W^*-DYNAMICS OF INFINITE QUANTUM SYSTEMS

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ABSTRACT. We prove that the dynamics of an infinite quantum system, as formulated in the Schrödinger representation within the framework constructed in an earlier work [1], corresponds to *-automorphisms of the W^*-algebra dual to the space of its *physical* states.

The dynamics of infinite quantum systems is generally taken to be given by some appropriate limit of that of corresponding finite ones [1] – [9]. Accordingly, it has been established, with the C^*-algebraic formulation of quantum theory, that in the simplest ones, e.g., lattice systems with finite range forces, the dynamics corresponds to automorphisms of the observables [10], [11]; while, more generally, this is not so [3]–[5], [9], [12], [13], and time-translations in the island* of a Gibbs state correspond in some cases to automorphisms [3], [4], [9], in others to non-automorphic isomorphisms [13], of the weak closure of the associated GNS representation of the algebra of observables.

The object of the present note is to show that the dynamics of an infinite quantum system, as formulated in the Schrödinger representational framework that we constructed some time ago [1], corresponds quite generally – not merely in the Gibbs islands – to automorphisms of the W^*-algebra, dual to the subset of states designated to be the physical ones. We shall summarise the essentials of the framework of Reference [1], and then present our theorem concerning the W^*-dynamics of infinitely extended systems. It will then be seen that the present scheme generalises previous results on the dynamics of such systems. The mathematical key to our result is the classic treatment by Kadison [14] of the relationship between the Schrödinger and Heisenberg pictures in quantum theory.

We formulate the observables of an assembly of particles occupying the space $\Gamma(= \mathbb{R}^\nu$ or $\mathbb{Z}^\nu)$ in the following standard manner [2], [15]. We define L to be the family $\{\Lambda\}$ of bounded open subsets of Γ, and we assign to each Λ $(\in L)$ a W^*-algebra $\mathscr{A}(\Lambda)$, corresponding to the observables for the region Λ, and possessing the natural isotony property that $\mathscr{A}(\Lambda) \supset \mathscr{A}(\Lambda')$ if $\Lambda \supset \Lambda'$. We then define $\mathscr{A}_L$, the normed *algebra of local observables, to be $\cup_{\Lambda \in L} \mathscr{A}(\Lambda)$; and $\mathscr{A}$, the C^*-algebra of quasi-local bounded observables, to be the norm completion of $\mathscr{A}_L$.

We assume that each of the local subalgebras, $\mathscr{A}(\Lambda)$, is equipped with a weakly continuous

*By the 'island' of a state ω on a C^*-algebra $\mathscr{A}$, we mean the family of states on $\mathscr{A}$ that correspond to density matrices in the GNS representation space of ω.

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one-parameter group $\{\alpha_t^{(\Lambda)} \mid t \in \mathbb{R}\}$ of inner automorphisms, corresponding to time translations of a system of particles of the given species confined to Λ. In order to specify the physical states and dynamics of the system in terms of the family of groups $\alpha^{(\Lambda)}$, we introduce the following definitions.

A set $\mathscr{F}$ of positive, normalised, linear functionals on $\mathscr{A}$ is termed a *folium* [17] if it is convex, norm-closed and stable under the transformations $\omega \to \omega_B$, with $\omega_B = \omega(B^*(.)B)/\omega(B^*B)$ and $B \in \mathscr{A}_L$. A pair $(\mathscr{F}, \tau)$ is termed a *dynamical folium* [1] if $\mathscr{F}$ is a folium and $\{\tau_t \mid t \in \mathbb{R}\}$ is a one-parameter group of affine transformations of $\mathscr{F}$. This pair is termed a *physical folium* [1] if, furthermore, $\mathscr{F}$ consists of locally normal states on $\mathscr{A}$ (i.e., states whose restrictions to the sub-algebras $\mathscr{A}(\Lambda)$ are normal), and τ is induced by the local dynamical groups $\{\alpha^{(\Lambda)}\}$ according to the following formula:

$$(\tau_t\omega)(A) = \lim_{\Lambda \uparrow} \omega(\alpha_t^{(\Lambda)}A) \quad \forall \omega \in \mathscr{F}, A \in \mathscr{A}_L, t \in \mathbb{R}, \tag{1}$$

where the limit is taken over an increasing absorbing sequence of elements of L. Thus, a physical folium $(\mathscr{F}, \tau)$ may be characterized by the following properties, which we regard as natural desiderata for the physical states and dynamics of a system.

(1) $\mathscr{F}$ is closed with respect to convex combinations, i.e., if ω_1 and $\omega_2 \in \mathscr{F}$, then mixtures of these states also belong to $\mathscr{F}$.
(2) $\mathscr{F}$ is stable under the localised modifications $\omega \to \omega_B$.
(3) $\mathscr{F}$ consists of locally normal states, i.e., [16] ones for which there is zero probability of finding an infinite number of particles in a bounded region of space.
(4) The dynamical law, given by τ, takes a 'natural' limiting form of that of corresponding finite systems.

The above definitions imply [1] that there is a unique maximal physical folium $(\tilde{\mathscr{F}}, \tilde{\tau})$ of the system, characterised by the properties that, if $(\mathscr{F}, \tau)$ is any physical folium, then $\mathscr{F} \subset \tilde{\mathscr{F}}$ and $\tau = \tilde{\tau}_{|\mathscr{F}}$. Accordingly, we take $\tilde{\mathscr{F}}$ to be the *physical state space* of the system, and we take $\tilde{\tau}$ to represent its dynamics.

We now note (cf. [17]) that the linear space $[\mathscr{F}]$ of any folium $\mathscr{F}$ is the predual of a W^*-algebra $[\mathscr{F}]^*$, which may be canonically identified with the weak closure $\overline{\pi(\mathscr{A})}$ of any representation π of $\mathscr{A}$ belonging to the quasi-equivalence class of that given by the direct sum $\oplus_{\omega \in \mathscr{F}} \pi_\omega$ of the GNS representations π_ω induced by the states in $\mathscr{F}$. Thus, each element, ω of $\mathscr{F}$ corresponds to a unique normal state, $\hat{\omega}$, on $[\mathscr{F}]^*$, with

$$\omega(A) = \hat{\omega}(\hat{A}) \quad \text{and} \quad \hat{A} = \pi(A), \tag{2}$$

and τ induces a group $\{\hat{\tau}_t \mid t \in \mathbb{R}\}$ of transformations of the normal linear functionals on $[\mathscr{F}]^*$ according to the formula

$$\hat{\tau}_t\hat{\omega} = \widehat{\tau_t\omega} \tag{3}$$

The following theorem tells us that, if $(\mathscr{F}, \tau)$ is a physical folium, then $\hat{\tau}$ induces a dual group $\{\hat{\alpha}_t \mid t \in \mathbb{R}\}$ of *-automorphisms of the W^*-algebra $[\mathscr{F}]^*$. We note here that Roos [18] obtained a similar result for the case where $\mathscr{F}$ is a the island of a KMS state, as described in the Schrödinger picture, and τ is a group of affine transformations of $\mathscr{F}$ with suitable continuity properties.

THEOREM. *Let $(\mathscr{F}, \tau)$ be a physical folium. Then, with the above definitions and assumptions, $\hat{\tau}$ is the predual of a one-parameter group of *-automorphisms of $[\mathscr{F}]^*$, i.e.,*

$$(\hat{\tau}_t\hat{\omega})(B) = \hat{\omega}(\hat{\alpha}_t B) \quad \forall \omega \in \mathscr{F},\ B \in [\mathscr{F}]^*,\ t \in \mathbb{R}, \tag{4}$$

where $\{\hat{\alpha}_t \mid t \in \mathbb{R}\} \subset \mathrm{Aut}\,[\mathscr{F}]^*$.

The following corollary is an immediate consequence of this theorem.

COROLLARY. *The dynamics of the entire system corresponds to a one-parameter group of *-automorphisms of $[\widetilde{\mathscr{F}}]^*$, dual to $\widetilde{\tau}$, where $(\widetilde{\mathscr{F}}, \widetilde{\tau})$ is its maximal physical folium.*

COMMENTS. We note here that previous results fit into the general picture provided by the above theorem and corollary in the following ways.

(1) Lattice systems for which the local algebras, $\mathscr{A}(\Lambda)$, are finitely generated, and the forces are sufficiently tempered to ensure that, for each A in $\mathscr{A}_L$, $\alpha_t^{(\Lambda)}A$ converges in norm to $\alpha_t A$, where α_t is thus a *-automorphism of $\mathscr{A}$ [10], [11]. In this case, $\widetilde{\mathscr{F}}$ consists of all positive normalised linear functions on $\mathscr{A}$, $[\widetilde{\mathscr{F}}]^*$ is the weak closure of the universal representation $\widetilde{\pi}$ of $\mathscr{A}$, $\widetilde{\tau}_t$ is the dual of α_t, while the restriction to $\widetilde{\pi}(\mathscr{A})$ of $\widetilde{\alpha}_t$, the dual of τ_t, is given by $\widetilde{\alpha}_t\widetilde{\pi}(A) = \widetilde{\pi}(\alpha_t A)$.

(2) Systems satisfying the assumptions of References [3] or [9] for a Gibbs state ω. These assumptions lead to a physical folium $(\mathscr{F}_\omega, \tau)$, with $\mathscr{F}_\omega$ the island of ω and τ the predual of a one-parameter group of automorphisms of $[\mathscr{F}_\omega]^* \equiv \pi_\omega(\mathscr{A})$, where π_ω is the GNS representation for ω.

(3) Systems for which the dynamics in the island $\mathscr{F}_\omega$ of a Gibbs state ω correspond to non-automorphic isomorphisms of $\overline{\pi_\omega(\mathscr{A})}$, of the type inferred by Narnhofer [13]. In this case, one has a physical folium $(\overline{\mathscr{F}_\omega}, \tau)$, where $\overline{\mathscr{F}_\omega}$ is the minimal folium containing $\{\tau_t \mathscr{F}_\omega \mid t \in \mathbb{R}\}$; and further $\mathscr{F}_\omega$ is not stable under τ. Correspondingly, $[\overline{\mathscr{F}_\omega}]^*$ is the weak closure of a representation $\overline{\pi}$ of $\mathscr{A}$, of which π_ω is a subrepresentation and $\overline{\pi_\omega(\mathscr{A})}$ is not stable under the dynamical group $\hat{\alpha}$, dual to τ.

PROOF OF THEOREM. Identifying $[\mathscr{F}]^*$ with the weak closure of a representation π of $\mathscr{A}$ in a Hilbert space $\mathcal{H}$, we define $\hat{\alpha}_t^{(\Lambda)}$ to be the *-automorphism of $\pi(\mathscr{A}(\Lambda))$ given by

$$\hat{\alpha}_t^{(\Lambda)}\pi(A) = \pi(\alpha_t^{(\Lambda)}A) \quad \forall A \in \mathscr{A}(\Lambda). \tag{5}$$

Thus, it follows from Equations (1), (3) and (5) that if f is a normal linear functional on $\overline{\pi(\mathscr{A})}$ and B belongs to $\pi(\mathscr{A}_L)$, then

$$(\hat{\tau}_t f)(B) = \lim_{\Lambda\uparrow} f(\hat{\alpha}_t^{(\Lambda)} B).$$

Hence, as $\hat{\alpha}_t^{(\Lambda)} \in \mathrm{Aut}\, \pi(\mathscr{A}(\Lambda))$,

$$|(\hat{\tau}_t f)(B)| < \|f\|\, \|B\|$$

for all linear functionals f on $\overline{\pi(\mathscr{A})}$ and $B \in \pi(\mathscr{A}_L)$. Further, the formula may be extended by continuity to all B in $\overline{\pi(\mathscr{A})}$, in view of the fact that the unit ball of $\pi(\mathscr{A})$, and thus of $\overline{\pi(\mathscr{A}_L)}$, is weakly dense in that of $\pi(\mathscr{A})$. Therefore, for any B in $\overline{\pi(\mathscr{A})}$, the mapping $f \to (\hat{\tau}_t f)(B)$ corresponds to a continuous linear functional on the predual of $\overline{\pi(\mathscr{A})}$ and thus to an element of that W^*-algebra. Hence $\hat{\tau}$ induces a one-parameter group $\{\hat{\alpha}_t \mid t \in \mathbb{R}\}$ of linear transformations of $\overline{\pi(\mathscr{A})}$ according to the formula

$$(\hat{\tau}_t \hat{\omega})(B) = \hat{\omega}(\hat{\alpha}_t B) \quad \forall \omega \in \mathscr{F},\ B \in \overline{\pi(\mathscr{A})}. \tag{6}$$

Hence, it follows from a theorem of Kadison [14, Corollary 4.7] that the transformations $\hat{\alpha}$ are Jordan*-automorphisms of $\overline{\pi(\mathscr{A})}$, i.e.,

$$\hat{\alpha}_t(B)^* = \hat{\alpha}_t(B^*) \quad \forall B \in \overline{\pi(\mathscr{A})} \tag{7}$$

and

$$\hat{\alpha}_t(B^2) = \hat{\alpha}_t(B)^2 \quad \forall B = B^* \in \overline{\pi(\mathscr{A})}. \tag{8}$$

Consequently, if B is an arbitrary self-adjoint element of $\pi(\mathscr{A}_L)$ and ω an element of $\mathscr{F}$, then it follows from Equations (1) – (3), (5), (6) and (8) that as $\hat{\alpha}_t^{(\Lambda)} \in \mathrm{Aut}\, \pi(\mathscr{A}(\Lambda))$,

$$\lim_{\Lambda\uparrow} \hat{\omega}(\hat{\alpha}_t^{(\Lambda)} B) = \hat{\omega}(\hat{\alpha}_t B) \quad \text{and} \quad \lim_{\Lambda\uparrow} \hat{\omega}((\hat{\alpha}_t^{(\Lambda)} B)^2) = \hat{\omega}((\hat{\alpha}_t B)^2).$$

Since these equations are valid for all $\omega \in \mathscr{F}$, they imply that

$$W - \lim_{\Lambda\uparrow} \hat{\alpha}_t^{(\Lambda)} B = \hat{\alpha}_t B \quad \text{and} \quad W - \lim_{\Lambda\uparrow} (\hat{\alpha}_t^{(\Lambda)} B)^2 = (\hat{\alpha}_t B)^2$$

and therefore that $s - \lim_{\Lambda\uparrow} \hat{\alpha}_t^{(\Lambda)} B = \hat{\alpha}_t B$ for all self-adjoint elements B of $\pi(\mathscr{A}_L)$ and therefore, by linearity, for all B in $\pi(\mathscr{A}_L)$. Hence, as $\hat{\alpha}_t^{(\Lambda)} \in \mathrm{Aut}\, \pi(\mathscr{A}(\Lambda))$,

$$\hat{\alpha}_t(B_1 B_2) = \hat{\alpha}_t(B_1)\hat{\alpha}_t(B_2) \quad \forall B_1, B_2 \in \pi(\mathscr{A}_L),\ t \in \mathbb{R}. \tag{9}$$

Finally, we see from Equation (6) that, for each $t \in \mathbb{R}$, $\hat{\alpha}_t$ is a weakly continuous transformation of $\overline{\pi(\mathscr{A})}$ and therefore we may extend Equation (9) by continuity to the form

$$\hat{\alpha}_t(B_1 B_2) = \hat{\alpha}_t(B_1)\hat{\alpha}_t(B_2) \quad \forall B_1, B_2 \in \overline{\pi(\mathscr{A})},\ t \in \mathbb{R}, \tag{10}$$

thereby establishing that the group $\hat{\alpha}$ of Jordan *-automorphisms consists indeed of *bona fide* *-automorphisms of $\overline{\pi(\mathscr{A})} \equiv [\mathscr{F}]^*$. QED

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