + 4$ in Task D7 by using the second derivative. This method works for the kinds of functions traditionally set on examination papers at 16 +. However, the method does not always work, and in later study specialists will meet many functions for which the procedure is inappropriate. The question is whether a method which is only applicable in the short-term can be justified. First methods may be retained by students and not replaced by more appropriate ones. The problem would be solved by not using the second derivative at this early stage in the study of differentiation. The more investigatory approach to differentiation advocated earlier would suggest that stationary points should be studied graphically and through the nature of the gradient of the curve on each side of the stationary point.

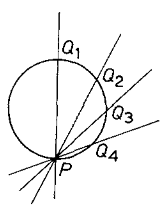
The symbols of differentiation and the approach to differentiation were clearly badly understood by the students. Unfortunately, the difficulties cannot be avoided if external examinations demand the use of the standard symbols. Early studies of gradients and tangents may incorporate the use of, perhaps, h and k, but eventually δx and δy need to be introduced in order to explain $\mathrm{d}y/\mathrm{d}x$. A careful programme of introducing symbols over a period of time is important, but so also is constant revision and reconsideration of the origin of the symbols. Where possible, it seems appropriate to leave the standard symbols until a second stage approach to calculus.

Finally, it is necessary to draw attention to the problems of algebra. There were clear indications from the study that the extent to which algebra is used in introducing aspects of calculus should be kept to a minimum. It certainly appeared that algebraic difficulties could be obscuring the ideas of calculus. Some of the confusion caused by algebraic situations in tasks of the study appeared to be caused by conceptual difficulties, but there was an even stronger impression that many students did have a reasonable understanding but they could not carry out the procedures they had in mind without error. It is perhaps over algebraic errors that foreknowledge on the part of the teacher is very valuable. Some errors may then be avoided by skilful teaching, others may be anticipated and quickly corrected.

APPENDIX: A SELECTION OF TASKS CONCERNING DIFFERENTIATION

TASK A5

The diagram shows a circle and a fixed point P on the circle. Lines PQ are drawn from P to points Q on the circle and are extended in both directions. Such lines across a circle are called *secants*, and some examples are shown in the diagram.

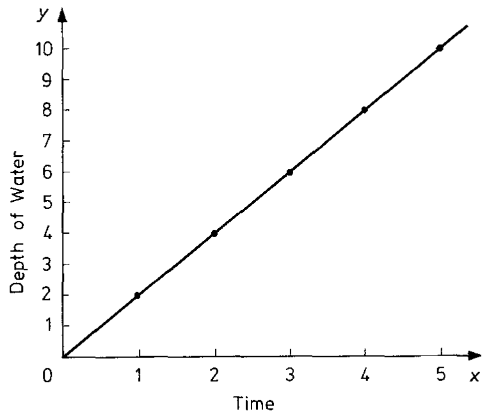


(a) How many different secants could be drawn in addition to the ones already in the diagram? (Item 3).
(b) As Q gets closer and closer to P what happens to the secant? (Item 4).

TASK C3

Water is flowing into a tank at a constant rate, such that for each unit increase in the time the depth of water increase by 2 units. The table and graph illustrate this situation.

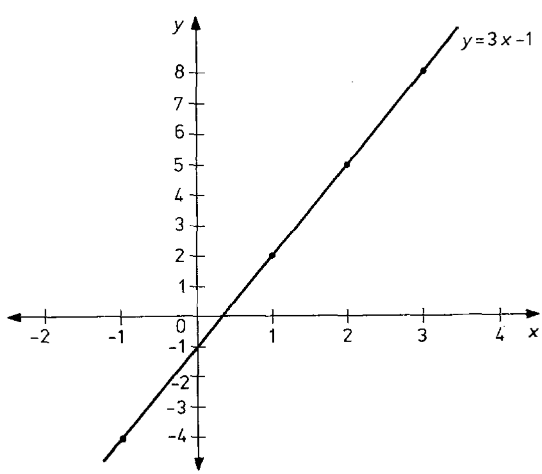
Time (x)	0		1		2		3		4		5
Depth (y)	0		2		4		6		8		10
1st difference (depth)		2		2		2		2		2	



(v) What is the rate of increase in the depth when $x = 2\frac{1}{2}$? when $x = T$? (Item 27).

TASK C4

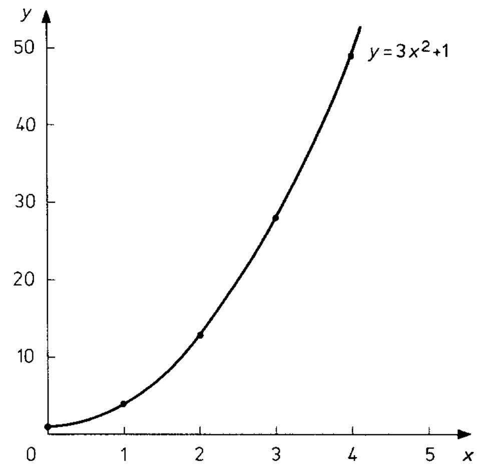
The graph below represents $y = 3x - 1$.



(i) What is the value of y when $x = a$? (Item 21).
(ii) What is the value of y when $x = a + h$? (Item 21).
(iii) What is the increase in y as x increases from a to $a + h$? (Item 21).
(iv) What is the rate of increase of y as x increases from a to $a + h$? (Item 28).
(vii) What is the rate of increase of y at $x = 2\frac{1}{2}$? at $x = X$? (Item 27).

TASK C6

The graph below represents $y = 3x^2 + 1$, from $x = 0$ to $x = 4$.



(i) What is the value of y when $x = a$? (Item 21).
(ii) What is the value of y when $x = a + h$? (Item 21).
(iii) What is the change in y as x increases from a to $a + h$? (Item 21).
(iv) What is the average rate of change in y in the x-interval a to $a + h$? (Item 28).
(v) Can you use the result of (iv) to obtain the rate of change of y at $x = 2\frac{1}{2}$? at $x = X$? If so, how? (Item 28).

TASK C7 (Item 32)

(ii) What is the formula for the rate of change for the equation $y = x^n$?
(iii) What is the rate of change formula for each of the following equations:

$$y = 3x^3?$$

$$y = 4?$$

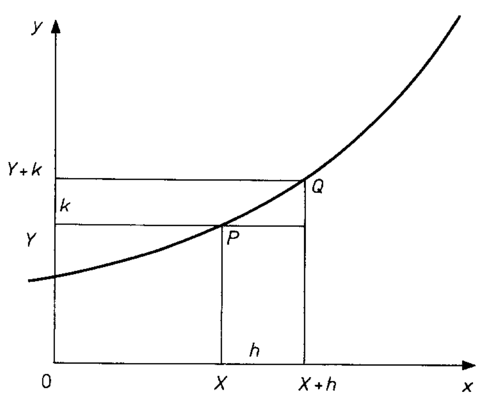
$$y = \frac{2}{x^2}?$$

TASK C8 (Item 33)

The diagram below is one that is commonly used to introduce the following definition for derivative or differentiation:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{k}{h}$$

(i) At which point or points of the graph does the formula measure the rate of change?
(ii) Explain why the formula defines this rate of change.

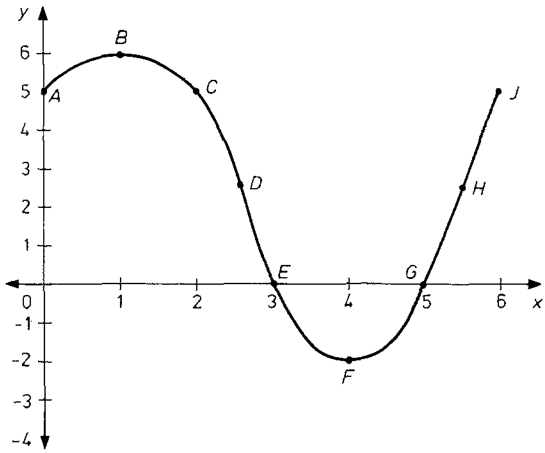


TASK D1 (Item 29)

The graph of y for a certain equation, for $x = 0$ to $x = 6$, is shown below.

What is the average rate of change of y with respect to x,

(vi) From A to B?
(vii) From B to E?
(viii) From A to J?



TASK D2 (Item 34)

Explain the meaning of each of the following symbols:

(i) δx,
(ii) δy,
(iii) $\dfrac{\delta y}{\delta x}$,
(iv) $\mathrm{d}x$,
(v) $\mathrm{d}y$,
(vi) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.
(vii) What is the relationship between $\dfrac{\delta y}{\delta x}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$?

TASK D3 (Item 35)

In each of the following, calculate the rate of change at the point indicated, and explain the significance of your answer:

(i) $y = x^2 - 4x + 1$ at $x = 1$,
(ii) $y = x^2 - 4x + 1$ at $x = 2$,
(iii) $y = \dfrac{1}{x}$ at $x = 0$.

TASK D7

Find the coordinates of the point or points on the curve

$$y = x^3 - 3x^2 + 4$$

at which there is a turning point or stationary point. Determine also what kind of point you have found. (Item 37).

Find also the gradient of the tangent to the curve when $x = 3$. (Item 36).

School of Education, University of Leeds

REFERENCES

Donaldson, M.: 1963, *A Study of Children's Thinking*, Tavistock Publications, London, pp. 183–185.

Hart, K. M.: 1981, *Children's Understanding of Mathematics: 11–16*, John Murray, London.

Janvier, C.: 1978, The interpretation of complex Cartesian graphs representing situations – studies and teaching experiments, Unpublished Ph.D. thesis, University of Nottingham.

Neill, H.: 1978, *The Use of Electronic Calculators in the Sixth Form*, Durham County Education Committee, Durham.

Orton, A.: 1977, 'Chords, secants, tangents and elementary calculus', *Mathematics Teaching* 78, 48–49.

Orton, A.: 1980, A cross-sectional study of the understanding of elementary calculus in adolescents and young adults, Unpublished Ph.D. thesis, University of Leeds.

Orton, A.: 1983, 'Students' understanding of integration', *Educational Studies in Mathematics* **14**, 1–18.

Shuard, H. and Neill, H.: 1977, *The Mathematics Curriculum: From Graphs to Calculus*, Blackie, Glasgow, pp. xii–xiii.