A COMPLETELY INTEGRABLE MECHANICAL SYSTEM

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ABSTRACT. We present a new completely integrable classical mechanical system, that of a particle constrained to a sphere with potential

$$U = \quad a_i x_i^2 + \beta \left[\sum \frac{x_i^2}{a_i}\right]^{-1}.$$

In recent years there has been a revived interest in completely integrable Hamiltonian systems. These systems are the exception rather than the rule, possessing sufficiently many conserved quantities so that it is possible (in principle) to integrate the equations of motion. The revival in interest has been brought about by the discovery of infinite sequences of conserved functionals associated with some partial differential equations (such as the Korteweg–de Vries, non-linear Schrödinger, Toda and sine-Gordon) which may be viewed as completely integrable systems with infinitely many degrees of freedom [1]. Such investigations have shown surprising links with scattering theory, and its application has led to several n-body one-dimensional systems being shown exactly integrable [2]. The hope is that such 1 + 1-dimensional systems may give insight into the 3 + 1-dimensional world, and that perhaps some realistic field theories have similar properties.

However, the number of known completely integrable systems in more than one dimension – be they many body, or single body systems – is few. Although they can be cast into the scattering theoretic guise developed for one-dimensional systems, their discovery has come by explicitly separating the variables in the Hamilton–Jacobi equations [3]. We present here another such potential with the hope that its hidden symmetries will give some insight into generalizing the one-dimensional theory.

We consider the motion of a mass-point on a sphere. Neumann [4] showed this completely integrable for a potential $U_1 = \Sigma a_i x_i^2$. Moser [5] discusses the integrable system of Rosochatius with potential $U_3 = \Sigma c_j x_j^{-2}$ showing how this and Jacobi's free motion of a particle on an ellipsoid are related to Neumann's system. The potential we consider is $U_2 = [\Sigma x_i^2/a_i]^{-1}$. This appears the generalization of the $1/r^2$ potential. Indeed, we show the potential $U = \alpha U_1 + \beta U_2$ is integrable.

A natural co-ordinate system for such problems is the generalization of sphero-conal coordinates. These are the Jacobi co-ordinates $\xi_1, \ldots, \xi_{n-1}$ and r, where

$$\sum_{i=1}^{n} x_1^2 = r^2 \tag{1a}$$

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$$\sum_{i=1}^{n} \frac{x_1^2}{a_i - \xi_p} = 0, \quad 1 \leqslant p \leqslant n-1 \tag{1b}$$

and we have taken $a_1 > a_2 \geqslant a_n \geqslant 0$. The ξ_p may be ordered $a_p > \xi_p > a_{p+1}$. Defining

$$P(\tau) = \prod_{i=1}^{n-1} (\tau + \xi_i) \tag{2a}$$

$$Q(\tau) = \prod_{i=1}^{n} (\tau + a_i) \tag{2b}$$

then

$$X_s^2 = r^2 P(-a_s)/Q'(-a_s) \tag{3}$$

and

$$\mathrm{d}s^2 = \mathrm{d}r^2 - \frac{r^2}{4} \sum_{p=1}^{n-1} \frac{P'(-\xi_p)}{Q(-\xi_p)}\, \mathrm{d}\xi_p^2. \tag{4}$$

In this coordinate system the Hamiltonian for the particle on the sphere $r = 1$ may be written

$$H = -2 \sum_{p=1}^{n-1} \frac{Q(-\xi_p)}{P'(-\xi_p)}\, v_p^2 + U \tag{5}$$

where v_p is the canonically-conjugate velocity, and U is the potential. Assuming U is time-independent, the Hamilton–Jacobi equation becomes

$$H\left(\xi_p, \frac{\partial W}{\partial \xi_p}\right) = \eta_1. \tag{6}$$

Here W, Hamilton's characteristic function, is the generating function [3] of the canonical transformation $(\xi, v) \to (u, \eta)$

$$v_p = \frac{\partial W}{\partial \xi_p}, \quad u_p = \frac{\partial W}{\partial \eta_p}. \tag{7}$$

The Hamilton–Jacobi theory leads us to solve the first-order partial differential (6); the $n-1$ constants of integration η_i becoming the transformed velocities; a system is completely integrable when this equation is separable. However, the task of finding U such that (6) is separable is not straightforward. The remainder of this letter describes a new potential, generalizing an existing one, such that this is so.

Suppose now

$$\tfrac{1}{2}U = \sum a_i x_i^2 + \frac{\beta}{a_1 \ldots a_n}\left[\sum \frac{x_i^2}{a_i}\right]^{-1}. \tag{8a}$$

Here we scale away any multiplicative factor to the first sum of the potential. This equation may be rewritten as

$$\tfrac{1}{2}U = \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} \xi_i\right) + \beta \sum_{i=1}^{n-1} \frac{1}{\xi_i} \cdot \frac{1}{p'(-\xi_i)}. \tag{8b}$$

The second term is the new one, and this may be obtained from (1) and (2) and using

$$\prod_{r=1}^{l} z_r^{-1} = \sum_{k=1}^{l} z_k^{-1} \prod_{i \neq k}^{l} (z_i - z_k)^{-1}. \tag{9}$$

The Hamilton–Jacobi equation (6) for this potential becomes

$$O = \sum_{p=1}^{n-1} \frac{1}{p'(-\xi_p)} \left\{-Q(-\xi_p)\left(\frac{\partial W}{\partial \xi_p}\right)^2 + \left(\sum_{i=1}^{n} a_i - \tfrac{1}{2}\eta_1\right)\xi_p^{n-2} - \beta_p^{n-1} + \frac{\beta}{\xi_p}\right\}. \tag{10}$$

This equation holds, if and only if

$$Q(-\xi_p)\left(\frac{\partial W}{\partial \xi_p}\right)^2 = -\xi_p^{n-1} + \left(\sum_{i=1}^{n} a_i - \tfrac{1}{2}\eta_2\right)\xi_p^{n-2} + \eta_2 \xi_p^{n-3} + \cdots + \eta_{n-1} + \frac{\beta}{\xi_p}$$
$$\equiv F(\xi_p). \tag{11}$$

Thus

$$W = \sum_{p=1}^{n-1} \int^{\xi_p} \sqrt{\frac{F(z)}{Q(-z)}}\, \mathrm{d}z. \tag{12}$$

Equation (10) shows that the Hamilton–Jacobi equation is separable for the potential $U = \alpha U_1 + \beta U_2$. The constants of integration η_i, $i \leqslant n-1$, may be taken as the transformed velocities. The hyperelliptic function W is the characteristic function for the system defined, and by using (7) the equations of motion may be written

$$\delta_{j1} = \tfrac{1}{2} \sum_{p=1}^{n-1} \left.\frac{z^{n-j-1}}{\sqrt{F(z)Q(-z)}}\right|_{z=\xi_p} \frac{\mathrm{d}\xi_p}{\mathrm{d}t}, \quad 1 \leqslant j \leqslant n-1. \tag{13}$$

We note that the characteristic function W has the same form as the $n+1$-dimensional Neumann problem with $a_{n+1} = 0$; it is also that of the particle on the ellipsoid with potential $U = k/x^2$ [6]. (The Hamilton–Jacobi equations giving W are, however, different.)

We have shown therefore, that a particle moving on a sphere with potential of the form

$$U = \sum a_i x_1^2 + \beta \left[\sum \frac{x_i^2}{a_i} \right]^{-1}$$

is completely integrable. This extends the class of completely integrable systems, generalizing the $1/r^2$ potential. The quantum mechanics of the problem is currently being examined.

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